Math 101 Fall 2001 Final Exam Solutions

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have three hours. Do all 12 problems. Please do all your work on the paper provided.
Please print your name clearly here.

Print name: __________________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

Grader’s use only:

1. _______ /15
2. _______ /30
3. _______ /20
4. _______ /30
5. _______ /20
6. _______ /10
7. _______ /10
8. _______ /15
9. _______ /15
10. _______ /10
11. _______ /10
12. _______ /15
1. [15 points] Evaluate the following limits, if they exist.

(a) \[ \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - x - 6} \]

The limit is indeterminate of type 0/0, therefore applying L’Hopital’s rule gives

\[ \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - x - 6} = \lim_{x \to 3} \frac{2x - 4}{2x - 1} = \frac{2}{5}. \]

or

\[ \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - x - 6} = \lim_{x \to 3} \frac{(x - 3)(x - 1)}{(x - 3)(x + 2)} = \lim_{x \to 3} \frac{x - 1}{x + 2} = \frac{2}{5}. \]

(b) \[ \lim_{x \to 0} \frac{e^{x^2} - 1}{1 - \cos x} \]

This limit is indeterminate of type 0/0, hence applying L’Hopital’s rule (twice) gives

\[ \lim_{x \to 0} \frac{e^{x^2} - 1}{1 - \cos x} = \lim_{x \to 0} \frac{2xe^{x^2}}{\sin x} = \lim_{x \to 0} \frac{2e^{x^2} + 4xe^{x^2}}{\cos x} = \frac{2}{1} = 2. \]

(c) \[ \lim_{x \to 1} \frac{x^2}{x-1} \]

This limit is indeterminate of type 1\(\infty\), therefore rewriting and applying L’Hopital’s rule gives:

\[ \lim_{x \to 1} \frac{x^2}{x-1} = \lim_{x \to 1} \exp \left( \frac{2 \ln x}{x - 1} \right) = \exp \left( \lim_{x \to 1} \frac{2 \ln x}{x - 1} \right) \]

\[ = \exp \left( \lim_{x \to 1} \frac{2/x}{1} \right) = \exp(2) = e^2. \]
2. [30 points] Compute the derivatives of the following functions.

(a) $f(x) = x^3 \ln(x + 3)$

$$\frac{df}{dx} = 3x^2 \ln(x + 3) + \frac{x^3}{x + 3}.$$

(b) $g(t) = \sec(e^t)$

$$\frac{dg}{dt} = \sec(e^t) \tan(e^t) \frac{d(e^t)}{dt} = e^t \sec(e^t) \tan(e^t).$$

(c) $h(w) = \arctan(3w^2)$

$$\frac{dh}{dw} = \frac{1}{1 + (3w^2)^2} \frac{d(3w^2)}{dw} = \frac{6w}{1 + 9w^4}.$$

(d) $G(x) = \int_1^{x^2} \frac{\sin t}{t} \, dt$

$$\frac{dG}{dx} = \frac{\sin(x^2)}{x^2} \frac{d(x^2)}{dx} = \frac{2\sin(x^2)}{x}.$$

(e) $H(z) = (2 + (1 + 3 \sin z)^3)^{1/2}$

$$\frac{dH}{dz} = \frac{1}{2}(2 + (1 + 3 \sin z)^3)^{-1/2} d(2 + (1 + 3 \sin z)^3) dz$$

$$= \frac{3}{2}(2 + (1 + 3 \sin z)^3)^{-1/2} (1 + 3 \sin z)^2 \frac{d(3 \sin z)}{dz}$$

$$= \frac{9}{2}(2 + (1 + 3 \sin z)^3)^{-1/2} (1 + 3 \sin z)^2 \cos z.$$
3. [20 points] You want to build a rectangular box with no top, a square base and a volume of 500 cm³. What dimensions will minimize the total surface area? Be sure to justify that your answer is really a global minimum.

