Math 101 Fall 2004 Exam 2 Solutions
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Tuesday, November 16, 2004

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have one hour and fifteen minutes. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.

Please print your name clearly here.

Print name: ________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

Grader’s use only:

1. _______ /10
2. _______ /10
3. _______/25
4. _______ /10
5. _______ /15
6. _______ /20
1. [10 points] Compute the first three derivatives of the following function.

\[ f(t) = (t^2 + 3t) \ln (t^2 + 3t) \]

\[ f'(t) = (2t+3) \ln (t^2 + 3t) + (t^2 + 3t) \frac {2t + 3}{t^2 + 3t} = (2t+3) \ln (t^2 + 3t) + 2t + 3. \]

\[ f''(t) = 2 \ln(t^2 + 3t) + (2t+3) \frac {2t + 3}{t^2 + 3t} + 2 = 2 \ln(t^2 + 3t) + \frac {(2t+3)^2}{t^2 + 3t} + 2. \]

\[ f'''(t) = \frac {4}{t^2 + 3t} + \frac {4(2t + 3)(t^2 + 3t) - (2t + 3)^2 \cdot (2t + 3)}{(t^2 + 3t)^2} \]

\[ = \frac {2(2t + 3)}{t^2 + 3t} - \frac {9(2t + 3)}{(t^2 + 3t)^2}. \]
2. [10 points] Evaluate the following limits, if they exist.

(a) \( \lim_{t \to 0} \frac{1 - \cos 3t}{t \sin t} \)

This limit is indeterminate of type 0/0 so applying L'Hôpital (twice) gives

\[
\lim_{t \to 0} \frac{1 - \cos 3t}{t \sin t} = \lim_{t \to 0} \frac{3 \sin 3t}{\sin t + t \cos t} = \lim_{t \to 0} \frac{9 \cos 3t}{2 \cos t - t \sin t} = \frac{9}{2}
\]

(b) \( \lim_{x \to 0} (1 - 3x)^{1/(2x)} \)

This limit is indeterminate of type 1\(^\infty\) so rearranging and using L'Hôpital gives

\[
\lim_{x \to 0} (1 - 3x)^{1/(2x)} = \lim_{x \to 0} \exp \left( \frac{\ln(1 - 3x)}{2x} \right) = \exp \left( \lim_{x \to 0} \frac{\ln(1 - 3x)}{2x} \right)
\]

\[
= \exp \left( \lim_{x \to 0} \frac{-3}{2} \right) = \exp(-3/2) = e^{-3/2}
\]
3. [25 points] Evaluate the following integrals:

(a) $\int (e^t + 1)^2 \, dt$

$$\int (e^t + 1)^2 \, dt = \int (e^{2t} + 2e^t + 1) \, dt = \frac{1}{2} e^{2t} + 2e^t + t + C.$$

(b) $\int x^2 \sec^2 (x^3 + 1) \, dx$

Substituting $u = x^3 + 1$ so $du = 3x^2 \, dx$ gives

$$\int x^2 \sec^2 (x^3 + 1) \, dx = \frac{1}{3} \int \sec^2 u \, du = \frac{\tan u}{3} + C = \frac{\tan (x^3 + 1)}{3} + C.$$

(c) $\int \sin^5 3z \cos 3z \, dz$

Substituting $u = \sin(3z)$, $du = 3\cos(3z) \, dz$ gives

$$\int \sin^5 3z \cos 3z \, dz = \frac{1}{3} \int u^5 \, du = \frac{1}{18} u^6 + C = \frac{1}{18} \sin^6(3z) + C.$$

(d) $\int_0^1 x(2 - x^2)^3 \, dx$

Substituting $u = 2 - x^2$, so $du = -2x \, dx$ and $x = 0$ means $u = 2$, $x = 1$ means $u = 1$ gives

$$\int_0^1 x(2 - x^2)^3 \, dx = -\frac{1}{2} \int_2^1 u^3 \, du = \frac{1}{2} \int_1^2 u^3 \, du = \frac{1}{8} u^4 \bigg|_1^2 = 2 - \frac{1}{8} = \frac{15}{8}.$$

