INTRODUCTION TO THE THEORY OF ADMISSIBLE REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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Preface

This draft of Casselman's notes was worked over by the Séminaire Paul Sally in 1992–93. In addition to Sally, participants included Jeff Adler, Peter Anspach, John Boller, Neil Chriss, Stephen DeBacker, Bill Graham, Jason Levy, Alan Roche, and Colin Rust. Although we made hundreds of changes, we did not alter any of the methods used or the numbering of any results. Casselman has neither approved nor disavowed the end product.

Several mathematical and stylistic questions have been put aside for another day, which has not yet arrived. These are all indicated in footnotes.

Contents

0. Introduction

Let k be a non-archimedean locally compact field and G the group of k-rational points of a reductive algebraic group defined over k. A (complex) *admissible representation* of G is a pair (π, V) where V is a vector space over $\mathbf{C}$ and π is a homomorphism from G to $\mathrm{GL}_{\mathbf{C}}(V)$ such that

(a) each $v \in V$ has an open isotropy subgroup — i.e., π is *smooth*, and
(b) for any open subgroup K, the space V^K of K-fixed vectors has finite dimension.

It is my intention in this paper to lay a part of the foundations of the theory of such representations (therefore complementing work of Harish-Chandra, Jacquet, and Langlands—[20], [22], [23]).

If P is a parabolic subgroup of G with Levi decomposition $P = MN$ and modulus character δ_P, and (σ, U) is an admissible representation of M, then σ determines as well a unique representation of P trivial on N since $M = P/N$. One defines the (normalized) *representation induced by* σ *from* P *to* G to be the right regular representation of G on the space $i_P^G\,\sigma$ of all locally constant functions $f\colon G \longrightarrow U$ such that $f(pg) = \sigma(p)\delta_P(p)^{1/2}f(g)$ for all $p \in P$ and $g \in G$. Because $P\backslash G$ is compact, it is not difficult to show that $i_P^G\,\sigma$ is an admissible representation (2.4.1).

If π is an admissible representation, one may define its *contragredient*, which is again admissible. The contragredient of $i_P^G\,\sigma$ is $i_P^G\,\widetilde{\sigma}$ (3.1.2).

There exists a form of *Frobenius reciprocity*: for any smooth G-representation π, one has a natural isomorphism of $\mathrm{Hom}_G(\pi, i_P^G\,\sigma)$ with $\mathrm{Hom}_P(\pi, \sigma\delta_P^{1/2})$ (2.4.1).

If (π, V) is any smooth G-representation and $P = MN$ a parabolic subgroup of G, one defines $V(N)$ to be the subspace of V generated by $\{\pi(n)v - v : n \in N, v \in V\}$, and one defines V_N to be $V/V(N)$. This is the space of a smooth representation π_N of M, and it has a universal property: if U is any space on which N acts trivially, then $\mathrm{Hom}_N(V, U) \cong \mathrm{Hom}_{\mathbf{C}}(V_N, U)$. This implies *a second form of Frobenius reciprocity*:

$$\mathrm{Hom}_G(\pi, i_P^G\,\sigma) \cong \mathrm{Hom}_M(\pi_N, \sigma\delta_P^{1/2}) \quad (3.2.4).$$

The V_N-construction was used to some extent in Jacquet-Langlands [23], but first seriously discussed in Jacquet's Montecatini notes [22], at least for $G = \mathrm{GL}_n$, where Jacquet falls only slightly short of proving that *when* π *is admissible, so is* π_N (3.3.1). Coupled with the more elementary observation that *if* π *is finitely generated so is* π_N, this becomes one of the cornerstones of the theory of admissible representations.

One basic fact is that *the functor* $V \rightsquigarrow V_N$ *is exact* (3.2.3).

For the group $G = \mathrm{PGL}_2$, it was probably first realized by Mautner that there are irreducible admissible representations of G which have no embeddings into an $i_P^G\,\sigma$ (where P is here the Borel subgroup of G, and σ may be assumed to be one-dimensional). Generalizing this phenomenon, one calls a finitely generated admissible representation of arbitrary G *absolutely cuspidal* if there are no non-trivial G-

morphisms into any representation of the form $i_P^G\sigma$ with P proper in G and σ an admissible representation of M. Equivalently, (π, V) is absolutely cuspidal if and only if $V_N = 0$ for all unipotent radicals N of proper parabolic subgroups.

The irreducible absolutely cuspidal representations may be characterized also as those whose matrix coefficients have compact support modulo the center of G (5.3.1). This fact may be used to show that *they are both projective and injective in a suitable category of* G*-modules* (5.4.1).

In [22], Jacquet did prove (for $G = \mathrm{GL}_n$) that *for any irreducible admissible representation* π *there exists at least one* P *and one irreducible absolutely cuspidal* σ *such that* π *may be embedded into* $i_P^G\sigma$ (5.1.2).

The V_N-construction also plays a role in the *asymptotic behavior of matrix coefficients.* This is best expressed like this: let N^- be the unipotent opposite to N. *Then there is a canonical non-degenerate pairing* $\langle\ ,\ \rangle_N$ *of* V_N *with* $\widetilde{V}_{N^-}$ which is characterized by the property that *for* $v \in V$ *and* $\widetilde{v} \in \widetilde{V}$ *with images* $u \in V_N$ *and* $\widetilde{u} \in \widetilde{V}_{N^-}$ *there exists* $\epsilon > 0$ *such that for all* $a \in A^-(\epsilon)$ (see §1.4 for notation) *one has* $\langle\pi(a)v, \widetilde{v}\rangle = \langle\pi_N(a)u, \widetilde{u}\rangle_N$ (§4.2).

This makes possible a *criterion for square-integrability* in terms of the spaces V_N (4.4.6), and also figures in the proof of the result mentioned earlier about the support of the matrix coefficients of an absolutely cuspidal representation.

The main results of this paper depend on almost everything mentioned so far, and are concerned with the composition series of the representation $i_P^G\sigma$. If A is the maximal split component of the center of M and W_A is the Weyl group of A, then the final result is this:

(a) *If* σ *is an irreducible absolutely cuspidal representation of* M*, then the length of* $i_P^G\sigma$ *is at most* $\mathrm{card}(W_A)$ (7.2.3);
(b) *if* π *is any irreducible composition factor of* $i_P^G\sigma$ *then there exists* $w \in W_A$ *and a* G*-embedding of* π *into* $i_P^G w\sigma$ (7.2.2).

(The latter makes sense because W_A is also $N(M)/M$, and for $m \in M$ one has $m\sigma \cong \sigma$. In this rather strong form, this result is due to Harish-Chandra, although others had proven weaker and related verions.) From this one deduces that *any finitely generated admissible representation has finite length as a* G*-module* (6.3.10).

The proof of this main result is rather long and complicated. I first prove a weaker result in §6 involving the associates of a parabolic, use this to prove the square-integrability criterion (also in §6), and use this in turn to prove the final version in §7. If G has semi-simple rank one, however, the argument is not so complicated, and it may be instructive to sketch it here:

(1) There is only one conjugacy class of proper parabolic subgroups, of the form $P = MN$ where M is compact modulo its center. Thus, the representation σ is finite-dimensional.
(2) Using the Bruhat decomposition $G = PwP \cup P$, where w is the non-trivial

element of the Weyl group, one has a filtration $0 \subsetneqq I_w \subsetneqq I = i_P^G \sigma$ of I as a P-space, where I_w is the subspace of functions in I vanishing along P. The corresponding filtration of I_N may be described explicitly; it fits into an exact sequence

$$0 \longrightarrow (w\sigma)\delta_P^{1/2} \longrightarrow I_N \longrightarrow \sigma\delta_P^{1/2} \longrightarrow 0$$

of M-spaces.

(3) If π is an irreducible composition factor of I, then π cannot be absolutely cuspidal, because if it were one could apply the projectivity of absolutely cuspidals to obtain an embedding of π into I. Thus by Jacquet's result, π has an embedding into some $i_P^G \rho$, and in particular, by Frobenius reciprocity, π_N has $\rho\delta_P^{1/2}$ as an M-quotient.

(4) But then by exactness of the functor $\pi \rightsquigarrow \pi_N$, $\rho\delta_P^{1/2}$ must be either $\sigma\delta_P^{1/2}$ or $(w\sigma)\delta_P^{1/2}$, proving part of the main result.

(5) The same argument shows that if $0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq I$ were a composition series, then on the one hand $(I_1)_N$, $(I_2/I_1)_N$, and $(I/I_2)_N$ would all be nonzero, but on the other hand only two are allowed to be non-trivial, a contradiction.

Another class of results contained in §6 concerns *intertwining operators and irreducibility of representations induced from parabolic subgroups.*

As a minor application of the general theory, I include in §8 a discussion of the *Steinberg representation* of G (thereby justifying claims made in an earlier paper [14]).

In §9 I work out an elementary example, that of *the unramified principal series of* SL_2, in some detail.

Very few of the results in this paper are entirely mine. A number of the basic ideas may be found in Jacquet-Langlands [23] and Jacquet [22]. Many results were discovered independently by Harish-Chandra (and given in a course of lectures at the Institute for Advanced Study, 1971-72; this course is partly summarized in [20], and details will presumably appear eventually[1]). Others are completely his. For example, the results of §§7.1–7.2 were communicated to me by him in correspondence; I would like to thank him for allowing me to include them here. The idea of the canonical liftings in §5 arose from my attempts to understand his theory of the constant term in [20]. And I have incorporated suggestions of his throughout.

Several other points were also discovered independently by Bernstein and Zelevinskii (see [1] and [2]), Baris Kendirli [24], Hideya Matsumoto ([26], [27]), Olshanskii [28], Allan Silberger [30], Graham Williams [31], and Norman Winarsky [32].

I would like to thank James Arthur, Armand Borel, Roger Howe, Hervé Jacquet, Robert Langlands, Rimhak Ree, and Allan Silberger for invaluable advice and suggestions. I hope I have given them the proper credit in the body of the paper. Finally,

[1]We should cite Silberger here.

thanks are due to both Matsumoto and Deligne for pointing out errors in earlier versions.

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1. Preparation

1.1. Let $\mathfrak{a}$ be any finite dimensional real vector space, and further let

$$\begin{aligned}
\mathfrak{a}^* &= \text{the real dual of } \mathfrak{a};\\
\Sigma &= \text{a reduced root system in } \mathfrak{a}^* \text{ (assumed to span } \mathfrak{a}^*);\\
\Sigma^+ &= \text{a choice of a set of positive roots in } \Sigma\\
&\qquad \text{(I write } \alpha > 0 \text{ for } \alpha \in \Sigma^+);\\
\Sigma^- &= -\Sigma^+;\\
\Delta &= \text{the simple roots in } \Sigma^+;\\
S &= \text{the set } \left\{ w_\alpha \,\middle|\, \alpha \in \Delta \right\} \text{ of reflections}\\
&\quad\ \text{corresponding to the elements of } \Delta;\\
W &= \text{the Weyl group of } \Sigma \text{ (acting on both}\\
&\quad\ \mathfrak{a} \text{ and } \mathfrak{a}^* \text{ and generated by } S);
\end{aligned}$$

For each $\Theta \subseteq \Sigma$, let $\mathfrak{a}_\Theta$ be $\left\{ x \in \mathfrak{a} \,\middle|\, \alpha(x) = 0 \text{ for all } \alpha \in \Theta \right\}$ (and if $\Theta = \{\alpha\}$, write $\mathfrak{a}_\alpha$, and call this a root hyperplane). We will write $\Theta > 0$ if $\alpha > 0$ for each $\alpha \in \Theta$. For $\Theta \subseteq \Delta$, let

$$\begin{aligned}
\Sigma_\Theta &= \text{the subset of } \Sigma \text{ of linear combinations of the roots in } \Theta;\\
\Sigma_\Theta^+ &= \Sigma^+ \cap \Sigma_\Theta;\\
\Sigma_\Theta^- &= \Sigma^- \cap \Sigma_\Theta;\\
W_\Theta &= \text{the subgroup of } W \text{ generated by } \left\{ w_\alpha \,\middle|\, \alpha \in \Theta \right\}.
\end{aligned}$$

Every element of Σ_Θ vanishes on $\mathfrak{a}_\Theta$, so that Σ_Θ may be identified with a subset of $(\mathfrak{a}/\mathfrak{a}_\Theta)^*$, and in fact it defines a root system in this space. Every element of W_Θ acts trivially on $\mathfrak{a}_\Theta$, and therefore acts naturally on $\mathfrak{a}/\mathfrak{a}_\Theta$, and in fact W_Θ is the Weyl group of Σ_Θ. (Use [10] for a general reference.)

For every $w \in W$, let Σ_w be the set $\left\{ \alpha \in \Sigma^+ \,\middle|\, w^{-1}\alpha < 0 \right\}$. The cardinality of Σ_w is also the length $\ell(w)$ of w — i.e., the length of a minimal expression for w as a product of elements of S [10, Cor. 2, p. 158]. Note that w is determined by Σ_w (by 5.2 of [5]).

Lemma 1.1.1. For any $w_1, w_2 \in W$ the following are equivalent:

(a) $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$;
(b) $\Sigma_{w_1 w_2} = w_1 \Sigma_{w_2} \cup \Sigma_{w_1}$;
(c) $\Sigma_{w_1} \subseteq \Sigma_{w_1 w_2}$ and $w_1 \Sigma_{w_2} > 0$.

Proof. The equivalence of (a) and (b) is Lemma 3.4 of [5]. That (b) implies (c) is immediate, and the converse is almost as elementary. □

The set Σ_w has a geometrical meaning: one knows that the open cone

$$C = \left\{ x \in \mathfrak{a} \,\middle|\, \alpha(x) > 0 \text{ for all } \alpha \in \Delta \right\}$$

is a fundamental domain for W, or more precisely that as w ranges over W the cone wC ranges over the connected components of $\mathfrak{a}^{\mathrm{reg}}$ (the *regular* points of $\mathfrak{a}$) $= \mathfrak{a} \setminus \cup_{\alpha\in\Sigma}\mathfrak{a}_\alpha$. Now each $\mathfrak{a}_\alpha$ is in fact determined by exactly one positive α, and the correspondence $\alpha \mapsto \mathfrak{a}_\alpha$ is a bijective correspondence between Σ_w and the set of root hyperplanes separating wC from C. In particular, the number of these separating hyperplanes is equal to $\ell(w)$.

Lemma 1.1.2. Let Θ be a non-empty subset of Δ. There exists in any left coset of W_Θ in W a unique element w characterized by any of these properties:

(a) For any $x \in W_\Theta$, $\ell(wx) = \ell(w) + \ell(x)$;
(b) $w\Theta > 0$;
(c) The element w is of least length in wW_Θ.

This is Proposition 3.9 of [5].

Proposition 1.1.3. Let Θ, Ω be non-empty subsets of Δ. In every double coset in $W_\Theta\backslash W/W_\Omega$ there exists a unique w characterized by any of these properties:

(a) The element w has least length in $W_\Theta w W_\Omega$;
(b) It has least length in $W_\Theta w$ and also in wW_Ω;
(c) $w^{-1}\Theta > 0$, $w\Omega > 0$.

Proof. Exercise 3, p. 57, of [10] says that every double coset has an element of least length, and also that (a) and (b) are equivalent. Lemma 1.1.2 implies that (b) and (c) are equivalent. □

The previous two results say that the projections from W to W/W_Ω, $W_\Theta\backslash W$, and $W_\Theta\backslash W/W_\Omega$ all have canonical splittings. Let $[W/W_\Omega]$, etc., be their images. Thus, $[W/W_\Omega] = \left\{ w \in W \,\middle|\, w\Omega > 0 \right\}$, $[W_\Theta\backslash W] = \left\{ w \in W \,\middle|\, w^{-1}\Theta > 0 \right\}$, and $[W_\Theta\backslash W/W_\Omega] = [W/W_\Omega] \cap [W_\Theta\backslash W]$.

For each $\Theta \subseteq \Delta$, let $w_{\ell,\Theta} = {w_{\ell,\Theta}}^{-1}$ be the longest element in W_Θ (and let w_ℓ be $w_{\ell,\Delta}$). The element $w_\ell w_{\ell,\Theta}$ clearly lies in $[W/W_\Theta]$, and in fact it is the longest element there. More precisely:

Proposition 1.1.4. Let Θ be a subset of Δ, write w_0 for $w_\ell w_{\ell,\Theta}$, and let $\overline{\Theta}$ be $w_0(\Theta) \subseteq \Delta$. Then

(a) $\Sigma_{w_0} = \Sigma^+ \setminus \Sigma^+_{\overline{\Theta}}$;
(b) For any $w \in [W/W_\Theta]$, $\ell(w_0) = \ell(w_0 w^{-1}) + \ell(w)$.

