1. Introduction

Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \). An algebraic geometer would commonly study (quasi-coherent) sheaves of \( \mathcal{O}_X \)-modules where \( \mathcal{O}_X \) is the sheaf of algebraic functions on \( X \) (notice that \( \mathcal{O}_X \) is a sheaf of rings).

A non-commutative algebraic geometry might be inclined to study sheaves of modules over a sheaf of non-commutative rings on \( X \). One natural non-commutative sheaf of rings is \( \mathcal{D}_X \) which is the sheaf of differential operators on \( X \). Notice that \( \mathcal{O}_X \subset \mathcal{D}_X \) is always a subring so any \( \mathcal{D}_X \)-module is naturally an \( \mathcal{O}_X \)-module.

**Example.** Let’s consider the variety \( X = \mathbb{A}^1 = \text{Spec}(\mathbb{C}[x]) \). The ring of global sections \( \Gamma(\mathcal{D}_X) \) is generated by \( x \) and \( \partial_x := \frac{d}{dx} \). Now \( \mathcal{O}_X \) is naturally a \( \mathcal{D}_X \)-module so taking global sections \( \Gamma(\mathcal{O}_X) = \mathbb{C}[x] \)

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is naturally a $\Gamma(D_X)$-module. For $f \in \mathbb{C}[x]$ we see that
\[(\partial_x \cdot x) \cdot f = (\partial_x)(xf) = f + x\partial_x f.\]
So we get the relation $\partial_x x = 1 + x\partial_x$. This is the only relation so we get
\[D_X \cong \mathbb{C}(x, \partial_x)/(\partial_x x = 1 + x\partial_x)\]
which is the classical Weyl algebra $W_1$.

**Example.** Let’s consider the variety $Y = \mathbb{P}^1$. We think of $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ and use the same local coordinate $x$ on $\mathbb{A}^1$. Then via restriction we have
\[\Gamma(D_Y) \hookrightarrow \Gamma(D_X) \cong \mathbb{C}(x, \partial_x)/(\partial_x x = 1 + x\partial_x) = W_1.\]
Now if we change local coordinates $x \mapsto 1/x$ so that around looking locally around $\infty \in \mathbb{P}^1$ then $\partial_x \mapsto x^2 \partial_x$ then the subalgebra of the Weyl algebra which extend to regular is generated (as an algebra) by
\[(1) \quad E := \partial_x, F := x^2 \partial_x \text{ and } H := 2x\partial_x.\]


We conclude that there is a map of algebras $U(\mathfrak{sl}_2) \to \Gamma(D_{\mathbb{P}^1})$ where $U(\mathfrak{sl}_2)$ is the universal enveloping algebra of $\mathfrak{sl}_2$. Why does this map exist? The reason is that $\mathbb{P}^1 \cong G/B$ where $G = SL_2(\mathbb{C})$ and $B$ is the standard Borel subgroup (upper triangular invertible matrices). So $G$ acts transitively on $\mathbb{P}^1$ and given a vector $v \in \mathfrak{sl}_2 = T_eG$ we can use the action of $G$ to translate $v$ and obtain a global vector field. In other words, we have a map
\[\mathfrak{sl}_2 \to \Gamma(T_{\mathbb{P}^1})\]
of Lie algebras which induces the map (1) above.

**Exercise.** Find an element in the kernel of this map.

**Exercise.** Show that if $G$ is a group acting on $X$ then it induces a map $\mathfrak{g} \to \Gamma(T_X)$ of Lie algebras.

**Question.** So why study modules over a sheaf of non-commutative rings?

- One reason is the construction above which generalizes to give a map $U(\mathfrak{g}) \to \Gamma(D_{G/P})$ for any reductive Lie algebra $G$ and parabolic subgroup $P$. Consequently one gets a map from $D_{G/P}$-modules to $U(\mathfrak{g})$ which is a very useful tool in geometric representation theory.

- Another reason is its relation to perverse sheaves. More precisely, inside the (derived) category $\mathcal{D}_{\mathrm{cont}}(X)$ of constructible sheaves on $X$ there is a special $t$-structure. The heart of this $t$-structure is (by definition) the abelian category of perverse sheaves. The advantage of this abelian category (over the standard abelian category of constructible sheaves) is that it’s much better behaved under standard operations like Verdier duality and pushforwards. In particular, the decomposition theorem of Beilinson-Bernstein-Deligne says that the direct image of a perverse sheaf decomposes into a direct sum of perverse sheaves (with shifts).

It turns out that there is a map, the DeRham functor $DR : \mathcal{D}_X \to \mathcal{D}_{\mathrm{cont}}(X) \to \mathcal{D}_{\mathrm{con}}(X)$ which is an equivalence of categories between (regular holonomic) $\mathcal{D}_X$-modules and perverse sheaves. In particular, this means that (regular holonomic) $\mathcal{D}_X$-modules are very well behaved under standard operations. This is remarkable since $\mathcal{O}_X$-modules are not so well behaved – the pushforward of an $\mathcal{O}_X$-sheaf under an arbitrary map can be an arbitrarily nasty complex of sheaves.

- The simplest examples of $\mathcal{D}_X$-modules are locally free sheaves with a flat connection $(E, \nabla)$. The connection is a map $\nabla : E \otimes T_X \to E$ which is the same as an action of $T_X$ on $E$. Flatness implies that $\nabla_{[\partial_{\theta_1}, \partial_{\theta_2}]} = [\nabla_{\partial_\theta_1}, \nabla_{\partial_\theta_2}]$ which is the necessary condition for $\nabla$ to induce a $\mathcal{D}_X$-module structure on $E$.

So, if $V$ is a local system (for example, one that comes from a variation of Hodge structures) then $V \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-module.
If one studies perverse sheaves in positive characteristic (using étale cohomology and so on) then one there is also Deligne’s theory of weights. This theory does not exist over \( \mathbb{C} \). However, one can enhance the category of \( D_X \)-modules to the category of Hodge \( D_X \)-modules where such a theory of weights has been developed by Saito. Understanding some of Saito’s theory is one (long term) hope for this seminar.

1.1. Differential Operators. Let \( U \subset X \) an open subset of \( X \). One can recursively define the sheaf \( D_X(U) \) of differential operators on \( U \) as follows. A \( \mathbb{C} \)-linear map \( D : \mathcal{O}_X(U) \to \mathcal{O}_X(U) \) is a differential operator of order \( k \) if and only if \([D,f]\) is a linear operator of order \( k - 1 \) for any function \( f \in \mathcal{O}_X(U) \). We denote by \( D_X \) the linear operators of order \( \leq i \).

One can check that if \( D_1 \) and \( D_2 \) are differential operators of orders \( k_1 \) and \( k_2 \) then \( D_1D_2 \) has order at most \( k_1 + k_2 \) while \([D_1,D_2]\) has order at most \( k_1 + k_2 - 1 \). This gives \( D_X \) the structure of a sheaf of filtered rings where the associated graded \( gr(D_X) \) is a sheaf of graded commutative \( \mathcal{O}_X \)-algebras.

For example, multiplication by any \( f \in \mathcal{O}_X(U) \) is a linear operator of order zero, while any vector field induces a linear operator of order one.

**Proposition 1.1.** \( D_X^1 \cong \mathcal{O}_X \oplus T_X \) while \( gr(D_X) \cong Sym_{\mathcal{O}_X}(T_X) \).

**Proof.** (sketch) We have a natural map \( Sym_{\mathcal{O}_X}(T_X) \to gr(D_X) \) induced by \( T_X \to D_X^1 \). One can then show by induction that \( gr^i(D_X) \cong Sym^i_{\mathcal{O}_X}(T_X) \). (see Prop 5.2 of [GA]). \( \square \)

**Proposition 1.2.** \( D_X \) is generated as a ring by \( f \in \mathcal{O}_X \) and \( \zeta \in T_X \) subject to the relations

\[
f_1 \ast f_2 = f_1 f_2, f \ast \zeta = f \zeta, \zeta_1 \ast \zeta_2 - \zeta_2 \ast \zeta_1 = [\zeta_1, \zeta_2], \zeta \ast f - f \ast \zeta = \zeta(f)
\]

where \( \ast \) is the multiplication in \( D_X \).

**Proof.** (sketch) The idea is to construct a map of filtered algebras \( A_X \to D_X \) where \( A_X \) is the algebra described above. Then to show this map is an isomorphism it suffices to show that the induced map \( gr(A_X) \to gr(D_X) \cong Sym_{\mathcal{O}_X}(T_X) \) is an isomorphism of graded algebras (see Prop 5.3 of [GA]). \( \square \)

**Remark 1.3.** Notice that the definition of \( D_X \) from \( \mathcal{O}_X \) works equally well to give a sheaf of differential operators starting from any sheaf of commutative rings. For example, you can start from the sheaf of smooth functions to get the sheaf of smooth differential operators. One can also define \( D_X \) if \( X \) is not smooth.

**Talk.** Describe the category of \( D_X \)-modules on \( X = \mathbb{A}^2 \) supported on the line \( x = 0 \). This should be the same as the category of \( D_{\mathbb{A}^1} \)-modules. Now describe the category of \( D_X \)-modules on \( \mathbb{A}^2 \) supported on \( xy = 0 \) and compare with the category of \( D_Y \)-modules where \( Y \) is the variety \( xy = 0 \). Do these categories differ?

2. Sheaves of non-commutative rings

Inspired by the description of \( D_X \) above we begin by studying general (sheaves of) non-commutative rings following closely the first part of [G].

\( A \) is a filtered ring if \( \ldots \cup A_i \subset A_{i+1} \subset \ldots \) with \( A_i A_j \subset A_{i+j} \) and \( 1 \in A_0 \) (also \( \cup_i A_i = A \)). For simplicity we assume \( i \in \mathbb{N} \) (more generally we can let \( i \in \mathbb{Z} \)). \( gr(A) \) denotes the associated graded ring \( \oplus_i (A_i/A_{i-1}) \). \( A \) is called almost commutative if \( gr(A) \) is commutative.

