1. Suppose you are given the differential equation \((t - 3)^2 y' = 3y\).

   (a) (7 points) Is the function \(y(t) = \frac{1}{(t-3)}\) a solution?

   **Solution:** We need to see if the function makes the ODE true. To do this we calculate LHS and RHS of the ODE using the function \(y(t) = \frac{1}{(t-3)}\).

   \[
   \text{LHS}= (t - 3)^2 y' = (t - 3)^2 \left( \frac{d}{dt} \left( \frac{1}{t-3} \right) \right) = (t - 3)^2 \left( \frac{-1}{(t-3)^2} \right) = -1.
   \]

   \[
   \text{RHS}= 3y = 3 \cdot \frac{1}{(t-3)} = \frac{3}{t-3}.
   \]

   Since LHS \(\neq\) RHS the function \(y(t) = \frac{1}{(t-3)}\) is NOT a solution of this ODE.

   (b) (3 points) What is the value of the slope of the tangent line of a solution passing through the point \((t, y) = (4, 1)\)?

   **Solution:** The slope of the tangent line of a solution is the value of the derivative of the solution. Thus if \(y = y(t)\) is our solution we need to calculate \(y'(4, 1)\). But the ODE says that

   \[
   y' = \frac{3y}{(t-3)^2}
   \]

   and so

   \[
   y'(4, 1) = \frac{3(1)}{(4-3)^2} = 3
   \]

   Thus the answer is 3.

   (c) (15 points) Solve the given differential equation with initial condition \(y(0) = 3e\) including the interval of existence (here \(e\) is the base of the natural logarithm i.e. \(e = e^1 = 2.71828\ldots\)).

   **Solution:** This ODE is separable and can be separated to:

   \[
   \frac{dy}{y} = \frac{3dt}{(t-3)^2}
   \]

   Finding the antiderivative of both sides we get:

   \[
   ln|y| = \frac{-3}{t-3} + C, C = \text{constant}
   \]

   Now solve for \(y\):

   \[
   |y| = e^{(-3/(t-3))+C}
   \]

   Then

   \[
   |y| = Ke^{3/(t-3)}, K = e^C
   \]
Then \( y = Ke^{-3/(t-3)} \) if we let \( K = \pm e^C \).

Now apply the initial condition \( y(0) = 3e \) to get \( 3e = Ke^{-3/(0-3)} \) i.e. \( K = 3 \).

Thus the formula for the solution is \( y = 3e^{-3/(t-3)} \).

We also need the interval of existence. During the method we divide by both \( y \) and \( t-3 \) so we need both these quantities to be non-zero. But for our solution, \( y > 0 \) for any value of \( t \) so this is not an issue. However we need \( t \neq 3 \) (in fact the solution is discontinuous at this \( t \)-value also). So our possible intervals are \( t < 3 \) or \( t > 3 \). Since our initial condition is at \( t = 0 \) we must choose the interval \( t < 3 \).

So the final solution is: \( y = e^{-3/(t-3)}, t < 3 \).

2. Suppose a mass of 4kg is moving under the influence of gravity (use \( g = 10 \) m/s\(^2 \)).

(a) (10 points) If the air-resistance is proportional to the velocity of the object with constant of proportionality \( r \) (where \( r > 0 \)), find the formula for the velocity of the object for \( t \geq 0 \) if the velocity at time \( t = 0 \) seconds is \( 20/r \) m/s (NOTE: this formula will involve \( r \)).

**Solution:**

Use Newton’s Law to model this motion. Let \( v = v(t) \) = velocity of the object (towards ground is negative) in m/s at time \( t \) in seconds. Then Newton’s law says: (mass)\( x \) (acceleration of object) = forces acting on it. There is the force of gravity acting downwards and there is air resistance acting in the opposite direction to velocity. So the equation is:

\[
m \frac{dv}{dt} = -mg - rv
\]

Using \( m = 4 \text{kg} \) and \( g = 10 \text{m/s}^2 \) and simplifying we get:

\[
\frac{dv}{dt} = -10 - rv/4
\]

We also have the initial condition: \( v(0) = 20/r \).

Now solve this ODE for \( v \): separate to get \( \frac{dv}{10+rv/4} = -dt \). Finding antiderivative of both sides gives:

\[
(4/r)\ln|10 + rv/4| = -t + C, C = constant
\]

Now solve for \( v \):

\[
|10 + rv/4| = Ke^{-rt/4}, K = e^{rC/4}
\]

Letting \( K = \pm e^{rC/4} \) gives

\[
10 + rv/4 = Ke^{-rt/4}
\]

and solving gives finally

\[
v = v(t) = (4K/r)e^{-rt/4} - 40/r
\]

Using the initial condition \( v(0) = 20/r \) gives \( K = 15 \) and so the equation is

\[
v = v(t) = (60/r)e^{-rt/4} - 40/r
\]
(b) (5 points) Suppose you also know that the terminal velocity of the object is -10 m/s. Find the time at which the object is at its highest point.