Proof. Since $w_0(\Theta) = \overline{\Theta}$, it is clear that $\Sigma_{w_0} \subseteq \Sigma^+ \setminus \Sigma^+_{\overline{\Theta}}$. But 1.1.2a, implies that $\ell(w_0) = \ell(w_\ell) + \ell(w_{\ell,\Theta})$, while that same result together with [10, Cor. 4, p. 20] (which says the length of an element in W_Θ is the same in W as in W_Θ) imply that this in turn is equal to the cardinality of $\Sigma^+ \setminus \Sigma^+_{\overline{\Theta}}$, and this proves (a).

To prove (b), apply 1.1.1(c). If $w\Theta > 0$, then $ww_0^{-1}(\overline{\Theta}) > 0$, so that $\Sigma_{w_0 w^{-1}} \subseteq \Sigma^+ \setminus \Sigma^+_{\overline{\Theta}} = \Sigma_{w_0}$, which is the first half of the criterion. To prove $w_0 w^{-1}\Sigma_w > 0$:

$\alpha \in \Sigma_w$ if and only if $\alpha > 0$ and $w^{-1}\alpha < 0$. But $w^{-1}\alpha < 0$ if and only if $w^{-1}\alpha \in \Sigma^- \smallsetminus \Sigma^-_\Theta$ or $w^{-1}\alpha \in \Sigma^-_\Theta$; if $w^{-1}\alpha \in \Sigma^-_\Theta$ then $\alpha \in w\Sigma^-_\Theta \subseteq \Sigma^-$, a contradiction. Thus $w^{-1}\alpha \in \Sigma^- \smallsetminus \Sigma^-_\Theta$ and $w_0 w^{-1}\alpha > 0$ by (a). $\square$

If Θ is a subset of Δ, the subset $\overline{\Theta} = w_\ell w_{\ell,\Theta}(\Theta) = w_\ell(-\Theta)$ is called its *conjugate* in Δ.

1.2. Continue the notation of §1.1. If Θ and Ω are subsets of Δ, they are called *associates* if the set $W(\Theta,\Omega) = \left\{ w \in W \,\middle|\, w\Omega = \Theta \right\}$ is not empty. For any $\Theta \subseteq \Delta$, let $\{\Theta\}$ be the set of its associates.

What I shall do now is describe the connected components of each $\mathfrak{a}^{\mathrm{reg}}_\Theta = \mathfrak{a}_\Theta \smallsetminus \cup_{\alpha\in\Sigma\smallsetminus\Sigma_\Theta}(\mathfrak{a}_\alpha \cap \mathfrak{a}_\Theta)$ and at the same time show how one can express the elements of $W(\Theta,\Omega)$ in a manner analogous to the way in which one expresses elements of W as a product of elements in S. These results, as far as I know, are essentially due to Langlands — see Lemma 2.13 of [25] — but I learned them, more or less in the form in which I present them, from James Arthur.

Let me first recall some extremely elementary ideas from [10] (exercises at the end of IV). Let V be a finite dimensional vector space, $\mathcal{H}$ a set of hyperplanes passing through the origin such that $V \smallsetminus \cup_{H\in\mathcal{H}} H$ is a union of simplicial cones. The connected components of $V \smallsetminus \cup_{H\in\mathcal{H}} H$ are called *chambers* of V associated to $\mathcal{H}$. An (irredundant) *gallery* in V is a sequence of chambers $C_0, C_1, \ldots, C_n$ such that no two successive chambers are the same, but the pair does share a common face. The integer n is the *length* of the gallery, and it is said to be a gallery between C_0 and C_n. A *minimal* gallery is one of least length between its ends. The *distance* between chambers is the length of a minimal gallery between them, and is also equal to the number of hyperplanes in $\mathcal{H}$ separating them.

I apply these ideas to the spaces $\mathfrak{a}_\Theta$ ($\Theta \subseteq \Delta$) and the hyperplanes of the form $\mathfrak{a}_\Theta \cap \mathfrak{a}_\alpha$ ($\alpha \in \Sigma \smallsetminus \Sigma_\Theta$). The cone $C_\Theta = \left\{ x \in \mathfrak{a}_\Theta \,\middle|\, \alpha(x) > 0 \text{ for all } \alpha \in \Delta \smallsetminus \Theta \right\}$ is a chamber in $\mathfrak{a}_\Theta$, and the *height* of any other chamber is defined to be its distance from C_Θ. The cone C_Θ lies on the boundary of the cone C defined after §1.1 (which now becomes $C_\emptyset$) and more generally the closure of C_Θ is the disjoint union of the C_Ω with $\Theta \subseteq \Omega$.

Observe that if $w \in W(\Theta,\Omega)$ then w takes $\mathfrak{a}_\Omega$ to $\mathfrak{a}_\Theta$ and chambers to chambers.

Proposition 1.2.1. Subsets $\Theta, \Omega \subseteq \Delta$ are associate if and only if there exists $w \in W$ with $w\mathfrak{a}_\Omega = \mathfrak{a}_\Theta$.

Proof. One way is of course trivial. For the other, suppose $w\mathfrak{a}_\Omega = \mathfrak{a}_\Theta$, and let w_0 be the element of least length in the coset $W_\Theta w W_\Omega$. Then since W_Ω acts trivially on $\mathfrak{a}_\Omega$, $w_0\mathfrak{a}_\Omega = \mathfrak{a}_\Theta$ as well, hence $w_0\Sigma_\Omega = \Sigma_\Theta$. Now by 1.1.2, $w_0\Omega \subseteq \Sigma^+_\Theta$, and hence $w_0\Sigma^+_\Omega \subseteq \Sigma^+_\Theta$; similarly, ${w_0}^{-1}\Sigma^+_\Theta \subseteq \Sigma^+_\Omega$, so that $w_0\Sigma^+_\Omega$ is actually equal to Σ^+_Θ. But Θ consists precisely of the indecomposable elements of Σ^+_Θ and similarly for Ω and Σ^+_Ω. $\square$

Proposition 1.2.2. Let Θ be a subset of Δ. Every chamber of $\mathfrak{a}_\Theta$ is equal to wC_Ω for a unique associate Ω of Θ and $w \in W(\Theta, \Omega)$.

Proof. First suppose that C is a chamber of $\mathfrak{a}_\Theta$ distinct from C_Θ but sharing a face with it. Since the closure of C_Θ is the union of the C_Φ with $\Theta \subseteq \Phi$, this face equals C_Φ where $\Phi = \Theta \cup \{\alpha\}$ for some $\alpha \in \Delta \setminus \Theta$. Let Ω be the conjugate of Θ in Φ and let w_0 be $w_{\ell,\Phi} w_{\ell,\Theta}$ (so that ${w_0}^{-1}\mathfrak{a}_\Omega = \mathfrak{a}_\Theta$). I claim that $C = {w_0}^{-1}(C_\Omega)$. In order to prove this I must show (i) ${w_0}^{-1}C_\Omega$ and C_Θ have the face C_Φ in common and (ii) ${w_0}^{-1}(C_\Omega) \neq C_\Theta$. The first holds simply because $w_0 \in W_\Phi$ and hence takes C_Φ to itself. For the second, let $x \in C_\Theta$ be given, so that $\alpha(x) > 0$. Since $\alpha \in \Phi \setminus \Theta$, $w_0\alpha < 0$ (1.1.4(a)), and hence $-w_0\alpha$ lies in $\Sigma^+ \setminus \Sigma^+_\Omega$. Therefore $(-w_0\alpha)(w_0x) = -\alpha(x) < 0$ and w_0x cannot lie in C_Ω and therefore $x \notin {w_0}^{-1}C_\Omega$.

The proof proceeds by induction on the height of the chamber C. If $\mathrm{ht}(C) > 1$, then there will exist C_1 sharing a face with C such that $\mathrm{ht}(C_1) = \mathrm{ht}(C) - 1$. By the induction assumption, there exist w_1 and Ω_1 such that $C_1 = w_1 C_{\Omega_1}$. But then $w_1^{-1}C_1 = C_{\Omega_1}$ and $w_1^{-1}C$ lies in $\mathfrak{a}_{\Omega_1}$ sharing a face with, but not equal to, C_{Ω_1}. By what I have just done, $w_1^{-1}C = wC_\Omega$ for suitable w and Ω. Hence $C = w_1 w C_\Omega$.

For uniqueness, suppose Ω_1 and Ω_2, w_1 and w_2 are such that $w_1 C_{\Omega_1} = w_2 C_{\Omega_2}$. Then $w_1^{-1}w_2 C_{\Omega_2} = C_{\Omega_1}$ and $w_1^{-1}w_2\Omega_2 = \Omega_1$. This implies that $w_1^{-1}w_2\alpha > 0$ for every $\alpha > 0$, which in turn implies that $w_1^{-1}w_2 = 1$ and $w_1 = w_2$. □

Corollary 1.2.3. If Θ is maximal proper in Δ, then Θ and $\overline{\Theta}$ are the only associates of Θ.

Proof. In this case $\mathfrak{a}_\Theta$ is a line; C_Θ is half of it and $(w_\ell w_{\ell,\Theta})^{-1}C_{\overline{\Theta}}$ is the other half. □

If Θ is a maximal proper subset of $\Omega \subseteq \Delta$, then I call the element $w_{\ell,\Omega} w_{\ell,\Theta}$ the corresponding *elementary conjugation*. The proof of 1.2.2 also shows:

Lemma 1.2.4. Let Θ be a subset of Δ. If $w_1 C_{\Omega_1}$ and $w_2 C_{\Omega_2}$ are neighboring but distinct chambers of $\mathfrak{a}_\Theta$, then Ω_1 is maximal proper in $\Phi = \Omega_1 \cup \Omega_2$, Ω_2 is its conjugate in Φ and $w_1^{-1}w_2$ is the corresponding elementary conjugation.

Let $C_\Theta = C_0, C_1, \ldots, C_n$ be a gallery in $\mathfrak{a}_\Theta$ with (say) $C_i = w_i C_{\Omega_i}$. If $x_i = w_{i-1}^{-1} w_i$ for $i > 0$, then each x_i is an elementary conjugation, by 1.2.4, and clearly $w_n = x_1 \ldots x_n$. The proof of 1.2.2 in fact even shows:

Proposition 1.2.5. For a given $w \in W(\Theta, \Omega)$, the above correspondence is a bijection between the set of galleries between C_Θ and wC_Ω and the representations of w as a product of elementary conjugations. In particular, w has such a representation of minimal length equal to the height of wC_Ω.

If $C_0, C_1, \ldots, C_n$ is a gallery between C_Θ and wC_Ω with (say) $C_i = w_i C_{\Omega_i}$, I call the corresponding representation of w *primitive* if Ω_i is never equal to Ω_{i-1}.

Corollary 1.2.6. If Θ and Ω are associates, then there exists at least one element in $W(\Theta,\Omega)$ with a primitive representation as a product of elementary conjugations.

Proof. If $w = x_1 x_2 \ldots x_n$ and $\Theta_i = \Theta_{i-1}$ for some i then $x_1 \ldots \hat{x}_i \ldots x_n$ also lies in $W(\Theta,\Omega)$. □

Does there ever exist more than one primitive element in $W(\Theta,\Omega)$?

For $w \in W(\Theta,\Omega)$, define the *height* $\mathrm{ht}(w)$ of w to be that of wC_Ω. (This depends on Θ, not just w or $W(\Theta,\Omega)$. Example: $\Theta = \Delta$ or $\emptyset$.)

Proposition 1.2.7. Let $\Theta_1, \Theta_2, \Theta_3$ be associates, $w = w_2 w_1$ with $w_1 \in W(\Theta_1,\Theta_2)$, $w_2 \in W(\Theta_3,\Theta_2)$. If $\mathrm{ht}(w) = \mathrm{ht}(w_2) + \mathrm{ht}(w_1)$ then $\ell(w) = \ell(w_2) + \ell(w_1)$.

Proof. It suffices to proceed by induction, and assume that w_1 is an elementary conjugation. □

For any w in some $W(\Theta,\Omega)$, let Ψ_w for the moment be the set of those hyperplaces in $\mathfrak{a}_\Theta$ separating C_Θ from wC_Ω. Then in analogy with 1.1.1, one has $\mathrm{ht}(w_2 w_1) = \mathrm{ht}(w_2) + \mathrm{ht}(w_1)$ if and only if $\Psi_{w_2} \cup w_2 \Psi_{w_1} \subseteq \Psi_{w_2 w_1}$. Applying 1.1.1 itself, what I want to show is that (i) any root hyperplane separating $w_2 C_\emptyset$ from $C_\emptyset$ also separates $w_2 w_1 C_\emptyset$ from $C_\emptyset$ and (ii) if H is a root hyperplane separating $w_1 C_\emptyset$ from $C_\emptyset$ then $w_2 H$ separates $w_2 w_1 C_\emptyset$ from $C_\emptyset$.

Lemma 1.2.8. Let Θ and Ω be associates in Δ, and let $w \in W(\Theta,\Omega)$.

(a) If $\alpha \in \Sigma^+ \setminus \Sigma^+_\Theta$ is such that $\mathfrak{a}_\Theta \cap \mathfrak{a}_\alpha$ separates wC_Ω from C_Θ, then $\mathfrak{a}_\alpha$ separates $wC_\emptyset$ from $C_\emptyset$.

(b) If $\alpha \in \Sigma^+$ is such that $\mathfrak{a}_\alpha$ separates $wC_\emptyset$ from $C_\emptyset$, then either $\alpha \in \Sigma^+_\Theta$ or $\alpha \in \Sigma^+ \setminus \Sigma^+_\Theta$ and $\mathfrak{a}_\Theta \cap \mathfrak{a}_\alpha$ separates wC_Ω from C_Θ.

This is elementary, and the proofs of claims (i) and (ii) follow directly from it.

One can prove similarly:

Proposition 1.2.9. Let $\Theta, \Omega \subseteq \Delta$ be associates, $w \in W(\Theta,\Omega)$. Then $\mathrm{ht}(w_\ell w_{\ell,\Omega}) = \mathrm{ht}(w_\ell w_{\ell,\Omega} w^{-1}) + \mathrm{ht}(w)$.

Note that this at least makes sense because $w_\ell w_{\ell,\Omega}$ takes Ω to its conjugate $\overline{\Omega}$ in Δ. The geometric interpretation of 1.2.9 is that $w_\ell w_{\ell,\Omega}$ takes C_Ω to $-C_{\overline{\Omega}}$.

1.3. In this section, suppose k to be any field. I shall refer to algebraic groups defined over k by boldface[2] letters with k as subscript, and the group of k-rational points of that group by the same letter in ordinary type, again with k as subscript. When confusion is unlikely, I shall drop the subscript. Thus G_k or G and $\mathbf{G}_k$ or $\mathbf{G}$.

Let G be a connected reductive group defined over k. If P is a parabolic subgroup, I shall let N_P be the unipotent radical of P, M_P a reductive subgroup of P with $P = N_P M_P$ a Levi decomposition, P^- the opposite of P, N_P^- the unipotent radical

[2]Between here and 1.3.1, I'm not sure just which letters should be in boldface.

of P^-. A_P the maximal split torus in the centre of M_P. I again often drop the subscripts.

If $P_\emptyset$ is a minimal parabolic of G and $A_\emptyset$ a maximal split torus of $P_\emptyset$, recall that a *root* of G with respect to $A_\emptyset$ is any non-trivial rational character α of $A_\emptyset$ such that the eigenspace $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \,\middle|\, \mathrm{Ad}(a)x = \alpha(a)x\} \text{ for all } a \in A_\emptyset \right\}$ in the Lie algebra $\mathfrak{g}$ of G is not trivial. It is said to be *positive* with respect to $P_\emptyset$ if $\mathfrak{g}_\alpha \subseteq \mathfrak{n}$, the Lie algebra of $N_\emptyset$. The roots may be considered to be embedded in the vector space $X(A_\emptyset) \otimes \mathbf{R}$, where $X(A_\emptyset)$ is the group of rational characters of $A_\emptyset$, and these form a root system. Let Σ be the reduced set of roots corresponding to this — i.e. those α such that $\alpha \neq \beta^2$ for any root β — and $\Sigma^+ = \Sigma^+_{P_\emptyset}$ the set of positive roots in Σ (with respect to $P_\emptyset$), Δ the set of simple roots in Σ^+, etc.

For $\Theta \subseteq \Delta$, let A_Θ be the connected component of the identity in $\cap_{\alpha\in\Theta} \ker(\alpha)$, $M_\Theta = Z_G(A_\Theta)$, P_Θ the standard parabolic corresponding to Θ, N_Θ its unipotent radical. Thus $P_\Delta = G$ and A_Δ is the maximal split torus in the centre of G. The Weyl group of Σ is naturally isomorphic to $N(A_\emptyset)/M_\emptyset$, and each W_Θ (notation as in §1.1) is naturally isomorphic to $(N(A_\emptyset) \cap M_\Theta)/M_\emptyset$. Note that M_Ω sits canonically in M_Θ for $\Omega \subseteq \Theta$.