**Example.** Let \( g \) be a Lie algebra. Then the universal enveloping algebra

\[
Ug = Tg/(x \otimes y - y \otimes x = [x, y])
\]

is a filtered algebra where \( g \) has degree one. The Poincare-Birkhoff-Witt theorem states that \( gr(Ug) \cong Symg \). So \( Ug \) is almost commutative. This can be seen quite directly since in \( gr(Ug) \) the relation \( x \otimes y - y \otimes x = [x, y] \) becomes \( x \otimes y - y \otimes x = 0 \) because \([x, y] \) has degree one.
Let $A$ be an almost commutative algebra. Suppose $A_0 = \mathbb{C}$ and $A$ is generated as a ring by $A_1$. Then $A \cong U\mathfrak{g}/I$ for some Lie algebra $\mathfrak{g}$ and ideal $I$.

Proof. Almost commutativity means that $ab - ba \in A_1$ if $a, b \in A_1$. Thus $A_1$ becomes a Lie algebra under the bracket $[a, b] = ab - ba$. By the universal property of $UA_1$ we have a map $UA_1 \to A$ which is surjective since $A$ is generated by $A_1$ (see Prop 1.1.5 of [G]).

Proposition 2.2. If $gr(A)$ is Noetherian then so is $A$ (but not vice-versa). If $gr(A)$ has no zero-divisors then $A$ has no zero-divisors.

A filtered module $M$ over a filtered ring $A$ is a sequence of modules $\cdots \subset M_i \subset M_{i+1} \subset \cdots$ such that $M_i = 0$ for $i \ll 0$. Define $gr(M) := \bigoplus_i M_i/M_{i-1}$ to be the associated graded $gr(A)$-module.

2.1. Rees ring. Define the Rees ring of $A$ by $\hat{A} := \sum t^i A_i$. If one thinks of this as a family of rings over $\mathbb{A}^1$ parametrized by $t$ then the fibre over $t \neq 0$ is isomorphic to $A$ while the fibre over $t = 0$ is isomorphic to $gr(A)$. To see the latter note that there is a map $gr(A) \to \hat{A}/t\hat{A}$ given by $A_i/A_{i-1} \ni a_i \mapsto t^i a_i$. This map is well defined because $t^i a_{i-1} = 0$ on the right hand side if $a_{i-1} \in A_{i-1}$. The inverse map is $t^i a_i \mapsto \hat{a}_i$.

Example. Consider $A = U\mathfrak{g}$ where $\mathfrak{g}$ is some Lie algebra. Then

$$\hat{A} \cong T\mathfrak{g}/(x \otimes y - y \otimes x = t[x, y]).$$

Similarly, define $\hat{M} := \sum t^i M_i$ for any filtered $A$-module $M$. Again we have that the fibre over $t = 0$ is isomorphic to $M$ while the fibre over $t = 0$ is isomorphic to $M/tM \cong gr(M)$.

Remark 2.3. In what way is $\hat{A}$ a flat family of rings over $\mathbb{A}^1$? $\hat{A}$ is a $\mathbb{C}[t]$-module via the natural map $\mathbb{C}[t] \to \hat{A}$. The module structure does not see the multiplicative structure on $\hat{A}$ and since all fibres of $\hat{A}$ are isomorphic once you pass to the associated graded this should mean the family is flat.

More precisely, $\hat{A}$ is a free $\mathbb{C}[t]$ module. Namely, it is isomorphic to $\mathbb{C}[t] \otimes_{\mathbb{C}} gr(A)$. To see this we have an inclusion $A_0 \otimes_{\mathbb{C}} \mathbb{C}[t] \to \hat{A}$ given by $t^i A_0 \mapsto t^i A_i$. The cokernel is isomorphic to $\hat{A}/A_0$. Since short exact sequence of $\mathbb{C}[t]$-modules split we have $\hat{A} \cong A_0 \otimes_{\mathbb{C}} \mathbb{C}[t] \oplus \hat{A}/A_0$. Now you repeat.

2.2. Characteristic variety. A filtration of $M$ is good if $gr(M)$ is a finitely generated $gr(A)$-module. From now on assume that $A$ is almost commutative and that $gr(A)$ is finitely generated as a $\mathbb{C}$-algebra.

Let $M$ be an $A$-module equipped with a good filtration. Then $gr(M)$ is a finitely generated $gr(A)$-module which means that $gr(M)$ is a coherent sheaf on $\text{Spec}(gr(A))$. We define the characteristic variety (or singular support) of $M$ to be the support of $gr(M)$ with its reduced scheme structure. We denote this $CV(M)$. For each component $S$ of $CV(M)$ we denote by $\text{mult}(M, S)$ the multiplicity of $S$ in the support of $gr(M)$.

Proposition 2.4 (J. Bernstein). Neither $CV(M)$ nor $\text{mult}(M, S)$ depend on the choice of a good filtration for $M$. Moreover, if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of $A$-modules then

$$\text{mult}(M, S) = \text{mult}(M', S) + \text{mult}(M'', S).$$

It is possible that you choose different good filtrations and obtain different non-reduced structures. Some properties of good filtrations:
• suppose $M' \subset M$ is an $A$-submodule. Then a good filtration on $M$ induces a good filtration on $M'$.
• the induced filtration on $M/M'$ is also good and there is a short exact sequence
  \[ 0 \to \text{gr}(M') \to \text{gr}(M) \to \text{gr}(M/M') \to 0 \]
which implies
  \[ CV(M) = CV(M') \cup CV(M/M') \]
• if $M \cong A/J$ then $CV(M)$ is the zero variety of $\text{gr}(J)$

**Example.** Let $X = \mathbb{A}^1$. Then $D_X \cong \mathbb{C}(x, \partial_x)/(\partial_x x = x \partial_x + 1)$ with $\text{gr}D_X \cong \mathbb{C}[x, \partial_x]$ where $\partial_x$ has degree one and $x$ degree zero. Then

- $CV(D_X) \cong \text{Spec}(\mathbb{C}[x, y])$ where $y$ has degree one and $x$ degree zero.
- Let $U_\lambda = D_X/D_X(x - \lambda)$. Then $U_\lambda$ as a left $D_X$-module is spanned over $\mathbb{C}$ by elements $\partial_x^i$. Now $x \cdot \partial_x \equiv \partial_x x = \lambda \partial_x$. Thus $CV(U_\lambda)$ is defined by $x - \lambda = 0$.
- Let $V_\lambda = D_X/D_X(\partial_x - \lambda)$. Then $V_\lambda$ as a left $D_X$-module is spanned by elements $x^i$. Now $\partial_x \cdot 1 = \partial_x = \lambda$ so the associated graded relation is $y = 0$. Thus $CV(V_\lambda)$ is defined by $y = 0$.

Notice that when $\lambda = 0$ $D_X/D_X \partial_x$ is the standard representation of the Weyl algebra (namely, the Weyl algebra acting on $\mathbb{C}[x]$). For instance,
  \[ \partial_x \cdot (x^2) = (x \partial_x + 1) \cdot x = x^2 \partial_x + x + x = 2x. \]

For general $\lambda$ we get a deformed version of the standard representation where $\partial_x \cdot 1 = \lambda$ and $\partial_x \cdot x = \lambda x + 1$ and $\partial_x \cdot x^2 = \lambda x^2 + 2x$ and so on. This action is the same as the action induced by $D_X \to D_X$ where $x \mapsto x$ and $\partial_x \mapsto \partial_x + \lambda$.

• Let $M_\lambda = D_X/D_X(x \partial_x - \lambda)$. Then $CV(M_\lambda)$ is given by $xy = 0$. The module is spanned by $x^i$ and $\partial_x^i$ ($i \in \mathbb{N}$). Notice that 1 generates the module so it is indecomposable. If $\lambda \notin \mathbb{Z}$ then $M_\lambda$ is irreducible because one can take any element of $M_\lambda$ and multiplying by $x$ and $\partial_x$ and using relations $x \partial_x = \lambda$ and $\partial_x x = 1 - \lambda$ one can obtain 1. In particular, this gives an example of an irreducible $A$-module whose $CV$ is reducible.

Now if $\lambda \in \mathbb{Z}$ and $\lambda \leq -1$ then $\{x^{-\lambda}, x^{-\lambda+1}, \ldots \}$ is invariant under the action of $D_X$. This is because $\partial_x \cdot x^{-\lambda} = 0$. Thus we get a short exact sequence
  \[ 0 \to V_0 \to M_\lambda \to Q_\lambda \to 0. \]
The quotient module $Q_\lambda$ is spanned by $1, \ldots, x^{-\lambda-1}$ and $\partial_x^i$ with $i \geq 1$.

If $\lambda \geq 0$ then $\{\partial_x^i, \partial_x^{\lambda+1}, \ldots \}$ is invariant and we get a short exact sequence
  \[ 0 \to U_0 \to M_\lambda \to Q_\lambda \to 0 \]
where the quotient $Q_\lambda$ is spanned by $1, \ldots, \partial_x^\lambda$ and $x^i$ with $i \geq 1$.

### 2.3. Gabber filtration.
We assume $\text{Spec} (\text{gr}(A))$ is smooth and connected. For any finite generated (and hence Noetherian) $A$-module $M$ define the Gabber filtration by
  \[ G_i(M) = \{ m \in M : \dim(CV(A \cdot m)) \leq i \}. \]
This is the same as the largest $A$-submodule of $M$ whose $CV$-support has dimension $\leq i$. To see that $G_i(M) \subset M$ is indeed a submodule suppose $m_1, m_2 \in G_i(M)$. Then $A \cdot (m_1 + m_2) \subset A \cdot m_1 + A \cdot m_2$ so it suffices to show that $\dim CV(A \cdot m_1 + A \cdot m_2) \leq i$. But from the surjective map $A \cdot m_1 \oplus A \cdot m_2 \to A \cdot m_1 + A \cdot m_2$ it suffices to show $\dim CV(A \cdot m_1 \oplus A \cdot m_2) \leq i$ which follows by assumption.