Solution: Need to find time \( t \) when \( v(t) = 0 \). To do this we need to know the value of \( r \).

When air-resistance is proportional to velocity we know that \( v_{\text{term}} = -mg/r = -40/r \). Alternatively, using our solution above, \( v_{\text{term}} = \lim_{t \to +\infty} v(t) = -40/r \). If \( v_{\text{term}} = -10/r \) this gives \( r = 4 \). Hence the actual velocity function for this object is

\[
v = v(t) = 15e^{-t} - 10
\]

Now solve \( v(t) = 0 \) i.e. \( 0 = 15e^{-t} - 10 \) giving \( t = -\ln(2/3) = \ln(3/2) \).

(c) (5 points) Find the height of the object at the highest point if it starts out at a height of 15 metres above ground level (do not simplify your expression for the height).

Solution: Let \( x(t) \) = height (in metres) of object above ground at time \( t \) in seconds. Need to find \( x(\ln(3/2)) \). To do this we need to find the formula for \( x = x(t) \). But

\[
\frac{dx}{dt} = v(t) = 15e^{-t} - 10
\]

Also we have the initial condition \( x(0) = 15 \). Solving this gives

\[
x(t) = -15e^{-t} - 10t + 30
\]

Then the required solution is

\[
x(\ln(3/2)) = -15e^{-\ln(3/2)} - 10(\ln(3/2)) + 30
\]

3. (20 points) Suppose a swimming pool of total volume 4000 litres initially has 1000 litres of a salt solution of concentration 2 kg/litre. If salt solution of concentration 0.4 kg/litre is being poured into the pool at a rate of 10 litres/minute and solution is leaving the pool at a rate of 5 litres/minute, find the total amount of salt in the pool at the time it becomes full (assume instantaneous mixing of all solutions).

(a) (5 points) Show that the differential equation modelling the total amount of salt, \( A = A(t) \) (measured in kg) in the pool at time \( t \) (minutes) is given by:

\[
\frac{dA}{dt} = 4 - A/(200 + t)
\]

Solution: Let \( A(t) \) = total amount (in kg) of salt in the pool at time \( t \) (minutes). Let \( V(t) \) = volume of salt solution in the pool at time \( t \).

We know that \( A(0) = (2)(1000) = 2000 \text{kg} \). Also \( V(0) = 1000 \).

Comparing the flow rates we see that there is a net INFLOW of 5 litres/minute. Thus we have \( V(t) = 1000 + 5t \).
Our model is: \[\frac{dA}{dt} = (\text{rate of salt coming in}) - (\text{rate of salt leaving}).\]
Note the units of this equation are kg/minute.
rate of salt coming in = (flow rate in) \times (\text{concentration coming in})
Thus rate of salt coming in = (10)(0.4) = 4\text{kg/minute}.
Similarly rate of salt leaving = (5)(A(t)/V(t)) = (5)(A/(1000+5t)) = A/(200 + t)
Thus our model is:
\[\frac{dA}{dt} = 4 - \frac{A}{200 + t}\]

(b) (12 points) Solve this differential equation to find a formula for \(A = A(t)\).

Solution: This is a linear ODE with \(a = a(t) = -1/(200 + t)\) and \(f = f(t) = 4\).
Solve this using either method:
Integrating factor method: \(u = e^{\int a(t)dt} = e^{\ln|200+t|} = |200+t|\). Since \(t \geq 0, 200 + t \geq 0\) so \(|200 + t| = 200 + t\). Thus \(u = 200 + t\).
Then \(A = A(t) = \frac{1}{u} \int uf(t)dt + C/u, C = \text{constant}\.
Solving gives \(A = A(t) = 400 + 2t + C/(200 + t)\). Using \(A(0) = 2000\) gives \(C = 320000\).
Variation of parameters: \(y_h = y_h(t)\) solves \(\frac{dA}{dt} = -A/(200 + t)\).
Solving gives \(y_h(t) = 1/(200 + t)\).
Then \(v = \int f/y_h dt + C = \int 4(200 + t)dt + C = 2(200 + t)^2 + C, C = \text{constant}\.
Then the solution is \(A = A(t) = vy_h = 400 + 2t + C/(200 + t)\). Using \(A(0) = 2000\) gives \(C = 320000\).
Thus \(A(t) = 400 + 2t + 320000/(200 + t)\) using either method.

(c) (3 points) Find the total amount of salt in the pool at the time it becomes full.

Solution: The pool is full when \(V(t) = 4000\). So we need to solve \(4000 = 1000 + 5t\). This gives \(t = 600\) minutes. Thus we are trying to find \(A(600)\).
Using our formula from part (b) we get \(A(600) = 2000\) (exactly the same value as we started with!).

