1. section 10.4, #56
The integral of $\frac{1}{1+t}$ is $\ln(1+x)$.
Integrating the right side we get
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n+1} + R_n$$
where
$$R_n = \int_0^x \frac{(-1)^{n+1} t^{n+1}}{1+t} dt \leq \int_0^x |(-1)^{n+1} t^{n+1}| dt = \int_0^x t^{n+1} dt = \frac{x^{n+2}}{n+2}$$
where, to get the first inequality, we used that $0 < x \leq 1$ so that $1+t \geq 1$.
Now
$$\lim_{n \to \infty} \frac{x^{n+2}}{n+2} = 0$$
if $0 < x \leq 1$ so that $R_n \to 0$. Thus we conclude that
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$$
if $0 < x \leq 1$.

2. section 10.4, #58
The argument in question 56 also works if $-1 \leq x \leq 0$. Substituting $-x$ for $x$ we then get
$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$
Then
$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$
where we just added the series for $\ln(1+x)$ and $\ln(1-x)$ term by term.

3. section 10.4, #59
If we substitute $x = 1$ in problem 56 the error is $\frac{1}{n+2}$ if we use $n$ terms to approximate $\ln(2)$. If we substitute $x = 1/3$ in problem 58 then we also get $\ln(2)$ but the error is now
$$\frac{2(1/3)^{n+2}}{n+2} = \frac{2}{3^{n+2}(n+2)}$$
(the factor of two accounts for the fact we have two error terms, one from the series for $\ln(1+x)$ and the other from the series for $\ln(1-x)$).
Clearly, the second estimate is much better if we use the same number of terms (since the error term is much smaller).
4. section 10.5, #2
\[ \frac{1}{(x+1)^{4/3}} \] is a decreasing function for \( x \geq 1 \). We also have
\[ \int \frac{1}{(x+1)^{4/3}} \, dx = \int (x+1)^{-4/3} \, dx = -3(x+1)^{-1/3} \]
and this converges as \( x \to \infty \). So by the integral test the given series also converges.

5. section 10.5, #6
\[ \frac{1}{x(x+1)} \] is a decreasing function for \( x \geq 1 \). We also have
\[ \int \frac{1}{x(x+1)} \, dx = \int \frac{1}{x} - \frac{1}{x+1} \, dx = \ln(x) - \ln(x+1) = \ln \frac{x}{x+1} = \ln(1+\frac{1}{x}) \]
and this converges as \( x \to \infty \). So by the integral test the given series also converges.

6. section 10.5, #8
\[ \frac{\ln x}{x} \] is a decreasing function for \( x \) large (just take its derivative). We also have
\[ \int \frac{\ln x}{x} \, dx = \frac{1}{2} \ln^2 x \]
and this diverges as \( x \to \infty \). So by the integral test the given series also diverges.

7. section 10.5, #18 \( \frac{x+1}{x} = \ln(1+\frac{1}{2}) \) is a decreasing function for \( x \geq 1 \) since \( 1/x \) is decreasing. We also have
\[ \int \ln \frac{x+1}{x} \, dx = \int_0^x \ln(x+1) \,dx - \int_0^x \ln x \,dx \\
= (x+1)\ln(x+1)-(x+1)-(x\ln x-x) \\
= x(\ln(x+1)-\ln(x))+\ln(x+1)-1 \\
\geq \ln(x+1)-1 \]
which diverges as \( x \to \infty \). So by the integral test the given series also diverges.

8. section 10.5, #20
\[ \ln \frac{2^{1/x}}{x} \] is a decreasing function for \( x \geq 1 \). We also have
\[ \int \frac{2^{1/x}}{x^2} \, dx = \int -2^u \, du = -\frac{2^u}{\ln 2} = -\frac{2^{1/x}}{\ln 2} \]
where we used the substitution \( u = 1/x \). Since \( 1/x \to 0 \) as \( x \to \infty \) this integral converges and so by the integral test the given series also converges.
9. section 10.5, #24

\( \frac{1}{x \ln^3 x} \) is a decreasing function for \( x \geq 1 \) since \( x \ln^3 x \) is increasing. We also have

\[
\int \frac{1}{x \ln^3 x} \, dx = \int \frac{1}{u^3} \, du = -\frac{1}{2u^2} = -\frac{1}{2 \ln^2 x}
\]

where we used the substitution \( u = \ln x \). Now \( \ln x \to \infty \) so this integral converges as \( x \to \infty \). Thus by the integral test the given series converges.

10. section 10.5, #32

The function we should use is \( f(x) = e^{-x} \sin x \). Now \( e^{-x} \) is decreasing to zero as \( x \) goes to infinity. But \( \sin(x) \) alternates sign between 1 and \( -1 \) so that \( f(x) \) is not decreasing (it alternates sign). This means that we cannot apply the integral test (which requires that \( f(x) \) be eventually decreasing).