Math 211  
Final Exam  
Spring 2003

You have **3 hours** to complete this test.

Make sure to show your work and justify your arguments.

**Calculator policy:** You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, $\ln(35/7)$, or $\sin(\pi/17)$). You may not use symbolic operations, numerical integration, or any graphing functions.

1. (10p) Solve the initial value problem

$$y' = \frac{t(t^2 + 1)}{4y^3}, \quad y(0) = -1/\sqrt{2}.$$  

**Solution** This is a separable equation, after separating the variables we obtain

$$4y^3y' = t(t^2 + 1),$$

and integration gives us

$$y^4 = \frac{(t^2 + 1)^2}{4} + C.$$  

At this point we can find $C$ by the initial value condition, we obtain $C = 0$ and the solution is

$$y = -\sqrt{\frac{t^2 + 1}{2}}.$$  

2. (10p) Solve the initial value problem

$$t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0.$$  

**Solution** We divide by $t^3$ to put the equation into normal form then find the integrating factor $e^{\int \frac{4}{t} dt} = t^4$ and the equation becomes

$$t^4y' + 4t^3y = te^{-t}.$$  

This implies that

$$t^4y = -te^{-t} - e^{-t} + C$$

and again we can find $C$ by the initial value condition to be $C = 0$. We obtain that

$$y = -\frac{1 + t}{t^4}e^{-t}.$$
3. (15p) Consider the periodically forced harmonic ordinary differential equation
\[ y'' + 2y' + 2y = 3 \cos 2t. \]
(a) Find a fundamental set of solutions of the associated homogeneous equation.
(b) Find the general solution of the inhomogeneous equation.
(c) Find the steady-state solution and determine its amplitude.
**Solution** (a) The characteristic equation is \( \lambda^2 + 2\lambda + 2 = 0 \), so the eigenvalues are \( \lambda_{1,2} = -1 \pm i \). This means that a fundamental set of solutions is
\[ \{ e^{-t} \cos t, e^{-t} \sin t \}. \]
(b) Using the complex method we try to find a solution in the form \( z = Ae^{2it} \), then check the real part. \( 2i \) is not an eigenvalue, so we will find a particular solution for the equation in this form.) After differentiating twice, we obtain that
\[ -4A + 4iA + 2A = 3, \]
which means \( A = \frac{3}{2 + 4i} = -\frac{3}{10} - \frac{3}{5}i \). So the particular solution we looked for is \(-\frac{3}{10} \cos 2t + \frac{3}{5} \sin 2t \) and the general solution is
\[ y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t - \frac{3}{10} \cos 2t + \frac{3}{5} \sin 2t. \]
(c) The steady-state solution is the previous particular solution, so the amplitude is
\[ \sqrt{\left( \frac{3}{10} \right)^2 + \left( \frac{3}{5} \right)^2} = \frac{3\sqrt{5}}{10}. \]

4. (15p) Find a fundamental set of solutions for the following system:
\[ x' = 3x - y \\
y' = x + y. \]
Also, find the solution to the initial value problem if \( x(0) = 1 \) and \( y(0) = 2 \).
**Solution** The characteristic polynomial is \( \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \), so 2 is a double eigenvalue. The nullspace of the matrix \( A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \) is one dimensional with basis \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). This means that one solution is
\[ \begin{bmatrix} x \\ y \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
The other solution can be found by any of the methods we discussed. Let’s choose \( w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then \((A - 2I)w = v\) and this \( w \) is a generalized eigenvector, and the second solution is
\[
\begin{bmatrix}
x \\
y 
\end{bmatrix} = e^{2t}(w + tv) = e^{2t} \begin{bmatrix} 1 + t \\ t \end{bmatrix}.
\]
These two are linearly independent, so a fundamental set of solutions is
\[
\{e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{2t} \begin{bmatrix} 1 + t \\ t \end{bmatrix}\}.
\]
The solution to the IVP can be found as a linear combination of these two solutions, say \( az_1(t) + bz_2(t)\). The initial value conditions give us the system \( a + b = 1 \) and \( a = 2 \), so \( b = -1 \) and the solution of the IVP is
\[
\begin{bmatrix}
x \\
y 
\end{bmatrix} = \begin{bmatrix} e^{2t} - te^{2t} \\ 2e^{2t} - te^{2t} \end{bmatrix}.
\]

5. (15p) Consider the system of ordinary differential equations
\[
\begin{align*}
x' &= -4y + 2xy - 8 \\
y' &= 4y^2 - x^2.
\end{align*}
\]
(a) Find all equilibrium points.
(b) Compute the Jacobian matrix.
(c) Classify the equilibrium points (as sinks, sources or saddles).