Proposition 1.3.1. If Θ, Ω are subsets of Δ, then one has a disjoint union decomposition

$$G = \coprod P_\Theta w P_\Omega$$

where w ranges over the set $[W_\Theta \backslash W / W_\Omega]$.

Recall from §1.1 that $[W_\Theta \backslash W / W_\Omega]$ is a particularly good choice of representatives in $N(A_\emptyset)$, but here this is of little importance.

Proof. One knows that $(G, P_\emptyset, N(A_\emptyset), S)$ form a Tits system. Lemma 1.1.3 and [10, Remark 2, p. 28] imply the proposition. □

Recall from section 3 of [4] that associated to each reduced root α is a subgroup N_α of G whose Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ (of course $\mathfrak{g}_{2\alpha}$ may be trivial) and such that $N_\emptyset$ is the product of all the N_α $(\alpha \in \Sigma^+)$, in any order. The unipotent radical of each P_Θ is equal to $\prod N_\alpha$ $(\alpha \in \Sigma^+ \smallsetminus \Sigma^+_\Theta)$ and $N_\emptyset \cap M_\Theta$ is equal to $\prod N_\alpha$ $(\alpha \in \Sigma^+_\Theta)$. For any $\Theta, \Omega \subseteq \Delta$ and $w \in W$ the canonical projection induces an isomorphism

$$\prod_{\substack{\alpha \in \Sigma^+ \smallsetminus \Sigma^+_\Omega \\ w^{-1}\alpha \notin \Sigma^+ \smallsetminus \Sigma^+_\Theta}} N_\alpha \xrightarrow{\sim} (wN_\Theta w^{-1} \cap N_\Omega) \backslash N_\Omega.$$

In the case where $\Theta = \Omega = \emptyset$, I write this last as N_w, which is also equal to $\prod N_\alpha$ $(\alpha \in \Sigma_w)$. It follows from the remarks of section 3.2 of [5] that for $w \in [W_\Theta \backslash W / W_\Omega]$

the product map induces an isomorphism

$$P_\Theta \times \{w\} \times \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma^+_\Omega \\ w^{-1}\alpha\notin\Sigma^+\smallsetminus\Sigma^+_\Theta}} N_\alpha \xrightarrow{\sim} P_\Theta w P_\Omega.$$

It follows immediately from the proceeding remarks and 1.1.1 that:

Proposition 1.3.2. If u, v are elements of W with $\ell(uv) = \ell(u)+\ell(v)$ and $x \in N(A_\emptyset)$ represents u, then the map $(n_u, n_v) \mapsto xn_vx^{-1}n_u$ is a bijection between $N_u \times N_v$ and N_{uv}.

It also follows immediately that:

Proposition 1.3.3. Let $\Theta, \Omega \subseteq \Delta$ be subsets of Δ, $w \in [W_\Theta\backslash W/W_\Omega]$.

(a) The subgroup $(w^{-1}P_\Theta w \cap P_\Omega)N_\Omega$ is the standard parabolic corresponding to $w^{-1}\Theta \cap \Omega$;

(b) Its radical is generated by N_Ω and $w^{-1}N_\Theta w \cap N_\emptyset$, and $w^{-1}N_\Theta w \cap N_\emptyset$, and its reductive component is $w^{-1}M_\Theta w \cap M_\Omega$;

(c) The group $w^{-1}P_\Theta w \cap M_\Omega$ is parabolic in M_Ω with radical $w^{-1}N_\Theta w \cap M_\Omega$ and reductive component $w^{-1}M_\Theta w \cap M_\Omega$.

Two parabolic subgroups of G are called *associates* if their reductive components are conjugate.

Proposition 1.3.4. Let Θ, Ω be subsets of Δ. The following are equivalent:

(a) The groups P_Θ and P_Ω are associate;

(b) The groups A_Θ and A_Ω are conjugate;

(c) The sets Θ and Ω are associate;

Proof. Conditions (a) and (b) are clearly equivalent. That (c) implies (b) is trivial. If (b) holds, and $gA_\Theta g^{-1} = A_\Omega$, then by [2, Corollary 4.22] one may assume $g \in N(A_\emptyset)$. Apply 1.2.1 to the image of g in W. □

It will be useful to observe that if $w \in W(\Theta, \Omega)$ then Σ_w is equal to the set $\Sigma^+\smallsetminus\Sigma^+_\Theta$, since $w^{-1}\Sigma^\pm_\Theta \subseteq \Sigma^\pm_\Omega$.

Proposition 1.3.5. Let Θ, Ω, Γ be associates in Δ, $u \in W(\Theta, \Omega), v \in W(\Omega, \Gamma)$, $\mathrm{ht}(uv) = \mathrm{ht}(u) + \mathrm{ht}(v)$. Then $P_\Theta uP_\Omega \cdot P_\Omega vP_\Gamma = P_\Theta uvP_\Gamma$.

Proof. This follows from the remark just made, one of the remarks made just after 1.3.1, and 1.3.2. □

1.4. For the rest of section 1, and indeed most of this paper, let k be a non-archimedean locally compact field, $\mathcal{O}$ its integers, and $\wp$ its prime ideal, and $\mathbf{G}$ a connected reductive group defined over k.

Let $P_\emptyset$ be a minimal parabolic in G (and assume notation as in section §1.3). For each ϵ in $(0,1]$ and each $\Theta \subseteq \Delta$ define $A_\Theta^-(\epsilon)$ to be

$$\Big\{a \in A_\Theta \Big| \; |\alpha(a)| \le \epsilon \text{ for all } \alpha \in \Delta \smallsetminus \Theta\Big\}.$$

I write simply A_Θ^- for $A_\Theta^-(1)$. Of course $A_\Theta^-(\epsilon_1) \subseteq A_\Theta^-(\epsilon_2)$ for $\epsilon_1 \le \epsilon_2$, and one has therefore in some sense a nested set of neighborhoods of A_Θ "at 0". If P is any parabolic of G, choose $g \in G$ such that $gPg^{-1} = P_\Theta$ for $\Theta \subseteq \Delta$, and define $A^-(\epsilon)$ to be $g^{-1}A_\Theta^-(\epsilon)g$. Since the conditions $gPg^{-1} = P$, $gAg^{-1} = A$ imply that $g \in M$, this definition is independent of the choice of g.

Lemma 1.4.1. If N is any unipotent group defined over k, then there exist in N arbitrarily large compact open subgroups.

Proof. This is clearly true of the subgroup of GL_n consisting of unipotent upper triangular matrices, and any other unipotent group has an embedding into this one for a suitable n. □

Proposition 1.4.2. Suppose ℓ to be a finite extension of k, $G_\ell = G_k \times \ell$, P a parabolic subgroup of G and $P_\ell = P_k \times \ell$, etc. Then $A_k^- = A_\ell \cap A_k$, and furthermore:

(a) For any ϵ_1 there exists ϵ_2 such that $A_k^-(\epsilon_2) \subseteq A_\ell^-(\epsilon_1) \cap A_k$;
(b) For any ϵ_1 there exists ϵ_2 such that $A_\ell^-(\epsilon_2) \cap A_k \subseteq A_k^-(\epsilon_1)$.

Proof. Define groups $A_\Theta^-(\epsilon)^*$ similar to the $A_\Theta^-(\epsilon)$:

$$A_\Theta^-(\epsilon)^* = \Big\{a \in A \Big| \; |\alpha(a)| \text{ for all } \alpha \in \Sigma^+ \smallsetminus \Sigma_\Theta^+\Big\}.$$

(They will only be used in this proof.) It is clear that $A_\Theta^-(1)^* = A_\Theta^-(1)$, that $A_\Theta^-(\epsilon)^* \subseteq A_\Theta^-(\epsilon)$, and that for every ϵ_1 there exists ϵ_2 such that $A_\Theta^-(\epsilon_2) \subseteq A_\Theta^-(\epsilon_1)^*$. Furthermore, because the restrictions of the α in $\Sigma^+ \smallsetminus \Sigma_\Theta^+$ are precisely the eigencharacters of the representations of A_Θ on $\mathfrak{n}_\Theta$ (the Lie algebra of N_Θ), it is clear that, in the present terminology, $A_{\Theta,k}^-(\epsilon)^* = A_{\Theta,\ell}^-(\epsilon)^* \cap A_k$. The proposition is immediate from these remarks. □

Proposition 1.4.3. If P is a parabolic subgroup of G and N_1 and N_2 are two open compact subgroups of N, then there exists $\epsilon > 0$ such that $a \in A^-(\epsilon)$ implies $aN_2a^{-1} \subseteq N_1$.

Proof. First assume G to be split over k, $P_\emptyset$ a minimal parabolic, $P = P_\Theta$ for some $\Theta \subseteq \Delta$. Then $N = \prod N_\alpha$ $(\alpha \in \Sigma^+ \smallsetminus \Sigma_\Theta^+)$, and since $A = A_\Theta$ acts via α on each N_α, the proposition is clear.

In general, let ℓ be a finite extension of k such that $G_\ell = G_k \times \ell$ is split; let $P_\ell = P_k \times \ell$, etc. By 1.4.2 one can find $N_{2,\ell}$ compact and open in N_ℓ containing N_2;

let $N_{1,\ell}$ be compact and open in N_ℓ such that $N_{1,\ell} \cap N_k \subseteq N_1$. By the preceding case one can find ϵ^* such that whenever $a \in A^-_\ell(\epsilon^*)$, $aN_{2,\ell}a^{-1} \subseteq N_{1,\ell}$. Apply 1.4.2(a). □

If P is a parabolic subgroup of G and K is a compact open subgroup, one says that K has an *Iwahori factorization* with respect to P if (i) the product map is an isomorphism of $N^-_K \times M_K \times N_K$ with K, where $N^-_K = N^- \cap K$, etc., and (ii) for every $a \in A^-$, $aN_Ka^{-1} \subseteq N_K$, $a^{-1}N^-_K a \subseteq N^-_K$.

Proposition 1.4.4. Let $P_\emptyset$ be a minimal parabolic subgroup of G. There exists a collection $\{K_n\}_{n\ge 0}$, which forms a neighborhood basis of the identity such that

(a) Every K_n is a normal subgroup of K_0;
(b) If P is parabolic subgroup containing $P_\emptyset$ then K_n has an Iwahori factorization with respect to P;
(c) If $P = MN$ is a parabolic subgroup containing $P_\emptyset$ then M_K has an Iwahori factorization with respect to $M \cap P_\emptyset$.

Proof. Assume first that $\mathbf{G}$ is split over k. Then according to [16, XXV.1.3], there exists a smooth group scheme $G_{\mathcal{O}}$ over $\mathrm{Spec}(\mathcal{O})$ such that $G \cong G_{\mathcal{O}} \times \mathrm{Spec}(k)$. If R is any ring supplied with a homomorphism from $\mathcal{O}$ to R, let $G(R)$ be the group of R-valued points on $G_{\mathcal{O}}$, and similarly for any group subscheme of $G_{\mathcal{O}}$. For each integer $n \ge 0$, let $G(\wp^n)$ be the kernel of the reduction homomorphism: $G(\mathcal{O}) \longrightarrow G(\mathcal{O}/\wp^n)$, and similarly for any group subscheme of $G_{\mathcal{O}}$. (I take $\wp^0$ to be $\mathcal{O}$.)

Let $P_{\emptyset,\mathcal{O}}$ be a minimal parabolic group subscheme of $G_{\mathcal{O}}$, K_0 the inverse image in $G(\mathcal{O})$ of $P_\emptyset(\mathcal{O}/\wp)$. It is essentially proven in [21] (in the proof of Theorem 2.5), that K_0 has an Iwahori factorization with respect to any parabolic P containing $P_\emptyset$. More precisely, $K_0 = N^-_1 M_0 N_0$ where $N^-_1 = N^-_1(\wp)$, $N_0 = N(\mathcal{O})$, and M_0 is the inverse image in M_0 of $(P_\emptyset \cap M)(\mathcal{O}/\wp)$. The same is true of every $K_n = G(\wp^n)$, with $N^-_n = N^-(\wp^n)$, etc., and therefore this sequence satisfies the conditions of 1.4.4. Since every parabolic subgroup of G is conjugate to one obtained from $G_{\mathcal{O}}$, this proves 1.4.4 in this case.

Now let G be the k-points of an arbitrary reductive group defined over k, P a minimal parabolic subgroup. Let ℓ/k be a finite Galois extension with Galois group $\mathrm{Gal}(\ell/k)$ such that $G \times \ell = G_\ell$ is split over ℓ. Let $\{K_{\ell,n}\}$ be a sequence satisfying 1.4.4 for $P_\ell = P \times \ell$, and define K_n to be $K_{\ell,n} \cap G$. The sequence $\{K_n\}$ is clearly a basis of the neighborhoods of 1 in G.

Proof of (a): for $k \in K_n$, one has $k = n^-mn$ with $n \in N_{\ell,n}$, etc. But then for $\sigma \in \mathrm{Gal}(\ell/k)$, $k = k^\sigma = (n^-)^\sigma m^\sigma n^\sigma$. Since P is defined over k, so is N^-, etc. Since $N^-_\ell \cap M_\ell N_\ell = \{1\}$ one has $(n^-)^\sigma = n^-$, etc. This implies that n^-, etc., are in fact in G, hence in $G \cap K_{\ell,n}$, and shows that K_n satisfies property (i) of an Iwahori factorization. The remainder of 1.4.4 is proved similarly (using 1.4.2 at one point). □

One can use results from [13] to obtain finer results, but at the cost of complication. Other elementary derivations of 1.4.4 have been given by Harish-Chandra and also by Deligne [17].

Lemma 1.4.5. Suppose that the center of G is anisotropic. Then there exists a maximal compact subgroup $K \subseteq G$ such that (a) $G = PK$ for any parabolic subgroup P; (b) $A_\emptyset(\mathcal{O}) \subseteq K$; and (c) $G = KA_\emptyset^- K$, with the map $a \mapsto KaK$ establishing a bijection between $K\backslash G/K$ and $A_\emptyset^-/A_\emptyset(\mathcal{O})$.

Proof. Set $A = A_\emptyset$ for convenience. Let $\widetilde{G}$ be the simply connected covering of the derived group of G, $\phi\colon \widetilde{G} \longrightarrow G$ the canonical projection, $\widetilde{A}$ the maximal split torus of $\widetilde{G}$ over A. By assumption $\phi|_{\widetilde{A}}$ is an isogeny. Define $\widetilde{\mathcal{N}}$ to be $N(\widetilde{A})/\widetilde{A}(\mathcal{O})$. Let B be an Iwahori subgroup in $\widetilde{G}$ compatible with $\widetilde{A}$ (see [12] and [13] for notation and statements I give here).

Since the bornology of $\widetilde{G}$ is that of compact subsets, 3.5.1 of [13] implies that ϕ is B-adapted (see 1.2.13 of [13]).

Since the canonical morphism from the direct product of the center of G and $\widetilde{G}$ into G is both central and surjective, 3.19 of [6] together with its proof imply that $Z(A)/\phi(Z(\widetilde{A}))Z \cong G/\phi(\widetilde{G})Z$, and this implies that ϕ is $\widetilde{\mathcal{N}}$-adapted (see 1.2.13 of [13]).

The group G acts on the building associated to $(\widetilde{G}, \widetilde{B}, \widetilde{N})$. Let $\mathcal{N}$ be the stabilizer of the apartment $\mathcal{A}$ corresponding to $\widetilde{A}$ — i.e., $\mathcal{N}$ is the normalizer of $\widetilde{A}$ in G, which amounts as well to the normalizer of $\phi(\widetilde{A})$ in G. But since the Zariski closure of $\phi(\widetilde{A})$ is A, and $\phi(\widetilde{A})$ is Zariski-dense in A, this is also the normalizer of A in G. This implies that ϕ is of connected type (see 4.1.3 of [13]).

The remarks in 4.4.5 of [13] together with the proposition in 4.4.6 of [13] imply that if K is the stabilizer of a special point in $\mathcal{A}$ (see 1.3.7 of [13]) it satisifies the conditions of the proposition. □

Proposition 1.4.6. Let G be arbitrary. There exists an open subgroup $\Gamma \subseteq G$ such that

(a) $G = \Gamma A_\emptyset^- \Gamma$;
(b) $A_\emptyset(\mathcal{O}) \subseteq \Gamma$; and
(c) $\Gamma/(\Gamma \cap Z)$ is compact.

Proof. Let $\overline{\mathbf{G}} = \mathbf{G}/\mathbf{A}_\Delta$, $\psi\colon G \longrightarrow \overline{G}$. Then ψ induces an isomorphism $G/A_\Delta \cong \overline{G}$ by 15.7 of [3], and the group $\overline{G}$ satisfies the hypotheses of 1.4.5. Let K be the group given there, and defined Γ to be $\psi^{-1}(K)$.