**Theorem 2.5.** $CV(G_i(M)/G_{i-1}(M))$ is a variety of pure dimension $i$. 

2.4. Poisson structures. Recall that a Poisson algebra consists of
• a commutative, associative algebra with unit
• a Lie bracket \{·, ·\}
• subject to the Leibniz identity \{ab, c\} = a\{b, c\} + b\{a, c\}.

If the algebra is graded then we allow \{·, ·\} to have some fixed degree \(d\). We then speak of graded Poisson algebras with bracket of degree \(d\).

Proposition 2.6. If \(A\) is filtered, almost commutative then \(\text{gr}(A)\) has a canonical structure of a graded Poisson algebra with bracket of degree \(-1\).

Proof. Define \(\{\bar{a}, \bar{b}\} := \overline{ab - ba}\). One can easily check this is well defined (the fact that \(\text{gr}(A)\) is commutative is used here). Checking the Lie bracket identity and Leibniz identity is immediate. \(\square\)

Notice that if \(A\) is commutative then the Poisson bracket is trivial. So the Poisson bracket measures the non-commutativity of \(A\).

Example. If \(\mathfrak{g}\) is a Lie algebra then \(U(\mathfrak{g})\) is an almost commutative filtered algebra. This induces a Poisson structure on \(\text{Sym}(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]\).

2.5. Some deformation theory. Consider the following diagram of relations. \(Y\) denotes \(\text{Spec}(\text{gr}(A))\) although we can certainly sheafify everything and work on some algebraic Poisson manifold \(Y\) where \(A\) is a sheaf of filtered almost commutative associative algebras whose associated graded is \(\mathcal{O}_Y\).

filtered almost commutative
associative algebra \(A\) \(\rightsquigarrow\) deformations of \(\text{gr}(A)\)

\((-1)\) graded Poisson algebra structure on \(\text{gr}(A)\)

\((P \in H^0(\wedge^2 T_Y)\) (of weight \(-1\)) such that \([P, P] = 0\) \(\longleftarrow\) flat first order deformation of \(\text{gr}(A)\) which extends to a second order deformation

The arrow in the top row associates to \(A\) the Rees ring \(\hat{A} = \oplus t^i A_i\) which is a one parameter deformation of \(\text{gr}(A)\).

The leftmost vertical arrow associates to \(A\) the graded Poisson algebra given by \(\{\bar{a}, \bar{b}\} = \overline{ab - ba}\).

The rightmost vertical arrow associates to a genuine one parameter deformation its first order deformation.

The leftmost bottom arrow associates to any \(P \in H^0(\wedge^2 T_Y)\) a bracket via
\[
\{f, g\} = P(df \wedge dg).
\]

This bracket immediately satisfies the Leibniz condition while \([P, P] = 0\) ensures that it also satisfies the Jacobi identity. Here \([P, P]\) denotes the Schouten-Nijenhuis bracket on polyvector fields. If \(P\) is non-degenerate then \(P\) corresponds to a non-degenerate 2-form \(\omega\) and the condition \([P, P] = 0\) translates to \(d\omega = 0\). This follows from the identity \([v_1, v_2] = -X_{\omega(v_1, v_2)}\) for any two vector fields \(v_1, v_2\) [Prop 3.1. [Mein]].

Conversely, given any Poisson structure, the map \(g \mapsto \{f, g\}\) is a derivation and induces a vector field \(X_f\). This vector field only depends on \(df\) so we get a map \(T_Y \to \mathcal{X}\) given by \(df \mapsto X_f\). The fact that \(\{f, g\} = -\{g, f\}\) means that this map is equivalent to a section \(P \in H^0(\wedge^2 T_Y)\).

In the graded picture we have a graded Poisson algebra structure on \(\text{gr}(A)\). This corresponds to the fact that \(P\) has weight \(-1\). Geometrically this means \(Y\) should have a \(\mathbb{C}^\times\)-action under which \(P\) has weight \(-1\). The standard example is \(Y = \text{Spec} \mathbb{C}[x, y]\) where \(P = \partial_x \wedge \partial_y\) and \(\mathbb{C}^\times\) acts by \((x, y) \mapsto (x, ty)\).
Example. Let us take $X = \mathbb{A}^1 = \text{Spec} \mathbb{C}[x]$ and $Y = T^*\mathbb{A}^1 = \text{Spec} \mathbb{C}[x, y]$ where $y$ has degree 1. We take $A = D_X = \mathbb{C}(x, \partial_x)/\partial_x x = x \partial_x + 1$. Let us compute $\{x^k, y^l\}$. We lift $x$ to $x$ and $y$ to $\partial_x$. Then

$$x^k \partial_y^l - \partial_x x^k = -klx^{k-1} \partial_x^{-1} + \text{lower order terms}$$

so that $\{x^k, y^l\} = -klx^{k-1}y^{l-1}$.

On the other hand, consider the standard symplectic form $\omega = dx \wedge dy$ or equivalently $P = \partial_x \wedge p_y$ (notice this has weight $-1$). Then

$$\{x^k, y^l\} = P(klx^{k-1}y^{l-1} dx \wedge dy) = klx^{k-1}y^{l-1}.$$}

So the two graded Poisson structures we get should agree (up to a sign).

A first order deformation of $gr(A)$ is a map $\mathbb{C}[t]/t^2 \to A'$ such that $A' \otimes_{\mathbb{C}[t]/t^2} \mathbb{C}[t]/t = gr(A)$. This gives us a short exact sequence

$$0 \to tA' \to A' \to gr(A) \to 0$$

and subsequently an isomorphism $A' \cong A \oplus tA'$ as $\mathbb{C}[t]/t^2$-modules via $f \mapsto (\bar{f}, tf)$. The fact that $A'$ is flat means $tA' \cong A$. Thus we get that a flat first order deformation of $gr(A)$ is of the form $gr(A) \oplus tgr(A)$ with some funny multiplication.

The rightmost bottom arrow associates to $P$ the first order deformation given by

$$(f_1, g_1) \ast (f_2, g_2) = (f_1f_2, f_1g_2 + g_1f_2 + P(df_1 \wedge df_2)).$$

Exercise. Check that this defines an associative product.

Conversely, any such deformation is of the form

$$(f_1, g_1) \ast (f_2, g_2) = (f_1f_2, f_1g_2 + g_1f_2 + \alpha(f_1, f_2)).$$

The associativity condition translates to

$$(2) \quad \alpha(f_1f_2, f_3) = f_1\alpha(f_2, f_3) - f_3\alpha(f_1, f_2) + \alpha(f_1, f_2f_3).$$

In this context, the condition $[P, P] = 0$ corresponds to the fact that this deformation can be lifted to second order.

2.6. Hochschild cohomology. The local version of Hochschild cohomology consists of a (graded) algebra $A$ and the chain complex

$$\ldots \overset{d}{\to} \text{Hom}_\mathbb{C}(A \otimes^n A, A) \overset{d}{\to} \text{Hom}_\mathbb{C}(A \otimes^{n+1} A, A) \overset{d}{\to} \ldots$$

where

$$d(\phi)(a_0, \ldots, a_n) = a_0\phi(a_1, \ldots, a_n) - \phi(a_0a_1, a_2, \ldots, a_n) + \cdots + (-1)^{n+1}\phi(a_0, \ldots, a_{n-1})a_n.$$  

Example. $Z^0(A, A)$ consists of maps $\phi : A \to A$ such that $d\phi(a) = a\phi(1) - \phi(1)a = 0$. So $HH^0(A, A) = A^A$ (i.e. elements $h = \phi(1)$ in the centre of $A$). If $A$ is commutative then $HH^0(A, A) = A$.

$Z^1(A, A)$ consists of maps $\phi : A \to A$ such that $\phi(f_1f_2) = f_1\phi(f_2) - \phi(f_1)f_2$. If $A$ is commutative then these are derivations of $A$ and the image $dC^0(A)$ is zero so $HH^1(A, A)$ consists of vector fields on $\text{Spec}(A)$.

If $A$ is commutative then the same argument shows that $HH^*(A, A)$ consists of polyvector fields on $\text{Spec}(A)$.

Now there is a cup product at the level of cochains which descends to a product on $HH^*(A, A)$. Moreover, there is a bracket on $HH^1(A, A)$ which is given by the commutator: recall that $C^1(A, A) = \text{Hom}_\mathbb{C}(A, A)$. This bracket extends via an equation just like (3) to a bracket on $HH^*(A, A)$. This gives $HH^*(A, A)$ the structure of a Gerstenhaber algebra (see the next section for a definition). If $A$ is commutative then this gives the usual Gerstenhaber structure on polyvector fields.
The non-local version consists of a (smooth) variety \( Y \) (not-necessarily affine) where the aim is to
deform the structure sheaf \( \mathcal{O}_Y \). One parameter deformations of \( \mathcal{O}_Y \) are parametrized by the Hochschild
cohomology group \( HH^2(\text{Coh}(Y)) \) of the category of coherent sheaves on \( Y \). This is the same as \( \text{Ext}^2_{X \times Y}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \) where \( \Delta \subset X \times Y \) denotes the diagonal. By the Hochschild-Kostant-Rosenberg we have
\[
\text{Ext}^2_{X \times Y}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong H^0(\wedge^2 T_Y) \oplus H^1(T_Y) \oplus H^2(\mathcal{O}_Y)
\]
(the content of the theorem is that the spectral sequence computing the left hand side degenerates). If \( Y \) is affine then the higher cohomology groups vanish so we are left with \( H^0(\wedge^2 T_Y) \) as above.

In general \( H^1(T_Y) \) correspond to first order deformations of \( Y \). To be precise, if \( \tilde{Y} \to \text{Spec} \ C[t] \) is a
deformation of \( Y \) then we get the standard short exact sequence
\[
0 \to T_Y \to T_{\tilde{Y}}|_Y \to \mathcal{O}_Y \to 0.
\]
The extension class of this sequence is an element in \( \text{Ext}^1(\mathcal{O}_Y, T_Y) = H^1(T_Y) \). This corresponds to a
defformation of \( \mathcal{O}_Y \) which does not split as \( \mathcal{O}_Y \oplus t\mathcal{O}_Y \).