Let \( x \) be the length of a side of the base of the box in cm and \( h \) the height in cm. The volume is \( V = x^2 h = 500 \), hence \( h = \frac{500}{x^2} \). Since \( x \) and \( h \) are lengths, they must be positive so \( 0 < x < \infty \). The surface area is \( S(x) = x^2 + 4xh = x^2 + \frac{2000}{x} \). Then \( S'(x) = 2x - \frac{2000}{x^2} = 2\left(x^3 - 1000\right)/x^2 \). The only critical point is when \( x = 10 \). Since \( S'(x) < 0 \) for \( x < 10 \) and \( S'(x) > 0 \) for \( x > 10 \) or because \( S''(x) = 2 + 4000/x^3 > 0 \) for all \( x \), \( x = 10 \) is a global minimum. Therefore the dimensions of the minimal surface area box are \( x = 10 \) cm and \( h = 5 \) cm.
4. [30 points] Let \( f(x) = \frac{x^2 - 3}{x^2 - 1} \).

(a) Find all horizontal and vertical asymptotes of the graph \( y = f(x) \). At the vertical asymptotes compute both the left and right hand limits of \( f(x) \).

\[
\lim_{x \to \infty} \frac{x^2 - 3}{x^2 - 1} = 1 \quad \text{and} \quad \lim_{x \to -\infty} \frac{x^2 - 3}{x^2 - 1} = 1
\]

so \( y = 1 \) is the only horizontal asymptote. The vertical asymptotes are at \( x = \pm 1 \) where we get constant/0. Since \( f > 0 \) for \(-1 < x < 1\) and \( f > 0 \) for \( 1 < x < \sqrt{3} \) and for \(-\sqrt{3} < x < -1\) we have

\[
\lim_{x \to -1^-} \frac{x^2 - 3}{x^2 - 1} = -\infty, \quad \lim_{x \to -1^+} \frac{x^2 - 3}{x^2 - 1} = \infty,
\]

\[
\lim_{x \to 1^-} \frac{x^2 - 3}{x^2 - 1} = \infty, \quad \lim_{x \to 1^+} \frac{x^2 - 3}{x^2 - 1} = -\infty
\]

(b) Find the intervals on which \( f(x) \) is increasing and those on which it is decreasing.

\[
\frac{df}{dx} = \frac{2x(x^2 - 1) - 2x(x^2 - 3)}{(x^2 - 1)^2} = \frac{4x}{(x^2 - 1)^2}
\]

Note \( f \) is discontinuous at \( x = \pm 1 \). The denominator is always nonnegative, the numerator is positive for \( x > 0 \) and negative for \( x < 0 \). Hence \( f \) is decreasing on \( (-\infty, -1) \) and on \( (-1, 0] \) and \( f \) is increasing on \([0, 1)\) and on \((1, \infty)\).

(c) Find the critical points of \( f(x) \) and determine if they are local maxima or local minima.

Since \( f'(x) \) exists for \( x \neq \pm 1 \) and \( f'(x) = 0 \) only when the numerator is zero, i.e., when \( x = 0 \), the only critical point of \( f \) is at \( x = 0 \). Since \( f \) switches from decreasing to increasing at \( x = 0 \), \( x = 0 \) is a local minimum by the First Derivative Test or from part (d) below \( f''(0) = 4 > 0 \) so by the Second Derivative Test, \( x = 0 \) is a local minimum.

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(d) Find the intervals on which \( f(x) \) is concave upward and those on which it is concave downward.

\[
f''(x) = \frac{4(x^2 - 1)^2 - 4x \cdot 4x(x^2 - 1)}{(x^2 - 1)^4} = \frac{-4(3x^2 + 1)}{(x^2 - 1)^3}.
\]

Since \( 3x^2 + 1 > 0 \), \( f''(x) < 0 \) for \( |x| > 1 \) and \( f''(x) > 0 \) for \(-1 < x < 1\), \( f \) is concave down on \( (-\infty, -1) \) and on \( (1, \infty) \) and \( f \) is concave up on \( (-1, 1) \).

(e) Sketch the graph of \( y = \frac{x^2 - 3}{x^2 - 1} \) using your results in parts (a)-(d).
5. [20 points] Evaluate the following integrals.

