Math 211  
First Midterm  
February 18, 2003

Make sure to show your work and justify your arguments.

**Calculator policy:** You may use calculators to evaluate standard functions on floating point numbers (like $\sqrt{3.12}$, ln(35/7), or sin(π/17)). You may not use symbolic operations, numerical integration, or any graphing functions.

1) Suppose that $y(t) = 2e^{-4t}$ is the solution of the initial value problem

$$y' + ky = 0, \quad y(0) = y_0.$$  

What are the constants $k$ and $y_0$? (14%)  

**Solution** $y' + ky = -8e^{-4t} + 2ke^{-4t} = 0$, so $k = 4$. Also, $y_0 = y(0) = 2$.

2) Consider the differential equation

$$y' = \frac{1 + y}{t^3}.$$  

a) Find the general solution. (9%)  

b) Find a particular solution with $y(1) = 0$ and identify its interval of existence. (6%)  

**Solution** Separable equation; separate the variables and integrate. We obtain

$$\ln(1 + y) = -\frac{1}{2}t^{-2} + C$$

which implies that the general solution is

$$y(t) = Ce^{-\frac{1}{2}t} - 1.$$  

If $y(1) = 0$ we obtain that $C = e^{1/2}$ and the particular solution is

$$y(t) = e^{-\frac{1}{2}t} + \frac{1}{2} - 1.$$  

The interval of existence is $(0, \infty)$.

3) Find a solution to the initial value problem

$$2y' + (\cos t)y = -3 \cos t, \quad y(0) = -4. \quad (14\%)$$

**Solution** This is a linear equation, divide by 2 both sides to get the normal form. The integrating factor is

$$e^\int \frac{\cos t}{2} dt = e^\frac{1}{2} \sin t.$$
We obtain that 
\[ (e^{\frac{1}{2} \sin t} y)' = -\frac{3}{2} e^{\frac{1}{2} \sin t} \cos t \]
which gives us 
\[ e^{\frac{1}{2} \sin t} y = -3e^{\frac{1}{2} \sin t} + C \]
and then 
\[ y(t) = Ce^{-\frac{1}{2} \sin t} - 3. \]
The initial condition gives \( C = -1 \) and the solution for the IVP is 
\[ y(t) = -e^{-\frac{1}{2} \sin t} - 3. \]

4) A tank originally contains 100 gal of fresh water. At time \( t = 0 \), a solution containing 0.2 lb of salt per gallon begins to flow into the tank at a rate of 3 gal/min and the well-stirred mixture flows out of the tank at the same rate.

(a) How much salt is in the tank after 10 min? (6%)

(b) Does the amount of salt approach a limiting value as time increases? If so, what is this limiting value? (6%)

**Solution**

The IVP is 
\[ S' = .6 - .03S, \quad S(0) = 0. \]

This is a separable equation, after integration we obtain that 
\[ S(t) = 20(1 - e^{-0.3t}). \]
Then \( S(10) = 5.1836 \) lb. The limit of \( S \) as \( t \to \infty \) is 20 lb.

5) A model for population growth is given by the equation 
\[ P' = rP \left( \frac{P}{\theta} - 1 \right) \left( 1 - \frac{P}{K} \right), \]
where \( r, \theta \) and \( K \) are given positive constants and \( 2\theta < K \).

(a) Sketch the graph of the function on the right hand side of this differential equation and identify the equilibrium points. (7%)

(b) Draw the phase line and analyze the stability near each equilibrium point. (7%)

(c) Consider the solution \( P(t) \) with initial value \( P(0) = K/2 \). Describe its behavior as \( t \to \infty \). Does it approach any of the equilibrium solutions? (7%)

**Solution**
The zeroes for the function are \( 0, \theta \) and \( K \). Also, the leading coefficient of the cubic is a negative number, so the function is negative between \( 0 \) and \( \theta \) and positive between \( \theta \) and \( K \). This means that \( 0 \) and \( K \) are stable equilibriums
and θ is unstable. If we start a solution at \( K/2 > \theta \), the solution will converge to \( K \) in a monotone increasing way.

6) Can you conclude anything about the existence and uniqueness of the solution(s) of the initial value problem

\[
y' = \frac{1 + y}{t^3}, \quad y(1) = 0 \quad (10\%)
\]

**Solution** The right hand side of the equation and the partial derivative of the right hand side are both continuous in a small enough rectangle around the point \((1,0)\), so we have a unique solution for the IVP. (What we found in the second problem.)

7) Consider the initial value problem

\[
y' = \frac{y - 2}{\sin t - 2} \cos t, \quad y(0) = 1.
\]

Show that the solution \( y(t) \) of this initial value problem satisfies

\[
\sin t < y(t) < 2 \text{ for every } t. \quad (14\%)
\]

**Solution** \( 2 \) and \( \sin t \) are solutions of the differential equation, the initial condition for our solution is between the values of these at the point \( t = 0 \), so by the uniqueness theorem \( \sin t < y(t) < 2 \) for all \( t \).