The third term \( H^2(\mathcal{O}_Y) \) corresponds to looking at the category of twisted sheaves. Fix a 2-cocycle \( \alpha_{i,j,k} \) with respect to some open cover \( \{U_i\} \) of \( Y \). Then a twisted sheaf is a pair \( (\mathcal{F}_i, \phi_{ij}) \) where \( \mathcal{F}_i \) is a sheaf on \( U_i \) and
\[
\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \overset{\sim}{\to} \mathcal{F}_j|_{U_i \cap U_j}
\]
are isomorphisms such that
- \( \phi_{ii} = 1 \)
- \( \phi_{ij} = \phi_{ji}^{-1} \)
- \( \phi_{ki} \circ \phi_{jk} \circ \phi_{ij} = \alpha_{i,j,k} \).
If all \( \alpha_{i,j,k} = 1 \) this is just the usual sheaf. If \( \alpha_{i,j,k} \in H^2(\mathcal{O}_Y^*) \) (i.e. does not vanish) then one obtains
the category of coherent twisted sheaves \( \text{Coh}(Y, \alpha) \).

2.6.1. Higher order deformations. If \( A \) is a (graded) algebra then a higher order deformation of \( A \) corresponds to a deformation of the product \( m \) and can be written as
\[
m_t = m + t\beta_1 + t^2\beta_2 + \ldots
\]
where \( \beta_1 \in HC^2(A, A) \) are Hochschild 2-cochains. In other words words \( \beta_1 : A \otimes_C A \to A \) where
\( d\beta_1 = 0 \). The condition that \( m + t\beta_1 \) is associative is precisely equation (2) which is equivalent to the
condition that \( d\beta_1 = 0 \) that
\[
d\beta(f_1, f_2, f_3) = f_1\beta(f_2, f_3) - \beta(f_1f_2, f_3) + \beta(f_1, f_2f_3) - \beta(f_1, f_2)f_3.
\]
The condition that \( m + t\beta_1 + t^2\beta_2 \) is associative is given by the Maurer-Cartan relation \( d\beta_2 + [\beta_1, \beta_1] = 0 \).
This condition is equivalent to saying that \( [\beta_1, \beta_1] \) vanishes in \( HH^3(A, A) \). If \( A \) is commutative then
\( HH^3(A, A) \) is the same as \( H^0(\wedge^3 T_Y) \) where \( Y = \text{Spec} A \).

If \( HH^3(A, A) = 0 \) then all the associativity conditions automatically hold and one can lift \( m + t\beta_1 \)
to arbitrary order.

The higher Hochschild cohomology groups come in if one considers deformations of an \( A_\infty \) algebra
\( A \). Recall that an \( A_\infty \) algebra is a graded vector space \( A \) with higher multiplications \( m_n : A^{\otimes n} \to A \)
(satisfying some relations). If \( m_n = 0 \) for \( n \geq 3 \) then \( A \) is just a differential graded algebra and if
\( m_1 = 0 \) as well then one obtains a graded algebra. See [K] for a brief introduction to \( A_\infty \) algebras.

2.7. (super) Gerstenhaber algebras. A reference for this subsection is [KS].

A Gerstenhaber algebra differs from a super graded Poisson algebra in that \( \{\cdot, \cdot\} \) has degree \(-1\)
instead of degree 0. To be precise, a Gerstenhaber algebra is a (differential) graded, associative algebra
with a bracket \( \{\cdot, \cdot\} \) satisfying:
- \( ab = (-1)^{|a||b|}ba \)
• \{a, bc\} = \{a, b\}c + (-1)^{(|a|-1)|b|}b\{a, c\}
• \{a, b\} = -(-1)^{(|a|-1)(|b|-1)}1\{b, a\}
• \{(a, b), c\} = \{a, \{b, c\}\} + (-1)^{(|a|-1)(|b|-1)}\{b, \{a, c\}\}

Standard examples of Gerstenhaber algebras include

• the algebra of polyvector fields \∧^*T_Y on a manifold Y where multiplication is given by wedging – the Lie bracket is given by extending the usual Lie bracket on T_Y. This gives the Schouten-Nijenhuis bracket and is defined by

\[ [a_1 \ldots a_m, b_1 \ldots b_n] = \sum_{i,j} (-1)^{i+j}[a_i, b_j]a_1 \ldots a_{i-1}a_{i+1} \ldots a_m b_1 \ldots b_{j-1}b_{j+1} \ldots b_n. \]

• the Hochschild cohomology \HH^*(A, A) of a graded algebra A

Let A be an associative, graded commutative algebra. Let \( f : A \to \text{End}(A) \) be a map of degree \(|f|\). If \( u, v \in \text{End}(A) \) we denote

\[ [u, v] := uv - (-1)^{|u||v|}vu. \]

For \( a, b \in A \) we set \( [a, b] = f(a)b \). This means \([A^i, A^j] \subseteq A^{i+j+|f|}\). We impose that

(i) graded Jacobi identity: \([f(a), f(b)] = f(f(a)b)\)
(ii) graded skew-symmetry identity: \([f(a)b] = (-1)^{(|a|+|f|)(|b|+|f|)}f(b)a\)
(iii) \( f(a) \) is a graded derivation of \( A \) of degree \(|a| + |f|\)

Then if \(|f|\) is even then by definition we have a super graded Poisson algebra, if \(|f|\) is odd we get a Gerstenhaber algebra. Notice that if you ignore the grading (\textit{i.e.} make everything degree zero) then you get the usual definition of a Poisson algebra.

\textbf{Example.} Suppose we start with a graded associative, almost super-commutative algebra \( B \) and consider \( A = \text{gr}(B) \). Then we can take \( f : A \to \text{End}(A) \) to be

\[ f(\hat{a})(\hat{b}) = ab - (-1)^{(|a|-1)(|b|-1)}ba \]

where \( a, b \in B \). In this case \(|f| = -1\).

\textbf{Exercise.} Check that this definition leads to a Gerstenhaber algebra.

If we start with \( \mathcal{D}_X \) then we do not get a Gerstenhaber algebra (a.k.a. a graded Poisson super algebra of odd degree) because \( \mathcal{D}_X \) is almost commutative (not almost super-commutative). Consequently we get only a graded Poisson algebra with bracket of degree \(-1\).

On the other hand, recall (Proposition 1.2) that \( \mathcal{D}_X \) is generated as a ring by \( f \in \mathcal{O}_X \) and \( \zeta \in T_X \) subject to some relations. One of these relations is \( \zeta_1 \ast \zeta_2 - \zeta_2 \ast \zeta_1 = [\zeta_1, \zeta_2] \). It is this relation that leads to an almost commutative algebra (instead of an almost super-commutative algebra). If we change this relation to

\[ \zeta_1 \ast \zeta_2 + \zeta_2 \ast \zeta_1 = [\zeta_1, \zeta_2] \]

then we get an almost super-commutative algebra. The associated graded is just \( \mathcal{D}^*T_X \) (instead of \( \text{Sym}^*T_X \)). The construction above then leads to the Gerstenhaber structure on the wedge algebra of polyvector fields.

3. Fedosov Quantization

Basic question: given a (holomorphic) Poisson structure \( P \) on a complex variety \( X \) can you deform \( \mathcal{O}_X \) along direction \( P \)?

More precisely, as we saw \( P \) defines a first order deformation of \( \mathcal{O}_X \). We would like to know if this first order deformation extends to a formal deformation and, if it does, then how many such deformations are there.
This question was studied in the smooth case by De Wilde-Lecomte and Fedosov when \( P \) is a symplectic structure (Deligne has a classic exposition of these results). We present an algebraic approach to this problem when \( X \) is (holomorphic) symplectic following [BK].

**Definition.** A quantization \( \mathcal{D} \) of \( \mathcal{O}_X \) is a sheaf of associative, flat \( \mathbb{C}[\hbar] \) algebras on \( X \) (complete in the \( \hbar \)-adic topology) and equipped with an isomorphism \( \mathcal{D}/\hbar \mathcal{D} \cong \mathcal{O}_X \).

As we learned already a quantization leads to a Poisson structure with associated bivector field \( P \). Conversely, any first order quantization which splits (i.e. the deformation is purely non-commutative) is given by a bivector field \( P \). We say \( \mathcal{D} \) is non-degenerate if \( P \) is non-degenerate (i.e. \( P \) induces an isomorphism \( T_X \rightarrow \Omega_X \)).

**Question.** Classify non-degenerate quantizations \( \mathcal{D} \) of \( \mathcal{O}_X \).

For example, we have the following classification of non-degenerate quantizations of the formal polydisc.

**Lemma 3.1.** Let \( X = \mathrm{Spec} \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \) denote the formal polydisc. Then a non-degenerate quantization is isomorphic to the (formal) Weyl algebra

\[
[x_i, x_j] = [y_i, y_j] = [h, x_i] = [h, x_j] = 0 \quad \text{and} \quad [x_i, y_j] = \delta_{ij}h.
\]

Notice that \( HH^2(\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]) \) has basis given by 2-vector fields. Those corresponding to non-degenerate first order deformations are the non-degenerate 2-vector fields. The result above basically says that any such non-degenerate 2-vector field is isomorphic (after changing coordinates) to \( \sum_i \partial_{x_i} \wedge \partial_{y_i} \) (which is a Darboux-type result).

**3.1. Notation and assumptions.** Let \( X \) be a smooth variety and denote by \( \Omega^n_X \) the DeRham complex

\[
0 \rightarrow \mathcal{O}_X \overset{\partial}{\rightarrow} \Omega^1_X \overset{\partial}{\rightarrow} \Omega^2_X \overset{\partial}{\rightarrow} \ldots
\]

The (hyper)cohomology of this complex is denoted \( H^*_{DR}(X) \). Now consider the exact sequence

\[
\Omega^{\geq 1}_X \rightarrow \Omega^*_X \rightarrow \mathcal{O}_X
\]

where the left hand term is the kernel of the second map. We denote by \( H^*_P(X) \) the (hyper)cohomology of \( \Omega^{\geq 1}_X \).