(a) \( \int (x^2 + 3)^2 \, dx \)

\[
\int (x^2 + 3)^2 \, dx = \int (x^4 + 6x^2 + 9) \, dx = \frac{1}{5}x^5 + 2x^3 + 9x + C.
\]

(b) \( \int_0^{\pi/2} \frac{\cos x}{2 + \sin x} \, dx \)

Substituting \( u = 2 + \sin x \), \( du = \cos x \, dx \), when \( x = 0, u = 2 \) and when \( x = \pi/2, u = 3 \)

\[
\int_0^{\pi/2} \frac{\cos x}{2 + \sin x} \, dx = \int_2^3 \frac{1}{u} \, du = \ln u \bigg|_2^3 = \ln 3 - \ln 2.
\]

(c) \( \int \sec^2 x e^{\tan x} \, dx \)

Substituting \( u = \tan x \), \( du = \sec^2 x \, dx \) so

\[
\int \sec^2 x e^{\tan x} \, dx = \int e^u \, du = e^u + C = e^{\tan x} + C.
\]

(d) \( \int \frac{x^3}{\sqrt{1 - 4x^2}} \, dx \)

Substituting \( u = 2x^4 \), \( du = 8x^3 \, dx \) gives

\[
\int \frac{x^3}{\sqrt{1 - 4x^2}} \, dx = \frac{1}{8} \int \frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{8} \arcsin u + C = \frac{1}{8} \arcsin(2x^4) + C.
\]
6. [10 points] Evaluate \( \int_0^2 (1 + 3x^2)dx \) by computing \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \). No credit will be given for computing the integral in any other manner.

We have \( a = 0, b = 2 \), hence \( \Delta x = (b - a)/n = 2/n \) and \( x_i = a + i\Delta x = 2i/n \). Thus

\[
\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n} \left(1 + \frac{12i^2}{n^2}\right)\frac{2}{n} = \frac{2}{n} \sum_{i=1}^{n} 1 + \frac{24}{n^3} \sum_{i=1}^{n} i^2
\]

\[= 2 + \frac{24}{n^2} \left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n\right) = 10 + \frac{12}{n} + \frac{4}{n^2}.\]

Therefore

\[
\int_0^2 (1 + 3x^2)dx = \lim_{n \to \infty} \left(10 + \frac{12}{n} + \frac{4}{n^2}\right) = 10.
\]
7. [10 points] Find the area of the region in the plane bounded by \( y = 4 - x^2 \) and \( y = x^2 - 2x - 8 \).

The two parabolas cross where \( 4 - x^2 = x^2 - 2x - 8 \), hence \( 2x^2 - 2x - 12 = 2(x - 3)(x + 2) = 0 \) or \( x = -2 \) and \( x = 3 \). For \(-2 < x < 3\), \( 4 - x^2 > x^2 - 2x - 8 \). Hence \( a = -2, b = 3, f(x) = 4 - x^2, g(x) = x^2 - 2x - 8 \) and the area is

\[
A = \int_{-2}^{3} ((4 - x^2) - (x^2 - 2x - 8))\,dx = \int_{-2}^{3} (12 + 2x - 2x^2)\,dx
\]

\[
= \left(12x + x^2 - \frac{2}{3}x^3\right)\bigg|_{-2}^{3} = (36 + 9 - 18) - \left(-24 + 4 + \frac{16}{3}\right) = \frac{125}{3}.
\]
8. [15 points] Suppose a particle on a line has velocity function \( v(t) = t^2 - 4t + 3 \) for \( 1 \leq t \leq 4 \). Find the net distance travelled by the particle between \( t = 1 \) and \( t = 4 \) and the total distance travelled between \( t = 1 \) and \( t = 4 \).

\[
\text{Net} = \int_1^4 v(t)dt = \int_1^4 (t^2 - 4t + 3)dt = \left( \frac{1}{3}t^3 - 2t^2 + 3t \right)|_1^4
\]
\[
= \left( \frac{64}{3} - 32 + 12 \right) - \left( \frac{1}{3} - 2 + 3 \right) = \frac{4}{3} - \frac{4}{3} = 0.
\]

Since \( t^2 - 4t + 3 = (t - 1)(t - 3) \leq 0 \) for \( 1 \leq t \leq 3 \) and \( t^2 - 4t + 3 = (t - 1)(t - 3) \geq 0 \) for \( 3 \leq t \leq 4 \),

\[
\text{Total} = \int_1^4 |v(t)|dt = \int_1^3 (-t^2 + 4t - 3)dt + \int_3^4 (t^2 - 4t + 3)dt
\]
\[
= \left( -\frac{1}{3}t^3 + 2t^2 - 3t \right)|_1^3 + \left( \frac{1}{3}t^3 - 2t^2 + 3t \right)|_3^4
\]
\[
\left\{ (-9 + 18 - 9) - \left( -\frac{1}{3} + 2 - 3 \right) \right\} + \left\{ \left( \frac{64}{3} - 32 + 12 \right) - (9 - 18 + 9) \right\}
\]
\[
= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.
\]
9. [15 points] Let \( R \) be the region in the plane bounded by \( x = 4y - y^2 \) and the \( y \)-axis. Let \( S \) be the solid of revolution that results from revolving \( R \) about the \( y \)-axis. Express the volume of \( S \) as a definite integral in TWO ways, using the method of washers and the method of shells. Evaluate ONE of the two integrals (your choice).