(e) $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$

Substituting $u = \sin x$, so $du = \cos x \, dx$ and $x = 0$ means $u = 0$, $x = \pi/2$ means $u = 1$ gives

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = \int_0^1 e^u \, du = e^u \bigg|_0^1 = e - 1.$$
4. [10 points] Evaluate the definite integral below directly from the definition. That is, compute \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \) for a regular partition of the given interval of integration. 

\[
\int_{0}^{3} (3x^2 + 1) \, dx
\]

The following formulas may be helpful

\[
\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{1}{2} n^2 + \frac{1}{2} n, \quad \sum_{i=1}^{n} i^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n.
\]

Since \( a = 0, b = 3 \) and \( f(x) = 3x^2 + 1 \) we have \( \Delta x = (b - a)/n = 3/n \) and \( x_i = a + i \Delta x = 3i/n \). Hence

\[
\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} \left( 3 \left( \frac{3i}{n} \right)^2 + 1 \right) \frac{3}{n} = \sum_{i=1}^{n} \left( \frac{81i^2}{n^3} + \frac{3}{n} \right)
\]

\[
= \frac{81}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 = \frac{81}{n^3} \left( \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \right) + \frac{3}{n}
\]

\[
= 27 + \frac{81}{2n} + \frac{27}{2n^2} + 3 = 30 + \frac{81}{2n} + \frac{27}{2n^2}.
\]

Hence

\[
\int_{0}^{3} (3x^2 + 1) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} \left( 30 + \frac{81}{2n} + \frac{27}{2n^2} \right) = 30.
\]
5. [10 points] Find the area of the region in the plane bounded by
\[ y = x^3 - 3x^2 + 2x \quad \text{and} \quad y = 2x. \]

The two curves intersect when \( y = x^3 - 3x^2 + 2x = 2x \), hence \( x^3 = 3x^2 \), so \( x = 0 \) or \( x = 3 \). Plugging in \( x = 1 \), we get \( y = 0 \) for the cubic and \( y = 2 \) for the line. Hence the line is higher and the region is below \( y = 2x = f(x) \), above \( y = x^3 - 3x^2 + 2x = g(x) \) between \( x = 0 = a \) and \( x = 3 = b \). So

\[
A = \int_{a}^{b} (f(x) - g(x)) \, dx = \int_{0}^{3} (2x - (x^3 - 3x^2 + 2x)) \, dx
\]

\[
= \int_{0}^{3} (3x^2 - x^3) \, dx = \left[ x^3 - \frac{1}{4}x^4 \right]_{0}^{3}
\]

\[
= \left( 27 - \frac{81}{4} \right) - 0 = \frac{27}{4}.
\]
6. [15 points] We define the plane region $R$ to be bounded by 

\[ y = x^2 \quad \text{and} \quad x = y^2. \]

Consider the volume $V$ generated by rotating the region $R$ around the $x$-axis.

(a) Using the method of cross-sections, compute the volume $V$ described above.

To do cross-sections, we need the region described in terms of $x$. The two parabolas intersect at $x = 0 = a$ and $x = 1 = b$. Between $x = 0$ and $x = 1$, the higher parabola is $y = f(x) = \sqrt{x}$ and the lower curve is $y = g(x) = x^2$. Hence the volume is

\[
V = \int_{a}^{b} \pi [f(x)^2 - g(x)^2] \, dx = \int_{0}^{1} \pi \left[ (\sqrt{x})^2 - (x^2)^2 \right] \, dx \\
= \pi \int_{0}^{1} (x - x^4) \, dx = \pi \left. \left( \frac{1}{2}x^2 - \frac{1}{5}x^5 \right) \right|_{0}^{1} = \pi \left( \frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10}.
\]

(b) Using the method of cylindrical shells, compute the volume $V$ described above. Note: You should get the same result as in part 6a.