We say that \( X \) is admissible if the induced maps \( H^i_{DR}(X) \rightarrow H^i(\mathcal{O}_X) \) are surjective for \( i = 1, 2 \). This is the case if \( X \) is affine (since \( H^i(\mathcal{O}_X) = 0 \) for \( i > 0 \)) or if \( X \) is projective (Hodge theory).

If \( X \) is admissible then

\[
H^0_{DR}(X) \cong H^2_P(X) \oplus H^2(\mathcal{O}_X).
\]

We have a canonical map \( H^2_P(X) \rightarrow H^2_{DR}(X) \) but no canonical map \( \mathrm{pr} : H^2_{DR}(X) \rightarrow H^2_P(X) \).

Notice also that for each \( n \) there is a map \( \Omega^n_X \rightarrow \Omega^\leq n_X \) which induces an isomorphism on (hyper)cohomology groups \( H^\leq n \). On the other hand there is a map \( \Omega^n_X [-n] \rightarrow \Omega^\leq n_X \) which induces a map

\[
H^0(\Omega^n_X) \rightarrow H^n(\Omega^\leq n_X) = H^0_{DR}(X).
\]

**3.2. Symplectic deformations.** We begin by considering the following commutative deformation problem: classify (commutative) deformations of the pair \( (X, \Omega) \).

**Theorem 3.2 ([KV] 2.6).** Let \( X \) be an admissible holomorphic symplectic variety with form \( \Omega \). Denote by \( [\Omega] \in H^2_{DR}(X) \) the corresponding class. Let \( \mathcal{D}(X, \Omega) \) denote the universal formal deformation space of \( (X, \Omega) \). Then there exists a natural embedding

\[
\text{Per} : \mathcal{D}(X, \Omega) \rightarrow H^2_{DR}(X)
\]

called the period map which remembers the associated cohomology class \( [\Omega_h] \in H^2_{DR}(X) \). Composed with any splitting \( \text{pr} : H^2_{DR}(X) \rightarrow H^2_P(X) \) this identifies \( \mathcal{D}(X, \Omega) \) with the formal completion of \( H^2_P(X) \) at the point \( [\Omega] \in H^2_P(X) \).
In particular, there exists a formal symplectic deformation over the formal completion of $H^2_\omega(X)$ at $[\Omega]$ such that any deformation over some Artin scheme $S$ is the pullback of this universal deformation via the period map $S \to H^2_\omega(X)$.

When $X$ is compact, Bogomolov and Beauville show that the period map immerses the coarse marked moduli space $M$ of $(X, \Omega)$ into $H^2_{DR}(X)$. The image is the quadric in $H^2_{DR}(X)$ cut out by the Bogomolov-Beauville form. Moreover, the period map $M \to H^2_{DR}(X)$ is locally an isomorphism.

### 3.3. Non-commutative deformations

The following theorem classifies non-degenerate quantizations of an admissible holomorphic symplectic variety.

**Theorem 3.3 ([BK] 1.8)** Let $X$ be an admissible holomorphic symplectic variety with form $\Omega$. Denote by $[\Omega] \in H^2_{DR}(X)$ the corresponding class. Let $Q(X, \Omega)$ denote the set of isomorphism classes of quantizations of $X$ extending the first order quantization induced by $\Omega$. Then there exists a natural injection

$$\text{Per} : Q(X, \Omega) \to H^2_{DR}(X)[[h]]$$

called the non-commutative period map. For $q \in Q(X, \Omega)$ the constant term of $\text{Per}(q)$ is $[\Omega]$. Moreover, any splitting $\text{pr} : H^2_{DR}(X) \to H^2_\omega(X)$ of the canonical embedding $H^2_\omega(X) \to H^2_{DR}(X)$ induces an isomorphism

$$\text{pr} \circ \text{Per} : Q(X, \Omega) \xrightarrow{\sim} \text{pr}([\Omega]) + hH^2_\omega(X)[[h]] \subset H^2_\omega(X)[[h]].$$

If $X$ is projective then Hodge theory provides a canonical splitting $\text{pr}$. A quantization $\mathcal{D}$ is canonical if its period map is the constant power series $[\Omega]$. A canonical quantization need not exist, even if $X$ is admissible.

Why do these theorems 3.2 and 3.3 look so similar? Consider the universal family $(\tilde{X}, \tilde{\Omega})$ over $\text{Def}(X, \Omega)$. By theorem 3.3 (which also holds in families) this has a quantization which gives a family over $\Delta \times \text{Def}(X, \Omega)$ where $\Delta$ is the formal disc. Now for any section $s : \Delta \to \Delta \times \text{Def}(X, \Omega)$ the pullback of the universal deformation via $s$ gives a quantization of $X$. Algebraically, any such section $s$ is given by a power series $P_s \in H^2_{DR}(X)[[h]]$.

**Lemma 3.4 ([BK], Lemma 6.4).** If the quantization of $(\tilde{X}, \tilde{\Omega}) \to \text{Def}(X, \Omega)$ is canonical then the non-commutative period map sends the quantization given by section $s$ to the formal power series $P_s$.

**Remark:** If $X$ is compact it would be interesting to obtain a Bogomolov-Beauville type result. In other words, one would like to identity explicitly the image of the moduli space of non-commutative deformations inside $H^2_{DR}(X)$.

### 4. More sheaves of non-commutative rings

**4.1. Gabber’s Theorem.** A subvariety $V \subset \text{Spec}(B)$ is coisotropic if for any $f, g \in I_V$ (where $I_V$ denotes the ideal defining $V$) one has $\{f, g\} \in I_V$.

**Theorem 4.1.** Suppose $A$ is almost commutative and $\text{gr}(A)$ Noetherian. Let $M$ be a finite generated $A$-module. Then $CV(M) \subset \text{Spec}(\text{gr}(A))$ is coisotropic (with respect to the natural Poisson structure on $\text{gr}(A)$).

**Proof.** The proof is not so easy and originally due to Gabber. In section 1.2.7 of [G] a simpler proof by Knop is presented.

If $M = A/J$ then $I_{CV(M)} = \sqrt{\text{gr}(J)}$. Proving this special case implies the theorem. It is actually easy to show that $\{f, g\} \in \text{gr}(J)$ if $f, g \in \text{gr}(J)$. But in general it is not true that $\{I, I\} \subset I$ implies $\{\sqrt{I}, \sqrt{I}\} \subset \sqrt{I}$ in a Poisson algebra.

**Remark 4.2.** If one translates the isotropic condition into the language of symplectic geometry then it’s equivalent to asking that the tangent space to $CV(M)$ be coisotropic with respect to the holomorphic symplectic form on $X$. 

Example. \( D_X \) is a sheaf of almost commutative filtered algebras. The associated graded is the sheaf \( \text{Sym}(T_X) \). Sections of this sheaf are the same as sections of the structure sheaf of the cotangent bundle \( T'_X \) of \( X \) (in general sections of \( \text{Sym}(E) \) are the same as functions on the total space \( E' \)). Then by the above \( T'_X \) carries a Poisson structure. This Poisson structure is induced by the natural 2-form on \( T'_X \).

More precisely, if \( \pi : E \to X \) is a vector bundle then there is a natural map \( \pi^*(E') \to \mathcal{O}_E \). Applying this to \( \pi : T_X \to X \) we get a natural map \( \pi^*(T_X) \to \mathcal{O}_{T'_X} \). Precomposing with the natural map \( T_{T'_X} \to \pi^*(T_X) \) we get a map \( T_{T'_X} \to \mathcal{O}_{T'_X} \) (in other words, \( T'_X \) carries a natural 1-form \( \gamma \)). Let \( \omega = d\gamma \). Then \( \omega \) is non-degenerate and hence induces an isomorphism \( T_X \cong T'_X \). In particular, we also have a natural section \( \alpha \in \Gamma(\wedge^2 T_{T'_X}) \) which induces the Poisson structure mentioned above.

4.2. Commutative localization. Recall that if \( A \) is a commutative ring then \( \text{Spec} \, A \) carries a sheaf of algebras. Namely to any open \( U \subset \text{Spec} \, A \) we associate the localization \( A_U := S^{-1}A \) of \( A \) at the multiplicative set \( S \) made up of functions invertible on \( U \). Moreover, to an \( A \)-module \( M \) we associate a sheaf of modules via \( U \mapsto M \otimes_A A_U \). This gives us the usual equivalence between finitely generated \( A \)-modules and coherent sheaves on \( \text{Spec} \, A \).

Next we would like to generalize this point of view to when \( A \) is a filtered almost commutative algebra.

4.3. Non-commutative localization. Let \( A \) be an associative ring with unit. \( S \subset A \) is multiplicative if \( 1 \in S, 0 \notin S \) and it is closed under multiplication. If \( A \) were commutative then we would define the localization \( A_S \) of \( A \) by formally inverting elements in \( S \).

When \( A \) is non-commutative there are problems with doing this. For instance, it is not clear if we should throw in elements of the form \( s^{-1}a \) or \( as^{-1} \) and then it seems difficult to multiply two elements of the form \( s^{-1}a \).

To fix this we impose the Ore conditions on \( S \):

- any \( s^{-1}a_1 \) can be written as \( a_2s^{-1} \) (and vice versa)
- if \( sa = 0 \) then \( as' = 0 \) for some \( s' \) (and vice versa)

Usually one avoids the second condition by asking that \( S \) contain no zero-divisors.

**Theorem 4.3** (Ore). If Ore’s condition is satisfied then it makes sense to define \( A_S \) as the ring of elements of the form \( s^{-1}a \) (and this \( A_S \) satisfies the universal property of localization).