Note that Γ fits into an exact sequence

$$1 \longrightarrow A_\Delta \longrightarrow \Gamma \longrightarrow K \longrightarrow 1$$

and that furthermore Γ even contains all of Z. □

1.5. If $P = MN$ is any parabolic subgroup of G, let δ_P be its modulus character: $P \longrightarrow \mathbf{C}^*$, $p \longmapsto |\det \mathrm{Ad}_{\mathfrak{n}}(p)|$, where $\mathfrak{n}$ is the Lie algebra of N. It is trivial on N, hence essentially a character of M.

Assign G a Haar measure.

Lemma 1.5.1. If P is a parabolic subgroup of G and K_0 a compact open subgroup with an Iwahori factorization with respect to P, then for $a \in A^-$ one has $\mathrm{meas}\, K_0 a K_0 = \delta_P^{-1}(a) \mathrm{meas}\, K_0$.

Proof. The map $k_1 a k_2 \longmapsto k_1$ induces an isomorphism between the sets $K_0 a K_0 / K_0$ and $K_0/(K_0 \cap a K_0 a^{-1})$. But since $a \in A^-$, $K_0 \cap a K_0 a^{-1} = N_0^- M_0 (a N_0 a^{-1})$ and $[K_0 : K_0 \cap a K_0 a^{-1}] = [N_0 : a N_0 a^{-1}] = \delta_P^{-1}(a)$. $\square$

Proposition 1.5.2. Let P be a parabolic of G, K any compact open subgroup. There exist constants $C_2 \geq C_1 > 0$ such that for any $a \in A^-$ one has

$$C_1 \delta_P^{-1}(a) \leq \mathrm{meas}(KaK) \leq C_2 \delta_P^{-1}(a)$$

Proof. Let K_0 be an open subgroup of K with an Iwahori factorization with respect to P, and let C be $[K : K_0]$. Assume for convenience that $\mathrm{meas}(K_0) = 1$.

First: $\mathrm{meas}(KaK) \leq C\delta_P^{-1}(a)$. This follows from

$$\begin{aligned} \mathrm{meas}(KaK) &= [K : K \cap aKa^{-1}] \\ &\leq [K : K_0 \cap aK_0a^{-1}] \\ &= [K_0 : K_0 \cap aK_0a^{-1}][K : K_0] \\ &= C\delta_P^{-1}(a) \end{aligned}$$

by 1.5.1.

Next: $\mathrm{meas}(KaK) \geq C^{-1}\delta_P^{-1}(a)$. One has an injection from $K_0/(K_0 \cap aKa^{-1})$ into $K/K \cap aKa^{-1}$, so that $\mathrm{meas}(KaK) \geq [K_0 : K_0 \cap aKa^{-1}]$. But one also has $1 \leq [K_0 \cap aKa^{-1} : K_0 \cap aK_0a^{-1}] \leq C$ which together with 1.5.1 implies the claim. $\square$

1.6. Let $\mathbf{P}_\emptyset$ be a fixed minimal parabolic of $\mathbf{G}$, etc.

A rational character of $\mathbf{G}$ is a k-morphism from $\mathbf{G}$ to $\mathbf{G}_m$, and of course determines a map on k-rational points $G \longrightarrow k^\times$. For any $\mathbf{G}$ let $X(\mathbf{G})$ be its group of rational characters.

Lemma 1.6.1. Any rational character of $\mathbf{G}$ is determined by its restriction to $\mathbf{A}_\Delta$.

Recall that $\mathbf{A}_\Delta$ is the maximal split torus in the center of $\mathbf{G}$.

Proof. Let $\mathbf{G}^{\mathrm{der}}$ be the derived group of $\mathbf{G}$, $\mathbf{T}$ the torus quotient $\mathbf{G}/\mathbf{G}^{\mathrm{der}}$, and $\mathbf{T}_s$ the maximal split quotient of $\mathbf{T}$ (see [18, XXII.6]). Any rational character of $\mathbf{G}$ factors through the projection $\mathbf{G} \longrightarrow \mathbf{T}_s$, and the restriction of this projection to $\mathbf{A}_\Delta$ is an isogeny. The map from $X(\mathbf{T})$ to $X(\mathbf{A}_\Delta)$ is thus an injection of one lattice into another. $\square$

For each $\Theta \subseteq \Delta$, let γ_Θ be the rational modulus character of $\mathbf{P}_\Theta$: $\gamma_\Theta(p) = \det \mathrm{Ad}_{\mathfrak{n}_\Theta}(p)$. If for each $\alpha \in \Sigma$ one lets $m(\alpha)$ be the dimension of the α-eigenspace $\mathfrak{g}_\alpha$ in $\mathfrak{g}$, then according to 1.6.1 and the definition of the roots, γ_Θ is characterized as the unique rational character of $\mathbf{M}_\Theta$ restricting to $\prod \alpha^{m(\alpha)}$ $(\alpha > 0)$ on $\mathbf{A}_\Theta$. Since $\mathbf{A}_\Theta$ is the connected component of $\cap_{\alpha\in\Theta} \ker(\alpha)$:

Proposition 1.6.2. For $\Theta \subseteq \Omega \subseteq \Delta$, the restriction of γ_Θ to $\mathbf{M}_\Omega$ is equal to γ_Ω.

A complex character of G is a continuous homomorphism from G to $\mathbf{C}^\times$. (If I write of a character without qualification, I shall mean a complex one. Note that I do not require a character to be *unitary* — i.e., have its image in the unit circle). If $\gamma\colon \mathbf{G} \longrightarrow \mathbf{G}_m$ is a rational character of G and $\beta\colon k^\times \longrightarrow \mathbf{C}^\times$ is a complex character of $k^\times$, then the composition is a complex character of G which I shall call $\beta\cdot\gamma\colon G \longrightarrow k^\times \longrightarrow \mathbf{C}^\times$. In particular one may choose β to be the modulus $x \mapsto |x|$, and one thus obtains the *norm* or *modulus* $|\gamma|$ of $\gamma\colon x \longmapsto |\gamma(x)|$. For example, the modulus δ_Θ of P_Θ is the usual modulus one refers to in connection with Haar measure on P_Θ.

One can describe the group of complex characters of G to some extent by means of results in [6]. Let G^u be the subgroup of G, hence of G^{der}, generated by the elements in the unipotent radicals of the minimal parabolics in G ([6] 6.2). Then the restriction of any complex character of G to G^u is trivial ([6] 6.4) so that it must factor through the projection: $G \longrightarrow G/G^u$.

Let $E = G/G^u$. I claim now that there exists in E a maximal compact subgroup E_0 such that the quotient E/E_0 is a free abelian group of rank equal to the dimension of the maximal split quotient T_s of G. First of all, one knows ([6] 6.14) that G^u is closed in G and that G^{der}/G^u is compact. Further, one has the exact sequences

$$1 \longrightarrow G^{\mathrm{der}} \longrightarrow G \longrightarrow T \longrightarrow 1$$
$$1 \longrightarrow T_0 \longrightarrow T \longrightarrow T/T_0 \longrightarrow 1$$

where T_0 is maximal compact in T, and T/T_0 is a lattice of rank equal to the dimension of T_s. Define E_0 to be the inverse image in E of T_0.

A complex character of G is said to be *unramified* if it is trivial on E_0. A choice of basis for E/E_0 gives an isomorphism of the group $X_{\mathrm{nr}}(G)$ of unramified characters of G with $(\mathbf{C}^\times)^r$ $(r = \dim T_s)$ so that $X_{\mathrm{nr}}(G)$ has naturally the structure of a complex analytic group of dimension r (and this structure is of course independent of the choice of basis). The unitary characters of $X_{\mathrm{nr}}(G)$ form a real analytic subgroup of this isomorphic to a product of r unit circles.

2. Elementary results about admissible representations

Throughout this section, let G be an arbitrary locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the identity. This condition is satisfied if and only if G has compact open subgroups and they are all profinite. It is also satisfied for any closed subgroup of G, in particular for its center Z.

See §3 of [3] for related matter.

2.1. Let F be an arbitrary field of characteristic 0. Let (π, V) be a representation of G on a vector space V defined over F. If K is any subgroup of G, define V^K to be $\left\{ v \in V \,\middle|\, \pi(k)v = v \text{ for all } k \in K \right\}$. Define (π, V) to be a *smooth* representation if every $v \in V$ lies in V^K for some open subgroup K. This is equivalent to the condition that π be continuous with respect to the discrete topology on V.

Define (π, V) to be *admissible* if it is smooth and if in addition V^K has finite dimension for every open subgroup K; *irreducible* if there are no proper G-stable subspaces; *finitely generated* if there exists a finite subset $X \subseteq V$ such that the smallest nonzero G-stable subspace containing X is all of V.

If H is a subgroup of G, then a representation of G is said to be *H-finite* if every vector is contained in a finite-dimensional H-stable subspace.

If X is any subset of V, then the *G-space generated by X* is the smallest G-stable subspace containing X, and is also the subspace of vectors in V of the form $\sum c_i \pi(g_i) x_i$, with $g_i \in G$ and $x_i \in X$.

If ϵ is an involution of F, then (π, V) is said to be *unitary with respect to ϵ* if there exists a G-invariant anisotropic form on V, Hermitian with respect to ϵ. If $F = \mathbf{C}$, then the involution will be understood to be conjugation.

There are a number of smooth representations associated to the action of the group on itself. Define $C^\infty(G, F)$ to be the space of all locally constant functions $f\colon G \longrightarrow F$; $C_u^\infty(G, F)$ to be those $f \in C^\infty(G, F)$ such that for some compact open subgroup K, $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$, $g \in G$ (these are the *uniformly* locally constant functions.) Define $C_c^\infty(G, F)$ to be $\left\{ f \in C^\infty(G, F) \,\middle|\, f \text{ has compact support} \right\}$. One has $C_c^\infty \subseteq C_u^\infty$, clearly. For $f \in C^\infty(G, F)$ and $g \in G$, define $R_g f$ and $L_g f$ by the respective formulas

$$(R_g f)(x) = f(xg) \quad \text{and} \quad (L_g f)(x) = f(g^{-1}x).$$

These operators define smooth representations of G, called respectively the *right regular* and *left regular representations,* on both $C_u^\infty(G, F)$ and $C_c^\infty(G, F)$.

From §3 on, I shall assume F to be $\mathbf{C}$ (or, occasionally, $\mathbf{R}$), but this is mostly a matter of convenience in notation, as will be explained later, and it is probably worth something to know that results may be formulated without this assumption. I should add that the possibility of using quite general fields of definition has not been, as far as I know, seriously exploited (see, however, [16] and [29]).

If (π, V) is a representation over F, then for any field extension E/F one obtains the obvious extended representation $(\pi, V \otimes_F E)$.

Proposition 2.1.1. If (π, V) is a representation of G over F and E/F is a field extension, then (π, V) is

(a) smooth;
(b) admissible;
(c) finitely generated;

if and only if $(\pi, V \otimes E)$ is.

Proof. The basic observation is that if K is any subgroup of G, then

$$(V \otimes E)^K = V^K \otimes E.$$

Certainly, the right-hand side is contained in the left. For the opposite inclusion, let $\sum v_i \otimes x_i$ be fixed by K. One may assume the x_i to be linearly independent over F. But then $\pi(k)(\sum v_i \otimes x_i) = \sum v_i \otimes x_i$ implies that $\sum(\pi(k)v_i - v_i) \otimes x_i = 0$, which in turn implies that $\pi(k)v_i = v_i$.

From this, the proposition is clear. □

A representation (π, V) is said to be *absolutely irreducible* if for every extension E/F, $(\pi, V \otimes E)$ is irreducible.

Proposition 2.1.2. If π is smooth, then it is K-finite for every compact open subgroup K.

Proof. For every $v \in V$, there exists a subgroup K_1 of finite index in K such that $v \in V^{K_1}$. Thus, $K \cdot v$ is a finite set, and generates a finite-dimensional K-stable subspace. □

Proposition 2.1.3. If H is any closed subgroup of G such that $H/H \cap Z$ is compact, then any admissible representation of G is H-finite. In particular, any admissible representation is Z-finite.

Proof. If (π, V) is an admissible representation of G, then for each compact open subgroup K, V^K is finite-dimensional and Z-stable. This proves the last statement. If H satisfies the hypotheses of the proposition and K is any compact open subgroup of G, then the image of $K \cap H$ in $H/H \cap Z$ has finite index. Thus the space spanned by the elements $\left\{ h \cdot v \,\middle|\, h \in H,\, v \in V^K \right\}$ is finite-dimensional. □

Assign G a *rational Haar measure,* i. e., one such that for some (hence any) compact open subgroup K one has $\text{meas}(K) \in \mathbf{Q}$. (Without further mention, all Haar measures will be assumed to be rational.) If (π, V) is a smooth representation of G and K is a compact open subgroup, define the operator $\mathcal{P}_K$ by the formula

$$\mathcal{P}_K(v) = \frac{\int_K \pi(k)v\,dk}{\text{meas}\,K}.$$

The smoothness of π implies that this is essentially a finite sum, hence makes sense. The operator $\mathcal{P}_K$ is the projection of V onto V^K. If $V(K)$ is the kernel of $\mathcal{P}_K$, then it may also be described as the space spanned by the vectors of the form $\pi(k)v - v$. One has $V = V^K \bigoplus V(K)$ as representations of K.

Proposition 2.1.4. Let (π, V) be a smooth representation and K a compact open subgroup of G. Then π is admissible if and only if the restriction of π to K is a direct sum of irreducible finite-dimensional representations, each isomorphism class occurring with finite multiplicity.

Proof. Assume π admissible. If K_1 is an open normal subgroup of K, one has $V = V^{K_1} \bigoplus V(K_1)$, each summand being K-stable. The group K_1 of course acts trivially on V^{K_1}, which may then be considered a representation of K/K_1, hence a direct sum with finite multiplicities of irreducible representations of K. An application of Zorn's Lemma then enables one to decompose V as a direct sum of irreducible finite-dimensional smooth representations of K.

Finite multiplicity follows from the fact that any given smooth finite-dimensional representation must have some normal open K_1 in its kernel.

The converse is clear. □

Proposition 2.1.5. If (π, V) is admissible and unitary, with Hermitian form (u, v), and U is any G-stable subspace of V, then $U^\perp = \left\{ v \in V \,\middle|\, (u, v) = 0 \text{ for all } u \in U \right\}$ is also G-stable, and $V = U \bigoplus U^\perp$.

The proof is straightforward.

Proposition 2.1.6. The categories of smooth and admissible representations of G are abelian categories.

This is trivial.

Proposition 2.1.7. Let (π_i, V_i) $(i = 1, 2, 3)$ be smooth G-representations, K a compact open subgroup of G. If $V_1 \longrightarrow V_2 \longrightarrow V_3$ is an exact sequence of G-morphisms, then the sequence $V_1^K \longrightarrow V_2^K \longrightarrow V_3^K$ is exact as well.

Proof. Given $v \in V_2^K$ whose image in V_3 is 0, choose $v_1 \in V_1$ with image v in V_2. Then $\mathcal{P}_K(v_1)$ lies in V_1^K and still has image v. □

Proposition 2.1.8. If (π_i, V_i) $(i = 1, 2, 3)$ are smooth representations of G and the sequence of G-morphisms

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is exact, then π_2 is admissible if and only if π_1 and π_3 are.

Proof. By 2.1.7. □

Define now a *character* of G to be a smooth one-dimensional representation. It thus amounts to a homomorphism from G to $F^\times$, with open kernel. (Since any homomorphism from G to $\mathbf{C}^\times$ continuous with respect to the usual topology on $\mathbf{C}$ has this property, the definition here does not contradict that in 1.6.2.) If ϵ is an involution of F, then the character $\chi\colon G \longrightarrow F^\times$ is unitary if and only if $\chi(G) \subseteq \left\{x \in F \,\middle|\, xx^\epsilon = 1\right\}$. If (π, V) is any smooth representation and χ is a character, one defines $(\pi \otimes \chi, V)$, or sometimes $(\pi \cdot \chi, V)$, to be the representation of G on the same space, taking g to $\pi(g) \cdot \chi(g)$.

If (π, V) is Z-finite and ω is a character of Z, then for each integer $n \geq 1$ define

$$V_{\omega,n} = \left\{v \in V \,\middle|\, (\pi(z) - \omega(z))^n v = 0 \text{ for all } z \in Z\right\},$$

and also define

$$\begin{aligned} V_{\omega,\infty} &= \bigcup_{n \in \mathbf{N}} V_{\omega,n}, \\ V_\omega &= V_{\omega,1}. \end{aligned}$$

Each $V_{\omega,n}$ is G-stable. The representation (π, V) is called an *ω-representation* if $V = V_\omega$.

Proposition 2.1.9. Assume that F is algebraically closed and that V is Z-finite. Then

(a) One has a direct sum decomposition $V = \bigoplus V_{\omega,\infty}$;
(b) If V is finitely generated, then there are only a finite number of ω with $V_{\omega,\infty} \neq 0$, and there exists n such that $V_{\omega,\infty} = V_{\omega,n}$ for each ω.

