1. section 10.8, #14
The ratio test gives us (after dropping the \((-1)^n\) term because we take the absolute value)

\[
\frac{4^{n+1}x^{n+1}/(n + 1) \ln(n + 1)}{4^n x^n / n \ln n} = \frac{4x \ln n}{(n + 1) \ln(n + 1)}
\]

Now as \(n \to \infty\) we have that \(\frac{n}{n + 1} \to 1\) and

\[
\lim_{n \to \infty} \frac{\ln n}{\ln(n + 1)} = \lim_{n \to \infty} \frac{1/n}{1/(n + 1)} = 1
\]

by L'Hôpital's rule. Thus the limit of \(\frac{4x \ln n}{(n + 1) \ln(n + 1)}\) is \(4x\). By the ratio test this means the series converges for \(-1/4 < x < 1/4\) and diverges if \(|x| > 1/4\).

Now when \(x = 1/4\) the series converges by the alternating series test. When \(x = -1/4\) the series is

\[
\sum \frac{1}{n \ln n}
\]

which diverges by the integral test. Namely, the integral of \(\frac{1}{x \ln x}\) is \(\ln \ln x\) which diverges.

2. section 10.8, #38
The series for \((1 + x)^{3/2}\) is

\[
1 + \frac{3}{2} x + \frac{3 \cdot 1}{2 \cdot 2!} x^2 + \frac{3 \cdot 1 \cdot (-1)}{2 \cdot 2 \cdot 3!} x^3 + \ldots
\]

plugging in \(x^2\) for \(x\) gives the power series for \((1 + x^2)^{3/2}\).

Now the radius of convergence of \((1 + x)^{3/2}\) is 1 i.e. it convergence when \(-1 < x < 1\) and diverges for \(|x| > 1\). So \((1 + x^2)^{3/2}\) converges for \(-1 < x^2 < 1\) which is equivalent to \(-1 < x < 1\) so the radius of convergence of \((1 + x^2)^{3/2}\) is also 1.

3. section 10.8, #40
We can write \(\frac{1}{\sqrt{9 + x^3}} = \frac{1}{3\sqrt{1 + \frac{x^3}{9}}}\). Now \(\frac{1}{3\sqrt{1 + x}}\) has power series

\[
\frac{1}{3} (1 + \frac{1}{2} x + \frac{1 - 1}{2 \cdot 2!} x^2 + \frac{1 - 1 - 3}{2 \cdot 2 \cdot 3!} x^3 + \ldots)
\]

plugging in \(\frac{x^3}{9}\) for \(x\) we get the power series we want.
Now the power series $\frac{1}{\sqrt{1+x}}$ converges for $-1 < x < 1$ so has radius of convergence 1. Thus our power series converges for $-1 < \frac{x^3}{9} < 1$ or equivalently $-9 < x^3 < 9$ or equivalently $-9^{1/3} < x < 9^{1/3}$ so the radius of convergence is $9^{1/3}$.

4. section 10.8, #42

We know from earlier computations in the chapter that

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$$

so that

$$x - \tan^{-1}(x) = \frac{1}{3}x^3 - \frac{1}{5}x^5 + \ldots$$

and dividing by $x^3$ we get

$$\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \ldots$$

Now the original power series for $\tan^{-1}(x)$ converges for $-1 < x < 1$ and subtracting $x$ or dividing by $x^3$ does not change this fact. So the radius of convergence is 1.

5. section 10.8, #44

We know that $\sin t$ has power series

$$t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots$$

so that $\frac{\sin t}{t}$ has power series

$$1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots$$

and integrating term by term we get

$$t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \ldots$$

6. section 10.8, #48

The power series for $\frac{1}{1+t}$ is

$$1 + t^2 + t^4 + t^6 + \ldots$$

so integrating term by term we get

$$t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \ldots$$
7. section 10.8, #50

We know that $\sum x^n = \frac{1}{1-x}$. Differentiating term by term we get that $\sum nx^{n-1}$ sums up to
\[
\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}
\]

Now differentiating again we get $\sum n(n-1)x^{n-2}$ sums up to
\[
\frac{d}{dx} \frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}
\]

Multiplying by $x^2$ we get that $\sum n(n-1)x^n$ sums up to
\[
\frac{2x^2}{(1-x)^3}.
\]

8. section 10.8, #52

We know that $\sum nx^n = \frac{x}{(1-x)^2}$ so plugging in $x = 1/2$ we get
\[
\sum \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2
\]

Similarly, we know that
\[
\sum n^2 x^n = \frac{x(1+x)}{(1-x)^3}
\]

so plugging in $x = 1/3$ we get
\[
\sum \frac{n^2}{3^n} = \frac{1/3(1+1/3)}{(1-1/3)^3} = \frac{4/9}{8/27} = 3/2
\]

9. page 814 #38

Using the ratio test we get
\[
\frac{x^{n+1} \ln(n+1)}{x^n \ln n} = \frac{x \ln n}{\ln(n+1)}
\]

which converges to $x$ as $n \to \infty$. Thus the series converges for $-1 < x < 1$ and diverges for $|x| > 1$.

Now when $x = -1$ the series $\sum \frac{(-1)^n}{n}$ converges by the alternating series test. When $x = 1$ we get $\sum \frac{1}{n}$ which diverges by the comparison test with $\sum \frac{1}{n}$ (which in turn diverges by the integral test).

10. page 814 #40

Again we use the ratio test to get
\[
\frac{(1 + \frac{1}{n+1})^{n+1}(x-1)^{n+1}}{(1 + \frac{1}{n})^n(x-1)^n}
\]
Now we know that $(1 + \frac{1}{n})^n$ tends to the constant $e$ as $n \to \infty$. So the limit above as $n \to \infty$ is equal to $e^{x-1} = x - 1$. So the series converges if $-1 < x - 1 < 1$ or equivalently if $0 < x < 2$. It diverges if $x - 1 > 1$ or $x - 1 < -1$ or equivalently if $x > 2$ or $x < 0$.

Finally, if $x = 0$ or $x = 2$ the series diverges since the individual terms in the series do not tend to zero (their absolute value tends to $e$).