**Solution** (a) The second equation gives us that \( 2y = x \) or \( 2y = -x \). If \( 2y = x \), then the first equation becomes \( x^2 - 2x - 8 = 0 \), which has two solutions, \( x = 4 \) and \( x = -2 \). If \( 2y = -x \), then the first equation becomes \( x^2 - 2x + 8 = 0 \) which has no real solutions. Thus, we obtained two equilibrium points:
\[
(x, y) = (4, 2) \quad \text{and} \quad (x, y) = (-2, -1).
\]
(b) The Jacobian matrix is
\[
J = \begin{bmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}
\end{bmatrix} = \begin{bmatrix}
2y & -4 + 2x \\
-2x & 8y
\end{bmatrix}.
\]
(c) The Jacobian at the equilibrium point \((4, 2)\) is \( \begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix} \), so the eigenvalues are 12 and 8 and this is a (nodal) source. The Jacobian at the equilibrium point \((-2, -1)\) is \( \begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix} \), so the eigenvalues are \(-5 + \sqrt{23}i\) and \(-5 - \sqrt{23}i\) and this is a (spiral) sink.
6. (10p) Find the nullspace of the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 2 & 1 & 0 \\
0 & 4 & 3 & -4
\end{bmatrix}.
\]

**Solution** The reduced row echelon form for this matrix is

\[
A = \begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

We can see from here that the pivots are in column 2 and 3. The other two are free variables, so if we use the coordinates \((x, y, x, v)\) then \(x = t\) is free, \(v = s\) is free and \(z = 4s\) and \(y = -2s\) from the first two rows. This means that the nullspace is

\[
\begin{bmatrix}
x \\
y \\
z \\
v
\end{bmatrix} = t \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
0 \\
-2 \\
4 \\
1
\end{bmatrix}.
\]

7. (10p) Consider the system of ordinary differential equations

\[
\begin{align*}
x' &= -x \\
y' &= 2y + x^2.
\end{align*}
\]

(a) Verify that the solution of the initial value problem where \(x(0) = c\) and \(y(0) = d\) is

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = \left( d + \frac{c}{4} \right) e^{2t} - \frac{c}{4} e^{-2t}.
\]

(b) Show that the set

\[
S = \{ \ (x, y) \in \mathbb{R}^2 : y = -\frac{x^2}{4} \}
\]

is invariant.

**Solution** (a) Differentiate and plug in. Also, \(x(0) = c\) and \(y(0) = d\).

(b) If we start a solution on \(S\), that means that \(y(0) = -x^2(0)/4\) or \(d = -c^2/4\). But then \(x(t) = ce^{-t}\) and \(y(t) = -c^2e^{-2t}/4\), so \(y(t) = -x^2(t)/4\) for every \(t\), so \((x(t), y(t)) \in S\) for every \(t\) and \(S\) is invariant.

8. (15p) Suppose that \(x, y\) and \(z\) are three linearly independent vectors in \(\mathbb{R}^3\). Prove that the vectors \(\{x + y, y + z, z + x\}\) are also linearly independent.
**Solution** Suppose that

$$a(x + y) + b(y + z) + c(z + x) = 0.$$  

(Where \(a, b, c\) are real constants.) This means that

$$(a + c)x + (a + b)y + (b + c)z = 0.$$  

We know these vectors are linearly independent, so \(a + c = 0\), \(a + b = 0\) and \(b + c = 0\). But the only solution this system has for \(a, b, c\) is \(a = 0\), \(b = 0\) and \(c = 0\). (Check this.) So we obtained that the assumption

$$a(x + y) + b(y + z) + c(z + x) = 0$$  

implies that \(a = b = c = 0\). That is the definition of linear independence and we are done.

9. (15p) Consider the system of ordinary differential equations

$$ \begin{align*}
    x' &= -ax + y + ay^2 \\
    y' &= (1-a)x + xy.
\end{align*}$$

(a is a real constant.) Determine the stability of the equilibrium solution (0, 0) for every \(a \in \mathbb{R}\).

**Solution** The Jacobian in this case is

$$J = \begin{bmatrix} -a & 1 + 2ay \\ 1-a+y & x \end{bmatrix}.$$  

At the origin, this matrix is

$$J = \begin{bmatrix} -a & 1 \\ 1-a & 0 \end{bmatrix}.$$  

So the eigenvalues are the roots of the characteristic equation,

$$\det(J - \lambda I) = \det \begin{bmatrix} -a - \lambda & 1 \\ 1-a & -\lambda \end{bmatrix} = \lambda^2 + a\lambda + (a-1) = 0.$$  

The roots of this are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4(a-1)}}{2} = \frac{-a \pm \sqrt{(a-2)^2}}{2} = \frac{-a \pm (a-2)}{2}.$$  

So we get two eigenvalues, \(\lambda = -1\) and \(\lambda = 1 - a\). By our linearization theorem if \(a > 1\), the origin is asymptotically stable, if \(a < 1\), the origin is
unstable. Now if $a = 1$, the linearization theorem does not work. But in this case the system of ODE’s is just

$$
x' = -x + y + y^2
$$
$$
y' = xy.
$$

We can find a solution of this fairly easily if we observe that if $x = y$, then both equations are just $x' = x^2$. But that has the solution $x(t) = -\frac{1}{t + C}$, so if we suppose $x(0) = y(0) = x_0 > 0$, then the solution is $x(t) = y(t) = \frac{x_0}{1 - t x_0}$, which goes to infinity as $t$ approaches $1/x_0$. So the origin is unstable if $a = 1$. (This last part is only for demonstration purposes; if you stated that at $a = 1$ we do not have a result from linearization, you got maximum credit.)