Proof. By standard facts about commuting families of operators on finite-dimensional spaces. □

Proposition 2.1.9(b) implies that there exists a finite filtration of V whose factors are ω-representations for certain ω.

For a given ω, the smooth and admissible ω-representations clearly form abelian categories, in analogy with 2.1.6.

If (π, V) is any smooth representation of G, define its *dual* $(\widehat{\pi}, \widehat{V})$ to be the representation ${}^t\pi(g^{-1})$ on the algebraic dual $\widehat{V}$ of V, and define its *contragredient* $(\widetilde{\pi}, \widetilde{V})$

to be the restriction of $\widehat{\pi}$ to the subspace $\widetilde{V}$ of elements of $\widehat{V}$ fixed by some open subgroup. Thus, $\widetilde{\pi}$ is smooth. For any compact open subgroup K, $\widehat{V}^K$ is the algebraic dual of V^K, and of course also equal to $\widetilde{V}^K$. This proves:

Proposition 2.1.10. The following are equivalent:

(a) π is admissible;
(b) $\widetilde{\pi}$ is admissible;
(c) the contragredient of $\widetilde{\pi}$ is isomorphic to π.

Proof. Clear. □

One has, of course, the canonical embedding of π into $\widetilde{\widetilde{\pi}}$.

Proposition 2.1.11. The functor $\pi \rightsquigarrow \widetilde{\pi}$ is contravariant and exact.

If (π, V) is smooth and U is any subset of V, define

$$U^\perp = \left\{ \widetilde{v} \in \widetilde{V} \;\middle|\; \widetilde{v}(x) = 0 \text{ for all } x \in U \right\}.$$

Note that we had previously defined $U^\perp$ to be a subspace of V in the case where π is unitary. It turns out that our new definition is compatible with the old one: if (π, V) is admissible and unitary, then the Hermitian form on V allows us to identify V with $\widetilde{V}$.

Proposition 2.1.12. Suppose (π, V) to be an admissible representation and U a G-stable subspace of V. Then $U^\perp \subseteq \widetilde{V}$ is isomorphic to the contragredient of V/U.

Corollary 2.1.13. The admissible representation π is irreducible if and only if $\widetilde{\pi}$ is.

The proofs are straightforward.

Proposition 2.1.14. Suppose that (π, V) is a unitary admissible representation of G. Then it is G-isomorphic to a direct sum of irreducible admissible unitary representations, each isomorphism class occurring with finite multiplicity. If G has a countable basis of neighborhoods at the identity, then the direct sum is countable.

Proof. Let K be a compact open subgroup, and first assume that V is generated by V^K. We prove by induction on the dimension of V^K that V is the direct sum of a finite number of irreducible admissible representations. Since V is finitely generated, an application of Zorn's Lemma guarantees that it has some irreducible quotient U, which is generated by U^K. The representation on V will then be (by Proposition 2.1.5) the direct sum of U_1 and $U_1^\perp$, where U_1 is the kernel of the natural map from V to U, and we may apply induction to U_1.

In the general case, the above implies that for each compact open subgroup K, the subspace of V generated by V^K is a finite direct sum of irreducible unitary admissible representations. Another application of Zorn's Lemma, letting K range over the set of compact open subgroups of G, will finish the proof of the first claim. The final remark is elementary. □

(The first application here of Zorn's Lemma is unnecessary; instead one may apply the considerations of §2.2.)

Proposition 2.1.15. If (π, V) is irreducible, unitary, and admissible, then (up to scalar multiplication) there is only one G-invariant Hermitian inner product on V.

2.2. For each compact open subgroup K define the *Hecke algebra* $\mathcal{H}_F(G,K)$ to be the space of all functions $f\colon G \longrightarrow F$ of compact support such that $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$, $g \in G$. Assign to it the convolution product

$$(f_1 * f_2)(g) = \int_G f_1(g g_0^{-1}) f_2(g_0)\, dg_0.$$

This algebra has as identity the element $(\operatorname{meas} K)^{-1} \operatorname{ch}_K$ (where ch_K is the characteristic function of K). Define $\mathcal{H}_F(G)$ to be $\bigcup_K \mathcal{H}_F(G,K)$. The convolution defines a product on this, but there is no identity unless G is discrete. The space $\mathcal{H}_F(G)$ is the same as $C_c^\infty(G,F)$.

For each character ω of Z, define $\mathcal{H}_{F,\omega}(G,K)$ to be the space of functions $f\colon G \longrightarrow F$ such that f has compact support modulo Z, f is bi-K-invariant, and $L_z f = \omega(z) f$ for all $z \in Z$. Convolution is defined by the formula

$$(f_1 * f_2)(g) = \int_{G/Z} f_1(g g_0^{-1}) f_2(g_0)\, dg_0.$$

(Note that this is well-defined.) Define $\mathcal{H}_{F,\omega}(G)$ to be $\bigcup_K \mathcal{H}_{F,\omega}(G,K)$.

If (π, V) is any smooth representation of G, then the space V becomes an $\mathcal{H}_F(G)$-module by the formula

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg,$$

which makes sense because the integral is essentially a finite sum. The algebra $\mathcal{H}_{F,\omega}(G)$ acts similarly on smooth ω-representations[3].

We shall often drop the reference to F if confusion is unlikely.

Whenever A and B are smooth G-representations, $\operatorname{Hom}_G(A,B)$ will denote the set of linear maps from A to B which commute with the action of G.

Proposition 2.2.1. If (π_i, V_i) $(i = 1, 2)$ are two smooth representations of G, then the natural map induces an isomorphism of $\operatorname{Hom}_G(V_1, V_2)$ with $\operatorname{Hom}_{\mathcal{H}(G)}(V_1, V_2)$. A similar statement is true for ω-representations and $\mathcal{H}_\omega$.

[3]From now on we will adopt the convention that if S is subset of G which is bi-invariant under some compact open subgroup K of G, then $\pi(S)$ denotes the action of ch_S as an element of $\mathcal{H}_F(G)$.

Proof. A G-morphism $f\colon V_1 \longrightarrow V_2$ is clearly an $\mathcal{H}(G)$-morphism as well. Conversely, suppose $f\colon V_1 \longrightarrow V_2$ is an $\mathcal{H}(G)$-morphism. Suppose $v \in V_1$ and $g \in G$. Choose a compact open K such that $v \in V_1^K$, $\pi_1(g)v \in V_1^K$, $f(v) \in V_2^K$, and $\pi_2(g)(f(v)) \in V_2^K$. Then $\pi_1(g)v = \text{meas}(KgK)^{-1}\pi_1(KgK)v$, and

$$\begin{aligned} f(\pi_1(g)v) &= \frac{f(\pi_1(KgK)v)}{\text{meas}(KgK)} \\ &= \frac{\pi_2(KgK)f(v)}{\text{meas}(KgK)} \\ &= \pi_2(g)f(v). \quad \square \end{aligned}$$

If (π, V) is a smooth representation of G, and K is a compact open subgroup, then the space V^K is stable under $\mathcal{H}(G, K)$.

Proposition 2.2.2. Suppose (π_i, V_i) $(i = 1, 2)$ are smooth representations of G, and K is a compact open subgroup of G. If

(i) V_1 is generated as a G-space by V_1^K and
(ii) every nonzero G-stable subspace of V_2 contains a non-zero vector fixed by K,

then

$$\text{Hom}_G(V_1, V_2) \cong \text{Hom}_{\mathcal{H}(G,K)}(V_1^K, V_2^K).$$

Proof. The map from left to right is the obvious one. The rest of the proof is word-for-word the same as in [15, pp. 33–34][4]. (I should mention that the argument there was inspired by the proof of [23, Lemma 7.1].) $\square$

A result similar to that in 2.2.2 holds for ω-representations and $\mathcal{H}_\omega(G, K)$.

The following may make condition (ii) in 2.2.2 more reasonable:

Lemma 2.2.3. Suppose that (π, V) is an admissible representation of G. Then V is generated by V^K as a G-space if and only if $\widetilde{V}$ satisfies the condition that every non-zero G-stable subspace of $\widetilde{V}$ contains a non-zero vector fixed by K.

Proof. Suppose that V is generated as a G-space by V^K, and let U be a G-stable subspace of $\widetilde{V}$ such that $U^K = 0$. If $U^\perp$ is the annihilator of U in $V \cong \widetilde{\widetilde{V}}$, then $(V/U^\perp)^K \cong \widetilde{U}^K = 0$. Thus by 2.1.7, $V^K = (U^\perp)^K$, and since V^K generates V, $V = U^\perp$, and $U = 0$. The converse argument is similar. $\square$

Proposition 2.2.4. Let (π, V) be a smooth representation of G, and K a compact open subgroup. Then

(a) If (π, V) is irreducible, then V^K is an irreducible module over $\mathcal{H}(G, K)$;
(b) If V satisfies the conditions (i) and (ii) of Proposition 2.2.2 and V^K is an irreducible $\mathcal{H}(G, K)$-module, then the representation (π, V) is irreducible.

[4]Should we include the proof?

Proof. Say (π, V) is irreducible. Let U be any non-zero $\mathcal{H}(G,K)$-stable subspace of V^K. Then U must generate V as a G-space, and every $v \in V$ is of the form $\sum \pi(g_i)u_i$, with $g_i \in G$, $u_i \in U$. Now if $u \in V^K$ and $g \in G$, then $\mathcal{P}_K(\pi(g)u)$ differs from $\pi(KgK)u$ only by a constant. Thus, with v as above,

$$\mathcal{P}_K(v) = \sum \mathcal{P}_K(\pi(g_i)u_i) = \sum(\text{constant}) \cdot \pi(Kg_iK)u_i,$$

which lies in U since U is $\mathcal{H}(G,K)$-stable. Thus $V^K \subseteq U$ and in fact $V^K = U$. This proves (a).

Conversely, assume (i) and (ii) to hold and V^K to be an irreducible $\mathcal{H}(G,K)$-space. If U is any non-zero G-stable subspace of V, then by (ii) $U^K \neq 0$. By hypothesis, $U^K = V^K$. But then by (i), $U = V$. This proves (b). □

Remark 2.2.5. The same reasoning shows that a smooth representation (π, V) is irreducible if and only if there exists a set of compact open subgroups $\{K_\alpha\}$ forming a basis of neighborhoods of the identity and such that each V^{K_α} is an irreducible $\mathcal{H}(G, K_\alpha)$-module.

Proposition 2.2.6. If F is algebraically closed and (π, V) is a smooth irreducible representation of G, then it is absolutely irreducible.

Proof. Let K be a compact open subgroup such that $V^K \neq 0$. Then V^K is an irreducible $\mathcal{H}(G,K)$-module by 2.2.4(a). If E/F is any field extension, then conditions (i) and (ii) of Proposition 2.2.2 hold for $V \otimes E$. Thus, by 2.2.4(b), in order to know $V \otimes E$ is irreducible, it suffices to show that $(V \otimes E)^K$ is irreducible over $\mathcal{H}_E(G,K)$. By [7, §1.2, Proposition 3, p. 9], the commutant of $\mathcal{H}_E(G,K)$ in $V^K \otimes E^K$ is E, since that of $\mathcal{H}_F(G,K)$ in V^K is F. By the proof of 2.1.1, $V^K \otimes E^K = (V \otimes E)^K$. Apply [7, §7.3, Theorem 2, p. 87] to finish the proof. □

This is due to A. Robert in [29].

The point of 2.2.6 is that from §3 on, where F will be $\mathbf{C}$, one doesn't have to worry about the distinction between irreducibility and absolute irreducibility.

2.3. If (π, V) is an admissible representation of G, then for every $f \in \mathcal{H}(G)$ the operator $\pi(f)$ has finite rank, and one may therefore speak of its trace. The functional on $\mathcal{H}(G)$ which takes f to the trace of $\pi(f)$ is called the *distribution character* of π, and referred to as ch_π. Of course, since the definition of $\pi(f)$ depends on the choice of a Haar measure for G, so does the definition of the distribution character.

Proposition 2.3.1. If $\{\pi_1, \pi_2, \ldots, \pi_n\}$ is a set of inequivalent irreducible admissible representations of G, then the functionals $\{\text{ch}_{\pi_1}, \ldots, \text{ch}_{\pi_n}\}$ are linearly independent.

Proof. This is [23, Lemma 7.1]. (Note that part of this proof already occurs in that of 2.2.2.) □

Corollary 2.3.2. If (π_1, V_1) and (π_2, V_2) are two irreducible admissible representations with the same distribution character, then they are isomorphic.

Proof. Clear. $\square$

Corollary 2.3.3. Let (π_1, V_1) and (π_2, V_2) be two admissible representations of finite length as G-spaces. Then they have the same irreducible composition factors (with multiplicities) if and only if their distribution characters are the same.

Proof. One way is clear, since it is straightforward to show that if

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0$$

is an exact sequence of admissible G-representations, then the distribution character of V is the sum of those of V_1 and V_2.

If (π, V) is now any admissible representation of finite length, define $(\pi^{\mathrm{ss}}, V^{\mathrm{ss}})$ to be the representation of G on the direct sum of the irreducible composition factors of V. (It is a semisimple G-space.) Then π and π^{ss} have the same characters, by what I have just remarked, and the conclusion of the corollary is merely that $\pi_1^{\mathrm{ss}} \cong \pi_2^{\mathrm{ss}}$. Thus, it suffices to assume that π_1 and π_2 are semisimple. But in this cases one may apply 2.3.2 and an inductive argument. (This is of course a rather well-known argument.) $\square$

2.4. If H is a closed subgroup of G and (σ, U) a smooth representation of H, define $\mathrm{Ind}_H^G \sigma$ to be the space of all functions $f\colon G \longrightarrow U$ such that

(i) $f(hg) = \sigma(h) f(g)$ for all $h \in H$, $g \in G$, and
(ii) for some compact open subgroup K of G, $R_k f = f$ for all $k \in K$.

Define c-$\mathrm{Ind}_H^G \sigma$ to be the subspace of $\mathrm{Ind}_H^G \sigma$ of functions with compact support modulo H. The group G acts on both of these by R (the right regular representation).

Theorem 2.4.1. Let H be a closed subgroup of G, (σ, U) a smooth representation of H. Then

(a) $\mathrm{Ind}_H^G \sigma$ and c-$\mathrm{Ind}_H^G \sigma$ are smooth representations of G;
(b) the maps $\Lambda\colon \mathrm{Ind}\,\sigma \longrightarrow U$ and $\Lambda_{\mathrm{c}}\colon$ c-$\mathrm{Ind}\,\sigma \longrightarrow U$, defined by $f \longmapsto f(1_G)$, are surjective H-morphisms;
(c) the restriction of Λ (or Λ_{c}) to any non-trivial G-subspace of $\mathrm{Ind}\,\sigma$ (or c-$\mathrm{Ind}\,\sigma$) is non-trivial;
(d) if $H\backslash G$ is compact and (σ, U) is admissible, then $\mathrm{Ind}\,\sigma =$ c-$\mathrm{Ind}\,\sigma$ is admissible;
(e) (Frobenius reciprocity) if (π, V) is any smooth G-representation, then composition with Λ induces an isomorphism of $\mathrm{Hom}_G(V, \mathrm{Ind}\,\sigma)$ with $\mathrm{Hom}_H(V, U)$.

Proof. (a) is immediate from the definitions.

For (b): One has $\Lambda(R_h f) = f(h) = \sigma(h)\Lambda(f)$. To see that Λ_c (hence Λ) is surjective, choose $u \in U$ and let K be a compact open subgroup of G such that $u \in U^{K\cap H}$. Define f on G by

$$f(g) = \begin{cases} \sigma(h)u & \text{if } g = hk,\ h \in H,\ k \in K; \\ 0 & \text{if } g \notin HK. \end{cases}$$

This lies in c-Ind σ and has u as its image under Λ_c.

For (c): If V is any nontrivial G-stable subspace of $\operatorname{Ind}\sigma$, choose $f \neq 0$ in V. Then $f(g) \neq 0$ for some $g \in G$, hence $\Lambda(R_g f) \neq 0$, but one also has $R_g f \in V$.

For (d): If K is any compact open subgroup of G, X a finite subset of G, and U_0 a finite-dimensional subspace of U, define

$$I(K, X, U_0) = \left\{ f \in (\operatorname{Ind}\sigma)^K \;\middle|\; f(X) \subseteq U_0; f \text{ has support in } HXK \right\}.$$

This space clearly has finite dimension. Assume (σ, U) to be admissible. Let K be a compact open subgroup of G, choose X such that $HXK = G$, let $L = \bigcap_{x\in X} xKx^{-1}$, and let $U_0 = U^{L\cap H}$. Then $(\operatorname{Ind}\sigma)^K \subseteq I(K, X, U_0)$, and hence $\operatorname{Ind}\sigma$ is admissible.