**Proof.** For instance, multiplication is defined in the obvious way as

\[
 s_1^{-1}a_1 \cdot s_2^{-1}a_2 = s_1^{-1}s_3^{-1}a_3a_2 = (s_3s_1)^{-1}a_3a_2.
\]

\( \Box \)

**Remark 4.4.** One can localize in a similar manner additive categories (note that a ring is an additive category with one object). In this setting \( A \) is an additive category, \( S \) is a subcategory where the objects are the same but we take a subset of morphisms which is closed under composition and contains all identity morphisms. Then the localization \( A_S \) is the category where the objects are the same and maps \( X \to Z \) are diagrams of the form \( X \to fY \leftarrow gZ \) where \( g \in S \).

Here are two conditions for verifying that \( S \) satisfies the Ore condition.

**Proposition 4.5.** If \( [s, \cdot] \) is a (locally) nilpotent operator for any \( s \in S \) then \( S \) satisfies the Ore condition.

**Proof.** Follows directly by writing out the nilpotency condition \( [s, [s, \ldots [s, a] \ldots]] = 0 \). \( \Box \)

**Proposition 4.6.** Suppose \( \text{gr}(A) \) is finitely generated and commutative. Given multiplicative \( \bar{S} \subset \text{gr}(A) \) with no zero divisors let \( S := \{ s \in A : \bar{s} \in \bar{S} \} \). Then \( S \subset A \) satisfies the Ore conditions.

We now assume $A$ has no zero divisors. So then $A_S$ exists for any multiplicative subset $\bar{S} \subset \text{gr}(A)$. We say that $s^{-1}a$ has degree $\le j-i$ if $\deg(\bar{a}) = j$ and $\deg(\bar{s}) = i$. Notice that this is a $\mathbb{Z}$-filtration on $S^{-1}A$ since there are non-trivial terms in all negative degrees.

**Exercise.** Check that this induces a well defined filtration on $A_S$.

We call the completion $A_S$ of $S^{-1}A$ in the topology defined by this filtration the *formal microlocalization* of $A$ at $S$. Given an $A$-module $M$ we define its formal microlocalization by $M_S := A_S \otimes_A M$.

**Properties:**

- have $A \rightarrow A_S$ where $A_S$ is flat over $A$ (since localization and completion are exact functors)
- $\text{gr}(A_S) = (\bar{S})^{-1}\text{gr}(A)$ (because taking associated grading does not see the completion)

Now, for any $\mathbb{C}^*$-equivariant open subset $U \subset \text{Spec gr}(A)$ (we think of $\text{Spec gr}(A)$ as having the Zariski cone-topology – i.e. open sets are cone-subsets) we denote by $A_U$ the microlocalization of $A$ with respect to the set of elements invertible on $U$. We define $M_U := A_U \otimes_A M$.

**Proposition 4.7.** $U \mapsto A_U$ defines a sheaf of algebras on $\text{Spec gr}(A)$ (thought of as a cone-scheme). Also, $M \mapsto M_U$ gives a sheaf of modules for any finitely generated $A$-module $M$.

For an $\mathcal{O}_X$-sheaf of filtered almost commutative algebras $A$ on $X$ we have $\text{Spec}_X(\text{gr}A) \rightarrow X$ (equipped with a $\mathbb{C}^*$ action fibrewise). The localization picture above gives us a sheaf of filtered almost commutative algebras on $\text{Spec}_X(\text{gr}A)$ whose associated graded is the structure sheaf. In other words, via the Rees construction we get a non-commutative deformation of the structure sheaf of $\text{Spec}_X(\text{gr}A)$.

For example, if we start with some variety $X$ and take $A = \mathcal{D}_X$ (the sheaf of differential operators) then $\text{Spec gr}A \cong T_X'$. The associated sheaf of algebras on $T_X'$ induces a first order (non-commutative) deformation of $T_X'$ which is encoded by an element in $H^0(\wedge^2T_X')$. This element is (the dual of) the natural symplectic form on $T_X'$.

4.4. **Duality and the Sato-Kashiwara filtration.** We will assume in this section that $A$ is almost commutative algebra with $X := \text{Spec gr}(A)$ smooth. The grading on $\text{gr}(A)$ induces a $\mathbb{C}^*$ action on $X$ whose fixed point set $X_0 := \text{Spec A}_0$ must also be is also smooth. (why?)

**Lemma 4.8.** Let $M$ be a filtered $A$-module such that

$$0 \leftarrow \text{gr}M \leftarrow \text{gr}(A)^{\oplus k} \leftarrow \text{gr}(A)^{\oplus l}$$

is exact. Then one can lift this exact sequence to an exact sequence

$$0 \leftarrow M \leftarrow A^{\oplus k} \leftarrow A^{\oplus l}$$

where the maps are strictly compatible with filtrations.

**Proof.** A map $f : N \rightarrow N'$ is strictly compatible with filtrations if $f(N_i) = f(N) \cap N'_i$. See Lemma 1.4.2 of [G] for a sketch (you get your hands dirty a bit).

**Corollary 4.9.** Given a projective resolution

$$0 \leftarrow \text{gr}M \leftarrow N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \ldots$$

one can (after possibly having to localize with respect on $X_0$) lift this resolution to a free resolution

$$0 \leftarrow M \leftarrow A^{k_0} \leftarrow A^{k_1} \leftarrow A^{k_2} \leftarrow \ldots$$

where the maps are strictly compatible with filtrations.
Proof. Notice that if $A$ is filtered almost commutative then for any $s \in A_0$ the operator $[s, \cdot]$ is nilpotent as a direct consequence of $\text{gr}A$ being commutative. So, in this case, any multiplicative subset of $A_0$ satisfies the Ore condition and we can localize. Thus we can shrink $X_0$ so that the resolution for $\text{gr}M$ becomes free. Then by the Lemma above we can lift this resolution. $\square$

**Corollary 4.10.** The homological dimension of $A$ is equal to $\dim X$.

Proof. We can lift any free resolution of $\text{gr}M$ to a free resolution of $M$. So the length of the minimal free resolution of $M$ and $\text{gr}M$ are the same. Hence $A$ and $\text{gr}A$ have the same homological dimension (namely $\dim X$).

If $M$ is a left (resp. right) $A$-module then $\text{Hom}_A(M, A)$ is a right (resp. left) $A$-module where the multiplication is on the $A$. If $M$ is filtered then $\text{Hom}_A(M, A)$ inherits a filtration: $\phi : M \to A$ is in the $l$th term if $\phi(M_i) \subset A_{l+i}$ for any $i$. One can check that $\text{gr} \text{Hom}_A(M, A) = \text{Hom}_\text{gr}A(\text{gr}M, \text{gr}A)$. More generally, this means that:

**Lemma 4.11.** $\text{Ext}^j_A(M, A)$ is a right (resp. left) $A$-module for $j \in \mathbb{Z}$ and $\text{gr} \text{Ext}^j_A(M, A)$ is a subquotient of $\text{Ext}^j_{\text{gr}A}(\text{gr}M, \text{gr}A)$.

Proof. Take a free resolution of $\text{gr}M$ and lift it to a free filtered resolution $M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$. Then $\text{Ext}^j_A(M, A)$ is computed using the complex

$$0 \to \text{Hom}_A(F_0, A) \to \text{Hom}_A(F_1, A) \to \ldots$$

Since $\text{gr} \text{Hom}_A(M, A) = \text{Hom}_{\text{gr}A}(\text{gr}M, \text{gr}A)$ there is a spectral sequence which computes the cohomology of this complex from $\text{Ext}^j_{\text{gr}A}(\text{gr}M, \text{gr}A)$. The result follows. $\square$

Applying $M \mapsto \text{Hom}_A(M, A)$ in the derived category induces duality functors

$$D : D^b(\text{mod} - A) \leftrightarrow D^b(\text{mod} - A)$$

To make sure that one lands in the bounded derived category it suffices that $A$ has finite cohomological dimension (which follows if $\text{gr}A$ is smooth).

Example: $DA = A$ while $D(A/\text{Aa}) = A/(aA)[{-1}]$ (the second example corresponds to a divisor in $X$).

**Lemma 4.12.** There is a functorial isomorphism $DD(M) \cong M$ in $D^b(A)$.

Given $M \in D^b(\text{mod} - A)$ we denote by $CV(M) := \cup_i CV(H^i(M))$. We also let $d(M) := \dim CV(M)$ and $j(M) := \min\{i : \text{Ext}^i_A(M, A) \neq 0\}$.

**Proposition 4.13.** If $M$ is a finitely generated $A$-module then

- $j(M) + d(M) = m$
- $d(\text{Ext}^i_A(M, A)) \leq m - i$
- $d(\text{Ext}^j_A(M, A)) = d(M)$

where $m = \dim \text{Spec} \text{gr}A$.

Proof. If we replace $A$, $M$ and $CV$ by $\text{gr}A$, $\text{gr}M$ and $\text{supp}$ then this is a standard result in commutative algebra. My personal mnemonic for this result is as follows. Let $i : Y \hookrightarrow X$ be a subvariety where $X$ is proper. Then for any sheaf $\mathcal{F}$ on $X$ we have

$$H^i(\mathcal{O}_Y \otimes \mathcal{F}) \cong \text{Hom}(\mathcal{O}_Y, \mathcal{F}[i]) \cong \text{Hom}(\mathcal{F}[i], \mathcal{O}_Y \otimes \omega_X[m]) \cong \text{Hom}(\mathcal{F}[i], i_*i^*\omega_X[m])$$

$$\cong \text{Hom}(i^*\mathcal{F}[i], i^*\omega_X[m]) \cong \text{Hom}(i^*\omega_X[m], i^*\mathcal{F}[i] \otimes \omega_Y[n])$$

$$\cong \text{Hom}(\omega_X[m], i^*\mathcal{F}[i] \otimes \omega_Y[n]) \cong \text{Hom}(\mathcal{O}_Y, \mathcal{F}[i] \otimes \omega_{Y/X}[n - m])$$

$$\cong H^i(\mathcal{F} \otimes \omega_{Y/X}[n - m])$$
where \( m = \dim X \) and \( n = \dim Y \). This means that \( \mathcal{O}_Y^\times \cong \omega_{Y/X}[\text{-codim} Y] \). In particular, we see that \( \mathcal{O}_Y^\times \) is supported in degree \( \text{codim}(Y) \) and has support \( Y \). The moral is that this commutative algebra result can be viewed as a formal consequence of Serre duality.