The \( y \)-axis is defined by \( x = 0 \) so the parabola crosses the \( y \)-axis at \((0, 0)\) and \((0, 4)\). For \( 0 \leq y \leq 4 \), \( 4y - y^2 \geq 0 \). Thus \( R \) is the region to the left of \( x = 4y - y^2 \), to the right of \( x = 0 \) between \( y = 0 \) and \( y = 4 \). Thus washers gives

\[
V = \pi \int_0^4 (16y^2 - 8y^3 + y^4) \, dy
\]

Solving for \( y \) as a function of \( x \), \( 4 - x = 4 - 4y + y^2 = (y - 2)^2 \) or \( y = 2 \pm \sqrt{4 - x} \). Thus \( R \) is the region below \( y = 2 + \sqrt{4 - x} \) above \( y = 2 - \sqrt{4 - x} \) between \( x = 0 \) and \( x = 4 \). Thus shells gives

\[
V = 2\pi \int_0^4 x \left( (2 + \sqrt{4 - x}) - (2 - \sqrt{4 - x}) \right) \, dx = 4\pi \int_0^4 x \sqrt{4-x} \, dx.
\]

Substituting \( u = 4 - x \), \( du = -dx \), when \( x = 0 \), \( u = 4 \) and when \( x = 4 \), \( u = 0 \) thus

\[
V = -4\pi \int_4^0 (4 - u) \sqrt{u} \, du = 4\pi \int_0^4 (4u^{3/2} - u^{3/2}) \, du
\]

\[
= 4\pi \left( \frac{8}{3} u^{5/2} - \frac{2}{5} u^{5/2} \right) \bigg|_0^4 = 4\pi \left( \frac{64}{3} - \frac{64}{5} \right) = \frac{512\pi}{15}.
\]
10. [10 points] Find the length of the curve \( y = 2(x - 2)^{3/2} \) from \( x = 2 \) to \( x = 9 \).

We compute \( \frac{dy}{dx} = 3(x - 2)^{1/2} \) hence

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + 9(x - 2) = 9x - 17
\]

and

\[
L = \int_2^9 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_2^9 (9x - 17)^{1/2} \, dx = \frac{2}{27} (9x - 17)^{3/2} \bigg|_2^9 = \frac{2}{27} ((64)^{3/2} - 1^{3/2}) = \frac{1022}{27}.
\]

11. [10 points] Express the area of the surface \( S \) obtained by revolving the curve \( y = x^2 \) for \( 0 \leq x \leq 3 \) about the \( x \)-axis as a definite integral, but do not attempt to evaluate the integral.

Since \( \frac{dy}{dx} = 2x \) we have

\[
A = \int_0^3 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^3 2\pi x^2 \sqrt{1 + 4x^2} \, dx.
\]
12. [15 points] The population of Houston in 1960 was 1 million people and in 2000 it was 2 million people. Assuming exponential growth, find the population of Houston as a function of time. What will be the population of Houston in 2050? When will the population of Houston be 3 million?

Let $t$ be the time in years since 1960 and $P(t)$ the population in millions at time $t$. By exponential growth $P(t) = Ce^{kt}$ for some constants $C$ and $k$. We have $P(0) = 1 = Ce^0 = C$ and hence $P(40) = 2 = e^{40k}$. Hence $\ln 2 = 40k$ or $k = \frac{\ln 2}{40}$.

Thus $P(t) = \exp(t \ln 2 / 40)$.

Hence the population in 2050 will be $P(90) = \exp(\frac{90 \ln 2}{40}) = \exp(\frac{9 \ln 2}{4}) = 2^{9/4}$ or about 4.75 million.

The population will be 3 million when $3 = P(t) = \exp(t \ln 2 / 40)$ or $t \ln 2 = 40 \ln 3$ or $t = 40 \ln 3 / \ln 2$ or sometime in 2023.