For (e): Composition with Λ gives a map from $\operatorname{Hom}_G(V, \operatorname{Ind}\sigma)$ to $\operatorname{Hom}_H(V, U)$. To define an inverse map, let $f\colon V \longrightarrow U$ be an H-morphism. Define the G-morphism Φ from V to $\operatorname{Ind}\sigma$ to be that which takes $v \in V$ to Φ_v, where $\Phi_v(g) = f(\pi(g)v)$. ($\Phi_v \in \operatorname{Ind}\sigma$ since π is smooth.) It is clear that $f \longmapsto \Phi$ works, since $\Lambda(\Phi_v) = f(v)$. □

Let δ_H be the modulus character $H \longrightarrow \mathbf{Q}^\times$. Assume that G is unimodular.

Theorem 2.4.2. Let H be a closed subgroup of G, (σ, U) a smooth representation of H. If $\pi = \text{c-Ind}_H^G \sigma$, then $\widetilde{\pi} \cong \operatorname{Ind}_H^G \widetilde{\sigma}\delta_H$.

Proof. For any $f \in C_c^\infty(G)$, define the operator $\mathcal{P}_\delta$ by the formula

$$(\mathcal{P}_\delta f)(g) = \int_H \delta_H(h)^{-1} f(hg)\, d_r h,$$

where $d_r h$ is a right Haar measure on H. The map $\mathcal{P}_\delta$ is clearly a surjection from $C_c^\infty(G)$ to $\text{c-Ind}_H^G \delta_H$. A slight modification of well-known results in [8] shows that there exists on $\text{c-Ind}_H^G \delta_H$ a unique G-invariant functional I_δ such that for all $f \in C_c^\infty(G)$,

$$\int_G f(g)\, dg = I_\delta(\mathcal{P}_\delta f).$$

Since $\widetilde{\sigma} \otimes \delta_H \cong \operatorname{Hom}_H(\sigma, \delta_H)$ as an H-space, there is a pairing of σ with $\widetilde{\sigma} \otimes \delta_H$ giving rise to an H-morphism

$$\sigma \otimes (\widetilde{\sigma} \otimes \delta_H) \longrightarrow \delta_H.$$

Let $\langle \omega, \widetilde{\omega} \rangle_*$ denote the image of $(\omega, \widetilde{\omega})$ under this pairing. If $\phi \in V = \text{c-Ind}\,\sigma$ and $\Phi \in \operatorname{Ind}(\widetilde{\sigma} \otimes \delta_H)$, then define $\langle \phi, \Phi \rangle_* \in \operatorname{Ind}\delta_H$ by $\langle \phi, \Phi \rangle_*(g) = \langle \phi(g), \Phi(g) \rangle_*$.

The map $(\phi, \Phi) \longmapsto \langle \phi, \Phi \rangle = I_\delta(\langle \phi, \Phi \rangle_*)$ is a G-invariant pairing, defining also a G-morphism from $\mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_H)$ to $\widetilde{V}$. To finish the proof, one may show quite easily that this induces an isomorphism of $(\mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_H))^K$ with $\widetilde{V}^K$ for each compact open K. □

Corollary 2.4.3. $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_H^G \sigma, F) \cong \mathrm{Hom}_H(\sigma, \delta_H)$.

Proof. This follows from the more general fact that if (σ, U) is any smooth H-representation, then

$$(\mathrm{Ind}_H^G \sigma)^G \cong U^H. \quad \square$$

The proofs of the following are trivial:

Proposition 2.4.4. Let H be a closed subgroup of G, (σ_i, U_i) $(i = 1, 2)$ smooth representations of H. To each H-morphism $f\colon U_1 \longrightarrow U_2$ is associated a canonical G-morphism $\mathrm{Ind}(f)\colon \mathrm{Ind}\,\sigma_1 \longrightarrow \mathrm{Ind}\,\sigma_2$. The map $\mathrm{Ind}(f)$ is a surjection, or an injection, if and only if f is. The functor $\sigma \rightsquigarrow \mathrm{Ind}\,\sigma$ is exact.

Proposition 2.4.5. Let $H_2 \subseteq H_1$ be closed subgroups of G, (σ, U) a smooth representation of H_2. Then $\mathrm{Ind}_{H_2}^G \sigma \cong \mathrm{Ind}_{H_1}^G(\mathrm{Ind}_{H_2}^{H_1} \sigma)$.

2.5. We have a natural pairing

$$\langle\ ,\ \rangle\colon V \otimes \widetilde{V} \longrightarrow F$$

given by $\langle v, \widetilde{v} \rangle = \widetilde{v}(v)$.

Let (π, V) be a smooth representation of G, $v \in V$, $\widetilde{v} \in \widetilde{V}$. The *matrix coefficient* of π associated to v and $\widetilde{v}$ is the function $c_{v,\widetilde{v}}(g) = \langle \pi(g)v, \widetilde{v} \rangle$.

Lemma 2.5.1. One has, for every $g \in G$:

$$c_{\pi(g)v,\widetilde{v}} = R_g c_{v,\widetilde{v}}$$
$$c_{v,\widetilde{\pi}(g)\widetilde{v}} = L_g c_{v,\widetilde{v}}$$

This is trivial.

Corollary 2.5.2. The function $c_{v,\widetilde{v}}$ is uniformly smooth. For a fixed $\widetilde{v} \in \widetilde{V}$, the map $v \longmapsto c_{v,\widetilde{v}}$ is a G-morphism from (π, V) to $(R, C_u^\infty(G))$, and for a fixed $v \in V$ the map $\widetilde{v} \longmapsto c_{v,\widetilde{v}}$ is a G-morphism from $(\widetilde{\pi}, \widetilde{V})$ to $(L, C_u^\infty(G))$.

Now assume F to be $\mathbf{R}$ or $\mathbf{C}$. Let $\omega\colon Z \longrightarrow F^\times$ be a character. If (π, V) is an admissible ω-representation of G, it is said to be *square-integrable modulo* Z (but later we will often just say "square-integrable") if $|\omega(z)| = 1$ for every $z \in Z$ and if for every $v \in V$ and $\widetilde{v} \in \widetilde{V}$, the function $|c_{v,\widetilde{v}}(g)|$ is square-integrable on G/Z.

Proposition 2.5.3. If (π, V) is an irreducible admissible ω-representation with $|\omega| = 1$, then in order for it to be square-integrable modulo Z, it is necessary and sufficient that for one nonzero $v_0 \in V$ and one nonzero $\widetilde{v}_0 \in \widetilde{V}$ the matrix coefficient $c_{v_0,\widetilde{v}_0}$ is square-integrable on G/Z.

Proof. Let V_0 be the space of all v such that $c_{v,\widetilde{v}_0}$ is square-integrable on G/Z. Then $V_0 \neq 0$ since $v_0 \in V_0$, and it is clearly G-stable, hence all of V. Treat $\widetilde{V}$ similarly. □

This is a well-known result.

Proposition 2.5.4. If (π, V) is an irreducible admissible square-integrable representation of G, then it is unitary.

Proof. Choose $\widetilde{v}_0 \neq 0$ in $\widetilde{V}$. Define an inner product by the formula

$$(u, v) = \int_{G/Z} \langle \pi(g)u, \widetilde{v}_0 \rangle \overline{\langle \pi(g)v, \widetilde{v}_0 \rangle} \, dg.$$

The integral converges by the Schwarz inequality, and clearly defines a G-invariant positive definite inner product. □

This is also well known.

2.6. If G_1 and G_2 are two topological groups, their direct product is locally profinite if and only if each G_i is. Assume this to be the case, and let $G = G_1 \times G_2$.

Lemma 2.6.1. If $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ are compact open subgroups and $K = K_1 \times K_2$, then $\mathcal{H}(G, K)$ is naturally isomorphic to $\mathcal{H}(G_1, K_1) \otimes \mathcal{H}(G_2, K_2)$.

This is straightforward.

Lemma 2.6.2. Let (π_1, V_1) and (π_2, V_2) be smooth representations of G_1 and G_2, respectively, and let $(\pi, V) = (\pi_1 \otimes \pi_2, V_1 \otimes V_2)$. Let $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ be compact open subgroups and $K = K_1 \times K_2$. Then the natural injection of $V_1^{K_1} \otimes V_2^{K_2}$ into V^K is an isomorphism.

Proof. To see this, use the fact that $V^K = \mathcal{P}_K(V)$. □

Proposition 2.6.3. If (π_1, V_1) and (π_2, V_2) are irreducible (resp. absolutely[5] irreducible) smooth representations of G_1 and G_2, respectively, then $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ is an irreducible (resp. absolutely irreducible) smooth representation of G.

[5] I still think this isn't the right way to say it. What Casselman proves is exactly the proposition with the word "absolute" deleted. His proof has nothing to do with absolute irreducibility. Maybe we should have a remark that this result implies a similar one about absolute irreducibility. This will also solve the problem of how to handle the numerous instances of "absolutely" in this and the next proof.

Proof. The smoothness is immediate.

If $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ are compact open subgroups and $K = K_1 \times K_2$, then by 2.6.2, $(V_1 \otimes V_2)^K \cong V_1^{K_1} \otimes V_2^{K_2}$ as a module over $\mathcal{H}(G, K) \cong \mathcal{H}(G_1, K_1) \otimes \mathcal{H}(G_2, K_2)$. By [7, §7.3, Theorem 2, p. 87], this is an irreducible module. Since the subgroups $K_1 \times K_2$ form a basis for the neighborhoods of 1 in G, one may apply Remark 2.2.5. □

Proposition 2.6.4. If (π, V) is an irreducible (resp. absolutely[6] irreducible) admissible representation of G, then there exist irreducible (resp. absolutely irreducible) admissible representations (π_1, V_1) and (π_2, V_2) of G_1 and G_2 (respectively) such that $\pi \cong \pi_1 \times \pi_2$.

Proof. Let $K = K_1 \times K_2$ be a compact open subgroup such that $V^K \neq 0$. By 2.2.4, this is an irreducible module over $\mathcal{H}(G, K)$. It is finite-dimensional, so that by [7, §7.3, Theorem 2, p. 87 and §7.7, Proposition 8, p. 93][7], there exists an irreducible module U_1 over $\mathcal{H}(G_1, K_1)$ such that $V^K \cong U_1 \otimes \mathrm{Hom}_{\mathcal{H}(G_1,K_1)}(U_1, V^K)$ as a module over $\mathcal{H}(G, K)$. Let $U_2 = \mathrm{Hom}_{\mathcal{H}(G_1,K_1)}(U_1, V^K)$.

Define (π_2, V_2) to be the obvious representation of G_2 on $\mathrm{Hom}_{\mathcal{H}(G_1,K_1)}(U_1, V^{K_1})$, and define (π_1, V_1) to be that of G_1 on $\mathrm{Hom}_{G_2}(V_2, V)$. Neither of these spaces is trivial. There is a canonical non-trivial map from $V_1 \otimes V_2$ to V.

It is immediate that since V is smooth, so is V_2. Furthermore, for any compact open subgroup $L_2 \subseteq G_2$, $V_2^{L_2} \cong \mathrm{Hom}_{\mathcal{H}(G_1,K_1)}(U_1, V^{K_1 \times L_2})$, which by the results from [7] already mentioned is an absolutely irreducible module over $\mathcal{H}(G_2, L_2)$, since $V^{K_1 \times L_2}$ is absolutely irreducible over $\mathcal{H}(G, K_1 \times L_2)$. By Remark 2.2.5, then, V_2 is absolutely irreducible.

Let $f\colon V_2 \longrightarrow V$ be any element of V_1, and choose $v_2 \in V_2$ nonzero. Then $f(v_2)$ lies in some V^{L_1} since V is smooth. But since V_2 is irreducible, v_2 generates V_2 as a G_2-module and $f(V_2) \subseteq V^{L_1}$ as well. Therefore V_1 is smooth, since $V_1^{L_1} \cong \mathrm{Hom}_{G_2}(V_2, V^{L_1})$.

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[6] See previous footnote.

[7] It's not obvious to me that the references say what Casselman says they say, although they are certainly close. Here are the relevant results:

Theorem 2. Let A and B be K-algebras. Let M and N be nonzero modules over A and B, respectively. Then

(a) $M \otimes N$ is simple (resp. semisimple) $\Rightarrow$ M and N are simple (resp. semisimple).
(b) Let M and N be simple, E and F the "field commutants" of M and N (fields aren't necessarily commutative), S and T the centers of E and F. Then $M \otimes N$ is simple $\Leftrightarrow$ $E \otimes F$ is a field.

Proposition 8. Let A and B be K-algebras, P a simple $A \otimes B$-module of finite dimension over K. Then there exist simple modules M and N (over A and B, respectively) such that P is isomorphic to a quotient of $M \otimes N$. M and N are uniquely determined up to isomorphism.

Casselman doesn't just use the existence of M and N; he assumes that N has a certain form.

In particular, $\mathrm{Hom}_{G_2}(V_2, V^{L_1}) \neq 0$ for suitably small L_1. Consider the restriction map from this space to $\mathrm{Hom}_{\mathcal{H}(G_2,K_2)}(U_2, V^{L_1\times K_2})$. It is an $\mathcal{H}(G_1, L_1)$-morphism. It is an injection, for if $f\colon V_2 \longrightarrow V^{L_1}$ were 0 on U_2 then $f = 0$, since V_2 is irreducible. But by [7] again, this latter $\mathcal{H}(G_1, L_1)$-module is absolutely irreducible. Since this is true for all suitably small L_1, Remark 2.2.5 implies that V_1 is absolutely irreducible.

Therefore $V_1 \otimes V_2$ is irreducible over G and since the canonical map from $V_1 \otimes V_2$ to V is non-trivial, it is an isomorphism. □

2.7. Let X be any complex analytic space.

A *holomorphic sheaf* of admissible representations of G over X, or an *analytic family* of such parametrized by X, consists of a pair $(\pi, \mathcal{V})$ where $\mathcal{V}$ is an analytic sheaf over X and π is a representation of G in the ring of analytic endomorphisms of $\mathcal{V}$ such that

(a) the sheaf $\mathcal{V}$ is the direct limit of the subsheaves $\mathcal{V}^K$, as K ranges over the compact open subgroups of G and
(b) each $\mathcal{V}^K$ is coherent.

In this situation, each stalk $\mathcal{V}_x$ is the direct limit of the stalks $\mathcal{V}_x^K$ and likewise the fibre $V_x = \mathcal{V}_x/\mathfrak{m}_x\mathcal{V}_x$ is the direct limit of the fibres $V_x^K = \mathcal{V}_x^K/\mathfrak{m}_x\mathcal{V}_x^K$. Each fibre is also, in the obvious way, the space of an admissible representation of G.

Each operator $\pi(g)$ defines an analytic morphism from $\mathcal{V}^K$ to $\mathcal{V}^{gKg^{-1}}$. For compact K, the projection operator $\mathcal{P}_K$ is an analytic morphism from $\mathcal{V}$ to $\mathcal{V}^K$. For each $f \in \mathcal{H}(G, K)$ the endomorphism $\pi(f)\colon \mathcal{V}^K \longrightarrow \mathcal{V}^K$ is an analytic morphism.

3. Representations induced from parabolic subgroups

From now on, k will be a fixed locally compact non-archimedean field, G the group of k-rational points of a connected reductive group $\mathbf{G}$ defined over k. Also, all of our vector spaces will be complex.

I remark that by density results of [3], the center of G consists of the k-rational points of the center of $\mathbf{G}$.

3.1. Let P be a parabolic subgroup of G, with Levi decomposition $P = MN$. If (σ, U) is a smooth representation of M, it defines as well a smooth representation of P, since $P/N \cong M$. In this situation, I shall define an induced representation which differs slightly from the one I defined in §2. This is because I shall not be concerned with the rationality of representations, and shall be concerned with a certain symmetry which is rather awkward to express in the old notation.

Therefore, let δ_P be the modulus character $P \longrightarrow M \longrightarrow \mathbf{C}^\times$ of P (so that $\delta_P(mn) = |\det \mathrm{Ad}_{\mathfrak{n}}(m)|$, where $\mathfrak{n}$ is the Lie algebra of N). *Define* $i_P^G\,\sigma$ *now to be what I defined as* $\mathrm{Ind}_P^G(\sigma\delta_P^{1/2})$ *in §2.4,* so that for every $f \in i_P^G\,\sigma$ one has $L_p f = \sigma^{-1}\delta_P^{-1/2}(p) f$ for every $p \in P$. This is known as *normalized induction.*

If (σ, U) is admissible, then so is $i_P^G\,\sigma$, by 2.4.1, since $P\backslash G$ is compact.

If P is a minimal parabolic subgroup and σ is irreducible (hence necessarily finite-dimensional) then $i_P^G\,\sigma$ is said to be a representation in the *principal series* of G.

Suppose K to be a good compact subgroup of G, so that one has an Iwasawa decomposition $G = PK$ (see [10, §4.4]). Let $K_P = K \cap P$, and let (s, U) be the restriction of (σ, U) to K_P.