The result when \( A \) is almost commutative uses that \( \text{gr} \text{Ext}_A^j(M, A) \) is a subquotient of \( \text{Ext}_{\text{gr}A}^j(\text{gr}M, \text{gr}A) \) (Lemma above).

Given a complex \( K \) denote by \( \tau_{\geq i}K \) the truncation of \( K \) so that \( \mathcal{H}^j(\tau_{\geq i}K) = 0 \) for \( j < i \). Now apply \( D \) to the canonical map \( \mathbf{D}M \to \tau_{\geq i}\mathbf{D}M \) to obtain a map

\[
\mathbf{D}(\tau_{\geq i}(\mathbf{D}M)) \to \mathbf{D}M \cong M.
\]

Define \( S_i M \) to be the image on zero cohomology

\[
H^0(\mathbf{D}(\tau_{\geq i}(\mathbf{D}M))) \to H^0(\mathbf{D}M) = M.
\]

Notice that \( S_i M = M \) if \( i < 0 \) while \( S_i M = 0 \) if \( i \) is greater than the cohomological dimension of \( A \). The filtration \( \{S_j M\} \) is called the Sato-Kashiwara filtration.

**Theorem 4.14.** \( G_i(M) = S_{m-i}(M) \) where the Gabber filtration \( G_i(M) \subset M \) is the largest \( A \)-submodule whose \( CV \)-support has dimension \( \leq i \).

**Proof.** For details see Appendix D of [HTT].

By the Proposition above \( G_i(M) \) is the same as the largest submodule of \( M \) satisfying \( j(M) \geq m-i \). One first uses this description to show that \( G_i(M)/G_{i-1}(M) \) is purely \( i \)-dimensional.

Next, from the definition of \( S_i \) one gets an exact sequence

\[
0 \to S_{i+1}(M) \to S_i(M) \to \mathcal{E}xt^i_A(\mathcal{E}xt^i_A(M, A), A)
\]

from which it follows \( j(S_i(M)) \geq i \) by a reverse induction on \( i \). Thus \( S_i(M) \subset G_{m-i}(S) \).

To get the opposite inclusion set \( N := G_{m-i}(M) \). Since \( j(N) \geq i \) we get

\[
\tau_{\geq i} \text{Hom}_A(N, A) = \text{Hom}_A(N, A)
\]

and hence \( N = S_i(N) \). By functoriality of \( S_i \) we get that \( N \hookrightarrow M \) implies \( N = S_i(N) \hookrightarrow S_i(M) \) and we are done. \( \square \)

**Remark 4.15.** The definition of the Gabber filtration is intuitive but hard to work with. The Sato-Kashiwara filtration is more abstract, but has good functorial properties.

**Lemma 4.16.** The Sato-Kashiwara filtration is functorial: if \( f : M_1 \to M_2 \) is a morphism of \( A \)-modules then \( f(S_j M_1) = S_j M_2 \).

**Proof.** Everything is functorial. \( \square \)

**Remark 4.17.** One of the properties a holonomic \( D \)-module \( M \) has is that its \( CV \) is Lagrangian. This means that \( \mathbf{D}M \) is a sheaf supported in degree \( m/2 \) where \( m = \dim \text{gr} A \). In particular, \( M \hookrightarrow \mathbf{D}M[m/2] \) maps \( D \)-modules to \( D \)-modules.

**4.4.1. Geometric version.** In some ways there is nothing remarkable in this section beyond what already happens in the commutative setting. Let \( X \) be a smooth (or Cohen-Macaulay) variety. Then any sheaf \( \mathcal{F} \) on \( X \) comes with a filtration \( \{S_j(\mathcal{F})\} \) where \( S_j(\mathcal{F}) \) consists of those sections whose support has dimension \( \leq j \).

If one disliked geometry one could give an alternative description of this filtration as \( S_j(\mathcal{F}) := \text{Im}(H^0(\mathbf{D}(\tau_{\geq j} \mathbf{D}(\mathcal{F})) \to \mathcal{F})) \) where \( \mathbf{D}(\mathcal{F}) := \text{Hom}(\mathcal{F}, \omega_X) \).

For example, let \( X = \mathbb{P}^2 \) and \( \mathcal{F} := \mathcal{O}_{S \cap L} \) where \( S \) is a plane and \( L \) a line in \( \mathbb{P}^3 \) intersecting in point \( p := S \cap L \). Then one has the natural exact triangle

\[
\mathcal{O}_L(-p) \to \mathcal{F} \to \mathcal{O}_S
\]
from which it follows that we get
\[ \omega_S[-1] \to \mathcal{F}^\vee \otimes \omega_X \to \omega_L(p)[-2]. \]
Then dualizing the natural cohomological filtration one gets \( S_1(\mathcal{F}) = \mathcal{O}_L(-p) \to \mathcal{F} = S_2(\mathcal{F}) \).

On the other hand, consider \( X = \mathbb{P}^4 \) and \( \mathcal{F} := \mathcal{O}_{P_1 \cup P_2} \) where \( P_1 \) and \( P_2 \) are two planes intersecting at a point \( p \). Then dualizing the short exact triangle
\[ \mathcal{F} \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to \mathcal{O}_p \]
we get
\[ \mathcal{O}_p[-4] \to \omega_{P_1}[-2] \oplus \omega_{P_2}[-2] \to \mathcal{F}^\vee \otimes \omega_X. \]
Dualizing the natural cohomological filtration we get \( \text{Im}(H^0(\mathcal{O}_p[-1]) \to \mathcal{F}) \) which is just zero. So the filtration is just \( S_1(\mathcal{F}) = 0 \) and \( S_2(\mathcal{F}) = \mathcal{F} \). In particular, we see that this filtration really just captures information about the dimension of supports of sections and not more subtle information like Cohen-Macaulayness.

5. Twisted Differential Operators

Let \( A \) be a commutative ring and \( M, N \) two \( A \)-modules. Then one can define \( \text{Diff}^0_A(M, N) := \text{Hom}_A(M, N) \) and \( f \in \text{Diff}^k_A(M, N) \in \text{Hom}_A(M, N) \) if \( [a, f] \in \text{Diff}^{k-1}_A(M, N) \) for any \( a \in A \).

Thus \( \text{Diff}_A(M, N) \subset \text{Hom}_A(M, N) \) is the largest subspace on which the adjoint action of \( A \) is nilpotent. Another way of saying this is the following. \( \text{Hom}_A(M, N) \) is naturally an \( A \)-\( A \)-bimodule. Since \( A \) is commutative it is a sheaf on \( \text{Spec } A \times \text{Spec } A \). Since \( \Delta \subset \text{Spec } A \times \text{Spec } A \) is generated by \( a \otimes 1 - 1 \otimes a \) this means that \( \text{Diff}_A(M, N) \) is the largest submodule of \( \text{Hom}_A(M, N) \) setwise supported on the diagonal. This in contrast to \( \text{Hom}_A(M, N) \) which is the largest submodule scheme-wise supported on the diagonal.

A sheaf \( \mathcal{D} \) of twisted differential operators (TDO) on \( X \) is a sheaf of filtered almost commutative algebras together with an isomorphism \( \text{gr} \mathcal{D} \cong \text{Sym} \mathcal{T}_X \) of Poisson algebras.

By definition \( \mathcal{D}^0 \cong \mathcal{O}_X \). Now any \( a \in \mathcal{D} \) defines a derivation \( [a, \cdot] \). Since \( \mathcal{D} \) is almost commutative this defines a map \( \mathcal{D}^{\leq 1} \to \mathcal{T}_X \) since \( [a, f] \in \mathcal{O}_X \) for any \( a \in \mathcal{D}^{\leq 1} \) and \( f \in \mathcal{O}_X \). A priori this map might be zero (for instance, if \( \mathcal{D} \) were commutative) but the fact that \( \text{gr} \mathcal{D} \cong \text{Sym} \mathcal{T}_X \) as Poisson algebras rules this out.

To see this note this map is precisely the Poisson bracket \( [a, f] = \{ a, f \} \) so if this is zero for all \( f \) it means that \( a \in \mathcal{D}^0 \). In particular, we get an exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{D}^{\leq 1} \to \mathcal{T}_X. \]
The fact that \( \mathcal{D}^{\leq 1} \to \mathcal{T}_X \) is onto follows since \( \{ a, \cdot \} : \text{Sym}^{\leq 1} \mathcal{T}_X \to \mathcal{T}_X \) is onto.

A TDO structure on a given sheaf of filtered algebras can be reconstructed from an isomorphism \( \mathcal{D}^0 \cong \mathcal{O}_X \) and a short exact sequence as above such that the induced map \( \text{Sym} \mathcal{T}_X \to \mathcal{D} \) is bijective (then it’s automatically a Poisson algebra isomorphism). Moreover, two TDOs \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are isomorphic if and only if there exists an isomorphism of sheaves \( \mathcal{D}_1 \cong \mathcal{D}_2 \) which takes \( \mathcal{O}_X \subset \mathcal{D}_1 \) to \( \mathcal{O}_X \subset \mathcal{D}_2 \).