To do shells, we need the region described in terms of $y$. The two parabolas intersect at $y = 0 = c$ and $y = 1 = d$. Between $y = 0$ and $y = 1$ the rightmost curve is $y = x^2$ or $x = \sqrt{y} = h(y)$ and the leftmost curve is $x = y^2 = k(y)$. Hence the volume is

\[
V = \int_{c}^{d} 2\pi y[h(y) - k(y)] \, dy = \int_{0}^{1} 2\pi y(\sqrt{y} - y^2) \, dy \\
= 2\pi \int_{0}^{1} (y^{3/2} - y^3) \, dy = 2\pi \left. \left( \frac{2}{5}y^{5/2} - \frac{1}{4}y^4 \right) \right|_{0}^{1} = 2\pi \left( \frac{2}{5} - \frac{1}{4} \right) = \frac{3\pi}{10}.
\]
7. [20 points] For the function \( f(x) = \frac{\sqrt{x^2+1}}{x+5} \), the first two derivatives are 

\[ f'(x) = \frac{5x-1}{(x+5)^2 \sqrt{x^2+1}} \]

and 

\[ f''(x) = \frac{(3-2x)(5x^2+6x+9)}{(x+5)^3 (x^2+1)^{3/2}} \]

YOU NEED NOT VERIFY THESE FORMULAS.

**Part (a)** Find (and justify) all horizontal and vertical asymptotes of the graph \( y = f(x) \). At any vertical asymptotes compute both the left and right hand limits of \( f(x) \).

Since 

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+5} = \lim_{x \to -\infty} \frac{\sqrt{1+x^{-2}}}{1+5/x} = \frac{\sqrt{1+0}}{1+0} = 1,
\]

and 

\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+5} = \lim_{x \to -\infty} \frac{\sqrt{1+x^{-2}}}{1+5/x} = -\frac{\sqrt{1+0}}{1+0} = -1,
\]

we see \( f \) has two horizontal asymptotes \( y = 1 \) approached at \( \infty \) and \( y = -1 \) approached at \( -\infty \). The only possible vertical asymptote is at the discontinuity \( x = -5 \). Since the numerator is positive \( f(x) > 0 \) for \( x > 5 \) and \( f(x) < 0 \) for \( x < 5 \). Hence

\[
\lim_{x \to -5^+} f(x) = +\infty, \quad \text{and} \quad \lim_{x \to -5^-} f(x) = -\infty.
\]

and \( x = -5 \) is a vertical asymptote.

(b) Find the intervals on which \( f(x) \) is increasing and those on which it is decreasing.

Note that \( f'(x) \) is undefined at the vertical asymptote \( x = -5 \). The denominator of \( f'(x) \) is always non-negative so \( f'(x) > 0 \) for \( x > 1/5 \) and \( f'(x) < 0 \) for \( -5 < x < 1/5 \) and for \( x < -5 \). Thus \( f \) is decreasing on \(( -\infty, -5) \) and on \((-5, 1/5] \) and \( f \) is increasing on \([1/5, \infty) \).

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(c) Find the critical points of $f(x)$ and classify them as local maxima, local minima or neither.

$f'(x)$ is undefined at $x = -5$, but this is the asymptote and $f(-5)$ is also undefined, hence it is not a critical point. Solving $f'(x) = 0$ gives $x = 1/5$ as the only critical point. Since by (b), $f'$ switches from negative to positive at $x = 1/5$, by the First Derivative Test, $x = 1/5$ is a local minimum.

(d) Find the intervals on which $f(x)$ is concave upward and those on which it is concave downward. (It may be helpful to notice that $5x^2 + 6x + 9 = 4x^2 + (x + 3)^2$ is positive for all $x$.)

The factors $5x^2 + 6x + 9$ and $(x^2 + 1)^{3/2}$ are both positive. $3 - 2x$ is positive for $x < 3/2$ and negative for $x > 3/2$. $(x + 5)^3$ is positive for $x > -5$ and negative for $x < -5$. Hence $f''(x)$ is negative on $(-\infty, -5)$ and on $(3/2, \infty)$ and $f''(x)$ is positive on $(-5, 3/2)$. Hence $f$ is concave down on $(-\infty, -5)$ and on $(3/2, \infty)$ and $f$ is concave up on $(-5, 3/2)$.

(e) On the next page sketch the graph of $y = \sqrt{\frac{x^2 + 1}{x+5}}$ showing the results of (a)-(d). (The following values may be helpful $f(1/5) = 1/\sqrt{26} \approx 0.196$, $f(3/2) = 1/\sqrt{13} \approx 0.277$.)