Proposition 3.1.1. The restriction of $i_P^G\,\sigma$ to K is isomorphic to $i_{K_P}^K\,s$.

Proof. The K-isomorphism is given by the restriction of an element of $i_P^G\,\sigma$ to K. This is surjective for obvious reasons, and injective because of the Iwasawa decomposition. □

Proposition 3.1.2. The contragredient of $i_P^G\,\sigma$ is isomorphic to $i_P^G\,\widetilde{\sigma}$.

Proof. This follows from 2.4.2 and the fact that $P\backslash G$ is compact. □

Let K be a good compact subgroup of G, $\langle u_1, u_2\rangle_*$ the pairing of U with $\widetilde{U}$ given by

$$(\sigma \otimes \delta_P^{1/2}) \otimes (\widetilde{\sigma} \otimes \delta_P^{1/2}) \longrightarrow \delta_P.$$

Proposition 3.1.3. For $f_1 \in i_P^G\,\sigma$, $f_2 \in i_P^G\,\widetilde{\sigma}$, one has

$$\langle f_1, f_2\rangle = \int_K \langle f_1(k), f_2(k)\rangle_*\,dk.$$

Proof. Refer back to the proof of Theorem 2.4.2, where the fact that $i_P^G\widetilde{\sigma}$ is the contragredient of $i_P^G\sigma$ was proven. From that proof, it clearly suffices to show that for $f \in i_P^G\delta_P^{1/2}$, $I_\delta(f) = \int_K f(k)\,dk$. But one may show easily that if one defines $I_\delta^*(f)$ to be $\int_K f(k)\,dk$ for any $f \in i_P^G\delta_p^{1/2}$, then for any $f \in C_c^\infty(G)$ one has

$$\int_G f(g)\,dg = I_\delta^*(P_\delta f),$$

and this shows that $I_\delta^* = I_\delta$. □

Proposition 3.1.4. The representation $i_P^G\sigma$ is unitary if σ is.

Proof. Since σ is unitary, σ is isomorphic to the contragredient of its conjugate, and 3.1.2 implies the same for $i_P^G\sigma$. The fact that I_δ is a positive functional implies that this isomorphism induces a positive definite Hermitian inner product on $i_P^G\sigma$. □

This explains the new normalization of induction.

3.2. I shall give now a new formulation of Frobenius reciprocity for P and G. First of all, of course, one must take into account the new normalization of $i_P^G\sigma$. But there is also a second and less obvious point to introduce and I digress slightly to make it.

Assume N to be for the moment any locally compact group such that the compact open subgroups form a basis of the neighborhoods of the identity, and possessing arbitrarily large compact open subgroups as well. This means that if X is any compact subset of N then there exists a compact open subgroup N_0 containing X. This condition is satisfied, for example, if N is the set of k-rational points of a unipotent group defined over k.

Let (π, V) be a smooth representation of N. For a compact subgroup $N_0 \subseteq N$, define $V(N_0)$ to be $\left\{v \in V \,\middle|\, \int_{N_0} \pi(n)v\,dn = 0\right\}$. Define $V(N)$ to be $\bigcup V(N_0)$, the union over all compact open subgroups N_0 of N. By the assumption on N, this is a subspace of V.

Proposition 3.2.1. The space $V(N)$ may also be characterized as the subspace of V spanned by the elements $\left\{\pi(n)v - v \,\middle|\, n \in N,\ v \in V\right\}$.

Proof. The subspace is contained in $V(N)$, for if $n \in N$ and $v \in V$, then there exists a compact open subgroup N_0 with $n \in N_0$, and $\pi(n)v - v \in V(N_0)$.

For the opposite inclusion, suppose $v \in V(N)$, and choose two compact open subgroups $N_0 \subseteq N_1$ such that $v \in V^{N_0} \cap V(N_1)$. Then

$$0 = \int_{N_1} \pi(n)v\,dn = (\text{constant}) \cdot \sum_{N_1/N_0} \pi(n)v$$

so that

$$v = (\text{constant}) \cdot \sum_{N_1/N_0} (\pi(n)v - v). \quad \square$$

Define V_N to be $V/V(N)$.

Corollary 3.2.2. If U is any space on which N acts trivially, then the canonical projection $V \longrightarrow V_N$ induces an isomorphism of $\mathrm{Hom}_N(V, U)$ with $\mathrm{Hom}_{\mathbf{C}}(V_N, U)$.

Proof. Clear. □

If P is some group in which N is normal, and (π, V) is a smooth representation of P, the space V_N becomes naturally the space of a representation (π_N, V_N) of P/N. One might call this the *Jacquet module* of (π, V) associated to P.

Proposition 3.2.3. If $U \longrightarrow V \longrightarrow W$ is an exact sequence of smooth N-spaces, then $U_N \longrightarrow V_N \longrightarrow W_N$ is also exact.

Proof. One may assume

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

exact. It is then elementary, applying 3.2.1, that

$$U_N \longrightarrow V_N \longrightarrow W_N \longrightarrow 0$$

is exact, and it suffices to show that $U_N \longrightarrow V_N$ is injective. This follows from the fact that, by its definition, $U(N) = U \cap V(N)$. □

This result really amounts to the claim that $H_1(N, U) = 0$, where U is a smooth N-space, since after all $U_N = H_0(N, U)$.

Let now P be a parabolic subgroup of G with unipotent radical N, and (σ, U) a smooth representation of $M \cong P/N$.

Theorem 3.2.4 (Frobenius reciprocity). If (π, V) is a smooth representation of G, then the P-morphism $\Lambda\colon (R, i_P^G\,\sigma) \longrightarrow (\sigma\delta_P^{1/2}, U)$ defined by $f \longmapsto f(1)$ induces an isomorphism of $\mathrm{Hom}_G(V, i_P^G\,\sigma)$ with $\mathrm{Hom}_M(V_N, U)$, where U is given the M-structure $\sigma\delta_P^{1/2}$.

Proof. This is immediate from 2.4.1(e) and 3.2.3. □

Corollary 3.2.5. If there exists a non-zero G-morphism from (π, V) to $i_P^G\,\sigma$, then $V_N \neq 0$.

Proof. This follows from 3.2.4, or also from 2.4.1(c) and 3.2.3. □

3.3. The following result is one of the cornerstones of the subject. It was suggested by a result of Harish-Chandra ([20, Theorem 4]); the original result leading to it is Jacquet's ([22, Theorem 5.1]); and the brief proof is due to Borel.

Theorem 3.3.1. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$, and let (π, V) be a smooth representation of G. If (π, V) is a finitely generated (resp. admissible) representation of G, then (π_N, V_N) is a smooth and finitely generated (resp. admissible) representation of M.

Proof. In either case, the smoothness of (π_N, V_N) is clear. Let X be a finite subset of V generating it as a G-space. Let K be a compact open subgroup such that $X \subseteq V^K$. Let Γ be a finite subset of G such that $P\Gamma K = G$. Then it is easy to see that since V is the linear span of $\pi(G)X$, V_N is generated as an M-space by the image of $\pi(\Gamma)X$. Hence, (π_N, V_N) is finitely generated. This proves the first claim.

As an amplification, suppose that K is a good compact subgroup of G.

Proposition 3.3.2. If (π, V) is any smooth representation of G, P a parabolic subgroup of G, and U a K-stable subspace of V generating it as a G-space, then the image of U generates V_N as an M-space.

Proof. Clear, from the fact that $G = PK$. □

Let us now prove the admissibility claim of 3.3.1. What I shall actually show is something more precise. The claim is implied by 1.4.4 and this:

Theorem 3.3.3. Let (π, V) be an admissible representation of G, K_0 a compact open subgroup of G with an Iwahori factorization $K_0 = N_0^- M_0 N_0$ with respect to P. Then the canonical projection from V^{K_0} to $V_N^{M_0}$ is surjective.

I begin the proof of this with a useful technical result.

Theorem 3.3.4 (Jacquet's First Lemma). With hypotheses as in 3.3.3, suppose that $v \in V^{M_0 N_0^-}$. Then $v_0 = \mathcal{P}_{K_0}(v)$ is also $\mathcal{P}_{N_0}(v)$, and $v - v_0 \in V(N_0)$.

Proof. Since K_0 is compact and isomorphic to $N_0 \times M_0 \times N_0^-$, one has

$$\begin{aligned} v_0 &= (\text{constant}) \cdot \int_{N_0} dn (\int_{M_0 N_0^-} \pi(nm) v \, dm) \\ &= (\text{constant}) \cdot \int_{N_0} \pi(n) v \, dn \end{aligned}$$

since $\pi(m)v = v$ for $m \in M_0 N_0^-$. The last claim follows immediately from the fact that $v_0 = \mathcal{P}_{N_0}(v)$. □

Corollary 3.3.5. Hypotheses as in 3.3.3. Then V^{K_0} has the same image in V_N as $V^{M_0 N_0^-}$.

Proof. This follows since v and v_0 have the same image in V_N. □

To return to the proof of 3.3.3: Let $\bar{U}$ be any finite-dimensional subspace of $V_N^{M_0}$, and let $U \subseteq V^{M_0}$ be any finite-dimensional subspace of V mapping onto $\bar{U}$. One can find a compact open subgroup $N_1^- \subseteq N^-$ such that $U \subseteq V^{M_0 N_1^-}$. Choose $a \in A$ such that $a^{-1} N_0^- a \subseteq N_1^-$ (by 1.4.3). Then $\pi(a)U \subseteq V^{M_0 N_0^-}$, since if $u \in U$ and $n \in N_0^-$, one has $\pi(n)\pi(a)u = \pi(a)\pi(a^{-1}na)u = \pi(a)u$. Hence the image of $\pi(a)U$, which is $\pi_N(a)\bar{U}$, is contained in the image of V^{K_0} by 3.3.5. This latter image, hence $\bar{U}$, has dimension bounded by that of V^{K_0}. Since (π, V) is admissible, this dimension is finite. Therefore, so is the dimension of $V_N^{M_0}$. Taking $\bar{U}$ to be $V_N^{M_0}$, we see that $\pi_N(a)V_N^{M_0}$ is in the image of V^{K_0}. But $\pi_N(a)V_N^{M_0} = V_N^{M_0}$. This proves 3.3.3 and also 3.3.1. □

It is often useful to know:

Proposition 3.3.6. Let (σ, U) be an irreducible admissible representation of M, $K_0 = N_0^- M_0 N_0$ a compact open subgroup with an Iwahori factorization with respect to P such that $U^{M_0} \neq 0$. If $V \neq 0$ is a G-stable subspace of $i_P^G \sigma$, then $V^{K_0} \neq 0$.

Proof. Apply 2.4.1(c) to see that $\Lambda\colon V \longrightarrow U$ is non-zero, hence a surjection. The map $\Lambda_N\colon V_N \longrightarrow U$ is therefore also a surjection, and by 3.3.3 the composition $V^{K_0} \longrightarrow V_N^{M_0} \longrightarrow U^{M_0}$ is surjective. Hence, $V^{K_0} \neq 0$. □

Corollary 3.3.7. With the hypotheses as in 3.3.6, the space $\widetilde{I} = i_P^G \widetilde{\sigma}$ is generated by $\widetilde{I}^{K_0}$.

Proof. This follows from 2.2.3, 3.1.2, and 3.3.6. □

3.4. Let P be a parabolic subgroup of G. Recall from §1.6 that the set $X_{\mathrm{nr}}(M)$ of unramified characters of M possesses a natural complex analytic structure.

If (σ, U) is any admissible representaton of M, define a sheaf I_σ over $X_{\mathrm{nr}}(M)$ by the condition that for any open $X \subseteq X_{\mathrm{nr}}(M)$ the space $\Gamma(X, I_\sigma)$ is that of all $f\colon X \times G \longrightarrow U$ such that

(a) for any $\chi \in X$, $g \in G$, $p \in P$ one has $f(\chi, pg) = \sigma\chi\delta^{1/2}(p)f(g)$;
(b) there exists an open $K \subseteq G$ such that for all $\chi \in X$, $g \in G$, $k \in K$ one has $f(\chi, gk) = f(\chi, g)$;
(c) if K is as in (b) then for every fixed $g \in G$ the function $\chi \longmapsto f(\chi, g) \in U^{gKg^{-1} \cap M}$ is analytic.

The group acts on this sheaf by right regular representations: $R_g f(\chi, x) = f(\chi, xg)$.

Proposition 3.4.1. The sheaf I_σ with this action of G defines a holomorphic sheaf of admissible representations of G whose fibre at $\chi \in X_{\mathrm{nr}}(M)$ is isomorphic to $i_P^G \sigma\chi$.

Proof. Straightforward. □

Since the representations $i_P^G\sigma\chi$ may be identified as K-representations (3.1.1), the subsheaves I_σ^K are not just coherent but in addition locally free, and isomorphic to $\mathcal{O}_{X_{\mathrm{nr}}(M)}(i_P^G(\sigma)^K)$.

Next comes what is in some sense a converse to this. If $(\pi, \mathcal{V})$ is any holomorphic sheaf of admissible representations of G with base some space X, then since the representation on each $\Gamma(U, \mathcal{V})$ is smooth, the sheaf $\mathcal{V}_N$ is defined by the formula $\Gamma(U, \mathcal{V}_N) = \Gamma(U, \mathcal{V})_N$, and defines a holomorphic family, at least, of smooth representations of M, whose fibre at $x \in X$ is isomorphic to $(V_x)_N$ (where V_x is the fibre of $\mathcal{V}$ at x). Substantially the same argument used to prove Theorem 3.3.1 together with standard facts about coherent sheaves will then prove:

Theorem 3.4.2. The holomorphic sheaf $(\pi_N, \mathcal{V}_N)$ of representations of M is admissible.

4. The asymptotic behavior of matrix coefficients

This section pursues a question suggested by the proof of Theorem 3.3.3 and some results of Harish-Chandra.

4.1. Fix a minimal parabolic subgroup $P_\emptyset$ of G, let P be any parabolic subgroup containing $P_\emptyset$, and let $P = MN$ be a Levi factorization. Let (π, V) be an admissible representation of G. Let K_0 be a compact open subgroup of G possessing an Iwahori factorization $K_0 = N_0 M_0 N_0^-$ with respect to P (according to 1.4.4, one can choose arbitrarily small such groups). Assume a Haar measure on G such that $\operatorname{meas}(K_0) = 1$.

Lemma 4.1.1. If $v \in V^{K_0}$ has image u in V_N, then for any $a \in A^-$ (as defined in §1.4), the element $\mathcal{P}_{K_0}(\pi(a)v)$ has image $\pi_N(a)u$.

Proof. One has $\pi(a)v \in V^{M_0 N_0^-}$. Therefore, by Jacquet's First Lemma (3.3.4), $\mathcal{P}_{K_0}(\pi(a)v) = \mathcal{P}_{N_0}(\pi(a)v)$, so that $\pi(a)v$ and $\mathcal{P}_{K_0}(\pi(a)v)$ have the same image in V_N. □

Theorem 4.1.2 (Jacquet's Second Lemma). Let N_0, N_1 be compact open subgroups of N, $v \in V(N_1)$, and $m \in M$ such that $mN_1m^{-1} \subseteq N_0$. Then $\mathcal{P}_{N_0}(\pi(m)v) = 0$.

Proof. One has

$$\begin{aligned} \mathcal{P}_{N_0}(\pi(m)v) &= (\text{constant}) \cdot \int_{N_0} \pi(n)\pi(m)v\, dn \\ &= (\text{constant}) \cdot \pi(m) \int_{m^{-1}N_0 m} \pi(n)v\, dn \\ &= 0 \end{aligned}$$

because $N_1 \subseteq m^{-1}N_0 m$. □

Suppose now that N_1 is a compact subgroup of N such that $V^{K_0} \cap V(N) \subseteq V(N_1)$.

Corollary 4.1.3. If $a \in A^-$ is such that $aN_1a^{-1} \subseteq N_0$, then $\mathcal{P}_{K_0}(\pi(a)v) = 0$ for all $v \in V^{K_0} \cap V(N)$.

Proof. By Jacquet's First Lemma (3.3.4), $\mathcal{P}_{K_0}(\pi(a)v) = \mathcal{P}_{N_0}(\pi(a)v)$. By his Second, this in turn is zero. □

For each $a \in A^-$, define $V_a^{K_0}$ to be $\mathcal{P}_{K_0}(\pi(a)V^{K_0}) = \pi(K_0 a K_0)V$. (To interpret the right-hand side, recall that by our convention from §2.2, we really mean the action via π of the characteristic function of $K_0 a K_0$.)

Proposition 4.1.4. If a is in A^-, then the projection from $V_a^{K_0}$ to $V_N^{M_0}$ is a surjection. If $aN_1a^{-1} \subseteq N_0$ then $V_a^{K_0} \cap V(N) = 0$ and the projection is an isomorphism.