4. Suppose a lake has a logistic model for the population of fish \(P = P(t)\) \((t=\text{time in days})\) given by the equation \(\frac{dP}{dt} = P(1 - P/600)\). Suppose also that people fish this lake and remove 10 percent of the fish population per day.

(a) (2 points) Write a model for the population of fish in the lake.

Solution: \(\frac{dP}{dt} = P(1 - P/600) - 0.1P\). Simplifying this gives
\[\frac{dP}{dt} = P(0.9 - P/600)\]

(b) (10 points) Find the phase-line for this model.
Solution: First locate the equilibrium solutions. These occur when RHS=0 i.e. solve \( 0 = P(0.9 - P/600) \) giving \( P = 0 \) or \( P = 540 \).

Now for each of the three regions \((P < 0, 0 < P < 540 \text{ and } P > 540)\) we need to calculate the sign of the derivative of solutions in these regions. To do this we use the differential equation.

When \( P < 0, \) \( P(0.9 - P/600) < 0 \) hence \( \frac{dP}{dt} = P(0.9 - P/600) < 0. \) Thus the solutions are decreasing in this region.

Similarly when \( 0 < P < 540, \) \( \frac{dP}{dt} > 0 \) so solutions are increasing in this region.

When \( P > 540, \) \( \frac{dP}{dt} < 0 \) so solutions are decreasing in this region.

Thus the phaseline looks like:

\[
\begin{array}{c}
\bullet & \rightarrow & \bullet \\
P=0 & \rightarrow & P=540
\end{array}
\]

(c) (4 points) Classify each equilibrium solution as stable or unstable.

Solution: For the equilibrium solution \( P = 0, \) solutions either side head away from \( P = 0 \) as time increases so this is an UNSTABLE equilibrium.

For the equilibrium solution \( P = 540, \) solutions either side head towards \( P = 540 \) as time increases so this is a STABLE equilibrium.

(d) (4 points) If at \( t = 0 \) the population is 200 fish, what value will the population of fish approach as \( t \to +\infty? \)

Solution: If \( P(0) = 200 \) this puts us in the region \( 0 < P < 540. \) Hence the solution will increase and in fact increases towards the stable equilibrium solution \( P = 540. \) Hence the population of fish starting at 200 will approach the value 540 as \( t \to +\infty. \)

5. Suppose you are given the differential equation \( y' = -e^{2t}y^3. \)

(a) (5 points) Find the values of \( K \) that make \( y(t) = Ke^{-t} \) a solution to this equation (where \( K \) is a constant).

Solution: For \( y(t) = Ke^{-t} \) to be a solution we need the LHS and RHS of the ODE to be equal when calculated using \( Ke^{-t}. \) So we calculate LHS and RHS.

\[
\text{LHS}=y' = \frac{d}{dt}(Ke^{-t}) = -Ke^{-t}, \\
\text{RHS}=-e^{2t}y^3 = -e^{2t}(Ke^{-t})^3 = -K^3e^{-t}.
\]

Hence if LHS=RHS we need \( -Ke^{-t} = K^3e^{-t} \) i.e. we need \( -K = K^3. \) Solving this gives \( K = 0, \pm 1. \)

(b) (10 points) If \( y = y(t) \) is also a solution and \( y(0) = 0.5 \) what is \( \lim_{t \to +\infty} y(t) ? \) (NOTE: you don’t necessarily need to solve the ODE to answer this question but you must justify your answer fully).

Solution: From part (a) we know that \( y(t) = e^{-t} \) and \( y(t) = 0 \) are solutions to the ODE. In particular at \( t = 0 \) they have values 1 and 0 respectively. The solution we are looking at starts at \( y(0) = 0.5 \) i.e. it starts between these two solutions. If we knew we had unique solutions for all points in the solution plane with \( t \geq 0 \) then the solution we are looking at would be forced to stay between \( y(t) = e^{-t} \) and \( y = 0 \) as \( t \to +\infty. \)

And since \( \lim_{t \to +\infty} e^{-t} = 0 \) this would mean that the solution we are interested in has limit 0 as \( t \to +\infty. \)
Thus we will be done if we know we have unique solutions at least for all times $t \geq 0$. We check that this is the case. We need to show that $-e^{2t}y^3$ and $\frac{\partial}{\partial y}(-e^{2t}y^3)$ are continuous for $t \geq 0$.

In fact $-e^{2t}y^3$ is continuous at any point $(t, y)$ and hence for any point with $t \geq 0$.

Calculating $\frac{\partial}{\partial y}(-e^{2t}y^3) = -e^{2t}(3y^2)$ and this is also continuous at any point $(t, y)$ and hence for any point with $t \geq 0$.

Thus we know that the ODE has unique solutions for $t \geq 0$ and so we know that $\lim_{t \to +\infty} y(t) = 0$ if $y(0) = 0.5$. 