Given a TDO \( \mathcal{D} \) the commutator on \( \mathcal{D}^1 \) gives the structure of a Lie algebra. One sees that
- there exists a short exact sequence of \( \mathcal{O}_X \) modules
  \[ 0 \to \mathcal{O}_X \to \mathcal{D}^1 \to \mathcal{T}_X \to 0 \]
- \( [a_1, f \cdot a_2] = (\bar{a}_1 f) \cdot a_2 + f \cdot [a_1, a_2] \) where \( \bar{a} \) denotes the image of \( a \in \mathcal{D}^1 \) in \( \mathcal{T}_X \)
- \( 1 \in \mathcal{O}_X \) is a central element of \( \mathcal{D}^1 \)

The first condition is explained above. Any derivation on an algebra takes \( 1 \to 0 \) which explains the third condition. To see the second condition note that the left side equals
\[ a_1 \cdot f \cdot a_2 - f \cdot a_2 \cdot a_1 \]
and the right side equals
\[(\bar{a}_1 \cdot f) \cdot a_2 + f \cdot a_1 \cdot a_2 - f \cdot a_2 \cdot a_1\]
so it suffices to show
\[a_1 \cdot f \cdot a_2 - f \cdot a_1 \cdot a_2 = (\bar{a}_1 \cdot f) \cdot a_2\]
which simplifies to showing
\[a_1 \cdot f - f \cdot a_1 = \bar{a}_1 \cdot f.\]
The left hand side is just \(\{a_1, f\}\) (the Poisson bracket) which equals the right hand side by the properties of the Poisson bracket.

A sheaf of \(\mathcal{O}_X\)-modules with a Lie bracket satisfying the conditions above is called an \textit{Atiyah algebra}.

**Lemma 5.1.** The set of isomorphism classes of Atiyah algebra forms a vector space \(\text{At}_X\). There is a linear map \(\text{At}_X \to \text{Ext}^1(T_X, \mathcal{O}_X)\) by forgetting the Lie bracket.

**Proof.** \(\lambda \in \mathbb{C}\) acts by multiplying the connecting map \(\text{Ext}^1(T_X, \mathcal{O}_X)\). The sum of two Atiyah algebras
is defined by via the Bauer sum of extensions – one needs to check that this explicit construction gives a Lie bracket on the sum. \(\square\)

**Example.** Given a line bundle \(L\) on \(X\) the image of \([\mathcal{D}_X(L, L)] \to \text{Ext}^1(T_X, \mathcal{O}_X) \cong H^1(\Omega_X^1)\) is the Chern class of \(L\).

**Example.** Given an extension \(\alpha \in \text{Ext}^1(T_X, \mathcal{O}_X)\) one can define a Lie bracket on \(\mathcal{A} := \text{Cone}(\alpha)\) via
\[[f_1 + \zeta_1, f_2 + \zeta_2] = \zeta_1 f_2 - \zeta_2 f_1 + [\zeta_1, \zeta_2]\]
locally on \(X\). Then one can show (need to check this) that this definition glues to give a bracket on \(\mathcal{A}\). Even more if \(\beta\) is a 2-form then one can define
\[[f_1 + \zeta_1, f_2 + \zeta_2] = \zeta_1 f_2 - \zeta_2 f_1 + \beta(\zeta_1, \zeta_2) + [\zeta_1, \zeta_2].\]
If \(\beta\) is closed then this satisfies the Jacobi identity. If it is locally exact then it turns out this is locally isomorphic to \(\mathcal{D}_X\) (locally trivial Atiyah algebra). Notice that in the holomorphic category any closed two form is locally exact so any \(\mathcal{A}\) is locally trivial (hence adding this extra \(\beta\) is only interesting in the algebraic category).

**Theorem 5.2.** Locally trivial Atiyah algebras are parametrized by \(H^1_{\text{cl}}(X, \Omega^1_X)\). Atiyah algebras in general are parametrized by
\[H^1_{\text{Zar}}(X, \Omega^1_X) \cong \text{Aut}(\mathcal{O}_X \to \mathcal{D}_X).\]

Then one shows that any \(\mathcal{A}\) is given by data \(\{\alpha_{ij}, \beta_i\}\) for some cover \(\{U_i\}\) where \(\alpha_{ij}\) is a 1-form and \(\beta_i\) a closed 2-form. Finally, one has the conditions
\[\beta_i - \beta_j = d\alpha_{ij} \text{ and } \alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0\]
which gives a class in \(H^1(\Omega^{\geq 1})\).

**Remark 5.3.** Notice that the locally trivial Atiyah algebras are precisely those associated to \(\mathcal{D}_X(L, L)\) where \(L\) is a line bundle (two line bundles with the same Chern class give isomorphic Atiyah algebras).

**Remark 5.4.** Standard maps of complexes induce the long exact sequence
\[
\ldots \to H^0(\Omega^1) \to H^0(\Omega^2_{\text{cl}}) \to H^1(\Omega^{\geq 1}) \to H^1(\Omega^1) \to \ldots
\]
as well as a map \(H^1(\Omega^1_{\text{cl}}) \to H^1(\Omega^{\geq 1})\) which embeds the space of locally trivial Atiyah algebras into the space of all Atiyah algebras.

**Theorem 5.5.** There is a natural bijections \(\{\text{TDO}\} \leftrightarrow \{\text{Atiyah algebras}\}\).
Proof. The arrow $\rightarrow$ is $\mathcal{D} \rightarrow \mathcal{D}^1$. The arrow $\leftarrow$ is $\mathcal{A} \rightarrow U_X(\mathcal{A})$. Here $U_X(\mathcal{A})$ denote the quotient of $U_C(\mathcal{A})$ (the universal enveloping algebra) by the relations

$$1_{U_C(\mathcal{A})} = 1_{\mathcal{A}} \text{ and } f \otimes a = f \cdot a \text{ where } f \in \mathcal{O}_X \text{ and } a \in \mathcal{A}.$$ 

This is supposed to mimic the universal enveloping algebra of $\mathcal{A}$ over $\mathcal{O}_X$. This is not possible directly since $\mathcal{O}_X \subset \mathcal{A}$ is not central. $\square$

Given any TDO, or equivalently any Atiyah algebra $\mathcal{A}$, one can form a family of filtered, associative algebras $\mathcal{A}_t \rightarrow \mathfrak{A}^1$ where $\mathcal{A}_t := t \cdot \mathcal{A}$. Thus $\mathcal{A}_t \cong \mathcal{A}$ for $t \neq 0$ while $\mathcal{A}_0 \cong \mathcal{A}_X$ where $\mathcal{A}_X$ is the Atiyah algebra associated with $\mathcal{D}_X$. This means that any Atiyah algebra is a deformation of $\mathcal{A}_X$. Applying $U_X(\cdot)$ one gets that any TDO is a deformation of $\mathcal{D}_X$.

A good picture to keep in mind is

$$\mathcal{A}_X \leftrightarrow \mathcal{D}_X \xrightarrow{\text{deformation}} \mathcal{A}_L \leftrightarrow \mathcal{D}_X(L,L) \xrightarrow{\text{s}}$$

where $(T^\chi_X)_L$ denotes the deformation of $T^\chi_X$ corresponding to the line bundle $L$. The vertical map denoted $S$ takes $\mathcal{A}_L$ to $\text{Sym}_C(\mathcal{A}_L)$ (the sheaf of symmetric algebras) modulo the relations

$$1_{\text{Sym}_C(\mathcal{A}_L)} = 1_{\mathcal{A}_L} \text{ and } f \otimes a = f \cdot a \text{ for any } f \in \mathcal{O}_X, a \in \mathcal{A}_L$$

which is a sheaf of filtered commutative algebras on $X$ (a “twisted symmetric algebra”). Taking Spec gives $(T^\chi_X)_L$ which is a “twisted cotangent bundle”.

Three descriptions of $(T^\chi_X)_L$:

(i) To construct $(T^\chi_X)_L$ let $Y$ denote the total space of the line bundle $L$ which comes equipped with the natural $\mathbb{C}^*$ action acting on the fibres. Let $\mu : T^\chi_X \rightarrow \mathbb{C}$ be the moment map. Then $\mu^{-1}(t)/\mathbb{C}^* \cong (T^\chi_X)_L$ if $t \neq 0$ while $\mu^{-1}(0)/\mathbb{C}^* \cong T^\chi_X$.

(ii) $(T^\chi_X)_L = \text{Conn}_X(L)$ is the space of connections on $L$. More precisely, a connection $\nabla$ takes a section $s$ of $L$ and a vector field $v$ and returns another section $\nabla_{v,s}$ of $L$. For any one form $\alpha$ one can define a new connection $\nabla'_{v,s}(s) := \nabla_{v,s}(s) + \alpha(v)s$. This way one can show that $\text{Conn}_X(L)$ is an affine $\Omega^1_X$ bundle. Note that $\text{Conn}_X(L) \not\cong \Omega^1_X$ since (in general) there is no section $X \rightarrow \text{Conn}_X(L)$. If $L \cong \mathcal{O}_X$ then there is a section, namely the connection $\nabla_{v,s} = ds(v)$ (this just says that vector fields naturally act on functions).

(iii) As a first order deformation of $T^\chi_X$ one can see $(T^\chi_X)_L$ as follows. A standard computation shows that $H^1(T^\chi_X) \cong H^1(T^\chi_X) \oplus H^1(\Omega^1_X) \oplus R$ where $R$ is the rest of the cohomology. In particular, for any class in $c_1(L) \in H^1(\Omega^1_X)$ we get a first order deformation $(T^\chi_X)_L$ of $T^\chi_X$.

Example. Take $X = \mathbb{P}^1$. Then (up to multiple) there is a unique line bundle $L := \mathcal{O}_{\mathbb{P}^1}(1)$. Then the deformation $(T^\chi_X)_L$ is the family $\{xy - z^2 = t\}$ over $t \in \mathbb{C}^1$. The projection map to $\mathbb{P}^1$ is $(x,y,z) \mapsto [x,z]$. It is easy to check the fibre is isomorphic to $\mathbb{C}^1$ but it is only an affine bundle (no zero section).

Point of view. $T^\chi_X$ has natural deformations $(T^\chi_X)_L$ parametrized by $c_1(L) \in H^1(\Omega^1_X)$. Just like $\mathcal{D}_X$ is a non-commutative deformation of $\mathcal{O}_{T^\chi_X}$ so is $\mathcal{D}_X(L,L)$ a non-commutative deformation of $\mathcal{O}_{(T^\chi_X)_L}$.

References


[K] B. Keller, A brief introduction to $A_{\infty}$ algebras.