Proof. Let u lie in $V_N^{M_0}$. Then $\pi_N(a^{-1})u$ also lies in $V_N^{M_0}$, and therefore by Theorem 3.3.3 there exists $v \in V^{K_0}$ with it as image in V_N. By 4.1.1, then, $\mathcal{P}_{K_0}(\pi(a)v)$ has image u in V_N.

Assume $aN_1a^{-1} \subseteq N_0$, and $v \in V_a^{K_0} \cap V(N)$, say $v = \mathcal{P}_{K_0}(\pi(a)v_0)$ with $v_0 \in V^{K_0}$. Then v lies in $V(N_1)$ and $v = \mathcal{P}_{N_0}(\pi(a)v_0)$, so that

$$\begin{aligned} 0 &= \int_{N_1} \pi(n_1)(\int_{N_0} \pi(n_0)\pi(a)v_0\, dn_0)\, dn_1 \\ &= (\text{constant}) \cdot \int_{a^{-1}N_0a} \pi(n)(\int_{N_0} \pi(n_0)\pi(a)v_0\, dn_0)\, dn \\ &= (\text{constant}) \cdot \pi(a) \int_{a^{-2}N_0a^2} \pi(n)(\int_{a^{-1}N_0a} \pi(n')v_0\, dn')\, dn \\ &= (\text{constant}) \cdot \pi(a) \int_{a^{-2}N_0a^2} \pi(n)v_0\, dn. \end{aligned}$$

Therefore v_0 lies in $V(N)$ as well, hence in $V(N_1)$, so that by 4.1.3,

$$v = \mathcal{P}_{K_0}(\pi(a)v_0) = 0. \quad \square$$

Lemma 4.1.5. For $a_1, a_2 \in A^-$ one has this identity in $\mathcal{H}(G, K_0)$:

$$\mathrm{ch}_{K_0a_1K_0} * \mathrm{ch}_{K_0a_2K_0} = \mathrm{ch}_{K_0a_1a_2K_0}\,.$$

Proof. As sets, one has $K_0a_1K_0 \cdot K_0a_2K_0 = K_0a_1a_2K_0$ because for $k_0 = n_0m_0n_0^-$ one has

$$a_1k_0a_2 = a_1n_0a_1^{-1} \cdot a_1a_2 \cdot m_0 \cdot a_2^{-1}n_0^-a_2$$

which is in $K_0a_1a_2K_0$. But by 1.5.1 the measures agree as well. (For my purposes here the agreement as sets is all that is required.) $\square$

Proposition 4.1.6. For all $a \in A^-$ with $aN_1a^{-1} \subseteq N_0$, the spaces $V_a^{K_0}$ are identical.

Proof. For all a_1, a_2 satisfying the hypothesis, the product a_1a_2 also satisfies it. By 4.1.5, $\pi(K_0a_1K_0)$ takes $V_{a_2}^{K_0}$ onto itself and into $V_{a_1a_2}^{K_0}$. Hence $V_{a_1a_2}^{K_0} \subseteq V_{a_2}^{K_0}$. By 4.1.4, the two spaces have the same dimension, hence are equal. Similarly, $V_{a_1a_2}^{K_0} = V_{a_1}^{K_0}$. $\square$

Following this, define $V_{A^-}^{K_0}$ to be the subspace of V^{K_0} equal to $V_a^{K_0}$ for all $a \in A^-$ with $aN_1a^{-1} \subseteq N_0$.

Lemma 4.1.7. For any $a \in A^-$, $\pi(K_0aK_0)$ is an isomorphism of $V_{A^-}^{K_0}$ with itself.

Proof. The space $V_{A^-}^{K_0}$ is stable under this map by 4.1.5. Since the space is finite-dimensional, it suffices to prove that the map is an injection. Say, then, that $\mathcal{P}_{K_0}(\pi(a)v) = 0$ for some $v \in V_{A^-}^{K_0}$. The image of $\mathcal{P}_{K_0}(\pi(a)v)$ in $V_N^{M_0}$ is 0 as well, but this is just $\pi_N(a)$ applied to the image of v by 4.1.1. Therefore the image of v in $V_N^{M_0}$ is 0 and by 4.1.4 v itself is 0. $\square$

According to 4.1.4 the canonical projection from V to V_N induces an isomorphism of $V_{A^-}^{K_0}$ with $V_N^{M_0}$. The inverse of this isomorphism is called the *canonical lifting* from $V_N^{M_0}$ to $V_{A^-}^{K_0}$. (Note that according to 1.4.3 there will exist $\epsilon > 0$ such that $a \in A^-(\epsilon)$ implies $aN_1a^{-1} \subseteq N_0$.)

This "canonical lifting" is not independent of the choice of K_0. More precisely, let $K_0' \subseteq K_0$ be two subgroups with Iwahori factorizations. Then it is not generally true that $V^{K_0} \cap V_{A^-}^{K_0'} = V_{A^-}^{K_0}$.

Proposition 4.1.8. If $v \in V_{A^-}^{K_0}$ and $v' \in V_{A^-}^{K_0'}$ have the same image in V_N, then

$$v = \mathcal{P}_{N_0}(v') = \mathcal{P}_{K_0}(v').$$

Proof. Let N_1' be chosen large enough so that $V(N) \cap V^{K_0'} \subseteq V(N_1')$, and choose $a \in A^-$ so that $aN_1'a^{-1} \subseteq N_0'$. According to 4.1.7, there exist $v_0 \in V_{A^-}^{K_0}$ and $v_0' \in V_{A^-}^{K_0'}$ such that $v = \mathcal{P}_{K_0}(\pi(a)v_0)$, $v' = \mathcal{P}_{K_0'}(\pi(a)v_0')$. If v_0 has image u_0 and v_0' has image u_0' in V_N, then v and v' have images $\pi_N(a)u_0$ and $\pi_N(a)u_0'$, by 4.1.1. Therefore $u_0 = u_0'$, and hence $v_0 - v_0' \in V(N) \cap V^{K_0'}$. But then by 4.1.3, $\mathcal{P}_{K_0'}(\pi(a)v_0 - \pi(a)v_0') = 0$, or

$$\mathcal{P}_{K_0'}(\pi(a)v_0) = \mathcal{P}_{K_0'}(\pi(a)v_0') = v'.$$

Since v_0 is fixed by K_0, $\pi(a)v_0$ and hence $\mathcal{P}_{K_0'}(\pi(a)v_0) = v'$ are both fixed by $M_0N_0^-$. Jacquet's First Lemma implies then that

$$\begin{aligned} \mathcal{P}_{N_0}(v') &= \mathcal{P}_{K_0}(v') \\ &= \mathcal{P}_{K_0}(\mathcal{P}_{K_0'}(\pi(a)v_0)) = v. \quad \square \end{aligned}$$

4.2. Let P be a parabolic subgroup of G and (π, V) an admissible representation of G.

Note that for the representation $(\widetilde{\pi}, \widetilde{V})$ and parabolic P^-, the set A^- must be replaced by $A^+ = (A^-)^{-1}$.

Lemma 4.2.1. If K_0 is a subgroup with an Iwahori factorization with respect to P, $v \in V_{A^-}^{K_0}$, and $\widetilde{v} \in \widetilde{V}^{K_0} \cap \widetilde{V}(N^-)$, then $\langle v, \widetilde{v} \rangle = 0$.

Proof. Choose N_1 so that $V^{K_0} \cap V(N) \subseteq V(N_1)$, N_1^- so that $\widetilde{V}^{K_0} \cap \widetilde{V}(N^-) \subseteq \widetilde{V}(N_1^-)$, and $a \in A^-$ so that $a^{-1}N_1^-a \subseteq N_0^-$. Choose $v_0 \in V_{A^-}^{K_0}$ such that $v = \mathcal{P}_{K_0}(\pi(a)v_0)$ (4.1.7). Then

$$\begin{aligned} \langle v, \widetilde{v} \rangle &= \langle \mathcal{P}_{K_0}(\pi(a)v_0), \widetilde{v} \rangle \\ &= \langle v_0, \mathcal{P}_{K_0}(\pi(a^{-1})\widetilde{v}) \rangle \\ &= 0 \end{aligned}$$

by 4.1.3 applied to $\widetilde{V}$. $\square$

Lemma 4.2.2. Suppose that $K_0' \subseteq K_0$ are two subgroups with Iwahori factorizations with respect to P, and $v' \in V_{A^-}^{K_0'}$, $\widetilde{v}' \in \widetilde{V}_{A^+}^{K_0'}$, $v \in V_{A^-}^{K_0}$, $\widetilde{v} \in \widetilde{V}_{A^+}^{K_0}$, and assume that $v - v' \in V(N)$ and $\widetilde{v} - \widetilde{v}' \in \widetilde{V}(N^-)$. Then

$$\langle v, \widetilde{v} \rangle = \langle v', \widetilde{v}' \rangle.$$

Proof. From 4.1.8, $v = \mathcal{P}_{N_0}(v')$. Therefore,

$$\begin{aligned} \langle v, \widetilde{v} \rangle &= \langle \mathcal{P}_{N_0}(v'), \widetilde{v} \rangle \\ &= \langle v', \widetilde{v} \rangle \end{aligned}$$

and

$$\begin{aligned} \langle v, \widetilde{v} \rangle - \langle v', \widetilde{v}' \rangle &= \langle v', \widetilde{v} - \widetilde{v}' \rangle \\ &= 0 \end{aligned}$$

by 4.2.1. □

One may thus define a *canonical pairing* of V_N with $\widetilde{V}_{N^-}$ according to the formula

$$\langle u, \widetilde{u} \rangle_N = \langle v, \widetilde{v} \rangle$$

where v, $\widetilde{v}$ are any two canonical lifts of $u, \widetilde{u}$.

Proposition 4.2.3. The canonical bilinear form on $V_N \times \widetilde{V}_{N^-}$ is characterized uniquely by the property that for any $v \in V$, $\widetilde{v} \in \widetilde{V}$ with images $u \in V_N$, $\widetilde{u} \in \widetilde{V}_{N^-}$ there exists $\epsilon > 0$ such that for any $a \in A^-(\epsilon)$ one has

$$\langle \pi(a)v, \widetilde{v} \rangle = \langle \pi_N(a)u, \widetilde{u} \rangle_N.$$

Proof. That the canonical bilinear form has this property follows from 4.2.1. To see that it is uniquely determined by this property, let B be a bilinear form on $V_N \times \widetilde{V}_{N^-}$ with this property, and suppose K_0 to be a compact open subgroup possessing an Iwahori factorization with respect to P. One can find $\epsilon > 0$ such that

(i) for all $v \in V^{K_0}$, $\widetilde{v} \in \widetilde{V}^{K_0}$ with images $u \in V_N$, $\widetilde{u} \in \widetilde{V}_{N^-}$, and $a \in A^-(\epsilon)$, one has $\langle \pi(a)v, \widetilde{v} \rangle = \langle \mathcal{P}_{K_0}(\pi(a)v), \widetilde{v} \rangle = B(\pi_N(a)u, \widetilde{u})$ and

(ii) $V_a^{K_0} = V_{A^-}^{K_0}$ for $a \in A^-(\epsilon)$.

But then $\mathcal{P}_{K_0}(\pi(a)v)$ is a canonical lift of $\pi_N(a)u$, and since $\pi_N(a)$ is surjective on $V_N^{M_0}$ this implies that for all $u \in V_N^{M_0}$ and $\widetilde{u} \in \widetilde{V}_{N^-}^{M_0}$ with canonical lifts $v \in V^{K_0}$ and $\widetilde{v} \in \widetilde{V}^{K_0}$ one has $B(u, \widetilde{u}) = \langle v, \widetilde{v} \rangle$. □

Theorem 4.2.4. The canonical bilinear form on $V_N \times \widetilde{V}_{N^-}$ is M-invariant and nondegenerate.

Proof. For any $m \in M$ the bilinear form $B(u, \widetilde{u}) = \langle \pi_N(m)u, \pi_N(m)\widetilde{u} \rangle_N$ has the characteristic property of 4.2.3, thus is the same as the canonical form.

For the non-degeneracy: Suppose $u \in V_N$ to be such that $\langle u, \widetilde{u} \rangle_N = 0$ for all $\widetilde{u} \in \widetilde{V}_{N^-}$. Let $v \in V_{A^-}^{K_0}$ be a canonical lift of u. Then 4.2.3 implies that $\langle v, \widetilde{v} \rangle = 0$ for every $\widetilde{v} \in \widetilde{V}^{K_0}$, hence $v = 0$. Therefore $u = 0$ as well. □

Corollary 4.2.5. The contragredient of the representation (π_N, V_N) of M is isomorphic to $(\widetilde{\pi}_{N^-}, \widetilde{V}_{N^-})$.

4.3. Let (π, V) be an admissible representation of G. Let $P_\emptyset$ be a minimal parabolic subgroup, Δ a set of simple roots associated to $A_\emptyset$ and $P_\emptyset$. Let K_0 be a compact open subgroup of G possessing an Iwahori factorization with respect to all the parabolics containing $P_\emptyset$ (arbitrarily small such K_0 exist according to 1.4.4).

It is immediate that for any $\Theta \subseteq \Delta$ and $\epsilon > 0$,

$$A_\Theta^-(\epsilon) \cdot A_\emptyset^- \subseteq \Big\{ a \in A_\emptyset^- \;\Big|\; |\alpha(a)| \le \epsilon \text{ for all } \alpha \in \Delta \setminus \Theta \Big\}.$$

Conversely:

Lemma 4.3.1. Let Θ be a subset of Δ. For any $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that $\Big\{ a \in A_\emptyset^- \;\Big|\; |\alpha(a)| \le \epsilon_2 \text{ for all } \alpha \in \Delta \setminus \Theta \Big\}$ is contained in $A_\Theta^-(\epsilon_1) \cdot A_\emptyset^-$.

Proof. Since the product morphism from $A_\Theta \times A_{\Delta\setminus\Theta}$ to $A_\emptyset$ is an epimorphism, the image of the product of the lattices $A_\Theta/A_\Theta(\mathcal{O}) \times A_{\Delta\setminus\Theta}/A_{\Delta\setminus\Theta}(\mathcal{O})$ in $A_\emptyset/A_\emptyset(\mathcal{O})$ has finite index. One may therefore find a finite set $\mathfrak{a}$ of representatives of the cokernel lying in $A_\emptyset^-$. If ϵ_0 is the minimum value of $|\alpha(a)|$ as α ranges over $\Delta \setminus \Theta$ and a over $\mathfrak{a}$, then the set $\Big\{ a \in A_\emptyset^- \;\Big|\; |\alpha(a)| \le \epsilon\epsilon_0 \text{ for all } \alpha \in \Delta \setminus \Theta \Big\}$ is contained in $A_\Theta^-(\epsilon) \cdot A_\emptyset^-$. □

Lemma 4.3.2. Let Θ be a subset of Δ, $P = P_\Theta$. For any $v \in V^{K_0}$ there exists $\epsilon > 0$ such that whenever $a \in A_\emptyset^-$ satisfies the condition $|\alpha(a)| \le \epsilon$ for all $\alpha \in \Delta \setminus \Theta$, then $\pi(K_0 a K_0)v \in V_{A^-}^{K_0}$.

Proof. Let $\epsilon_1 > 0$ be small enough so that $V_a^{K_0} = V_{A^-}^{K_0}$ for $a \in A^-(\epsilon_1)$, and let ϵ_2 be as in 4.3.1. Then for $a \in A_\emptyset^-$ such that $|\alpha(a)| \le \epsilon_2$ for all $\alpha \in \Delta \setminus \Theta$,

$$\begin{aligned} \pi(K_0 a K_0)v &\in \pi(K_0 A^-(\epsilon_1) K_0)\pi(K_0 A_\emptyset^- K_0)V \qquad \text{(by 4.1.5)} \\ &\subseteq \pi(K_0 A^-(\epsilon_1) K_0)V = V_{A^-}^{K_0}. \quad \square \end{aligned}$$

Theorem 4.3.3. Let Θ be a subset of Δ, $P = P_\Theta$, and let $v \in V$, $\widetilde{v} \in \widetilde{V}$ be given with images $u \in V_N$, $\widetilde{u} \in \widetilde{V}_{N^-}$. There exists $\epsilon > 0$ such that whenever $a \in A_\emptyset^-$ satisfies $|\alpha(a)| \le \epsilon$ for all $\alpha \in \Delta \setminus \Theta$, then $\langle \pi(a)v, \widetilde{v} \rangle = \langle \pi_N(a)u, \widetilde{u} \rangle_N$.

Proof. This follows immediately from the construction of the canonical pairing $\langle\ ,\ \rangle_N$ and 4.3.2. □

For each $\Theta \subseteq \Delta$ and ϵ in $(0,1]$, define ${}_\Theta A_\emptyset^-(\epsilon)$ to be

$$\left\{ a \in A_\emptyset \,\middle|\, \begin{array}{rl} |\alpha(a)| \le \epsilon & \text{for } \alpha \in \Delta \smallsetminus \Theta, \\ \epsilon < |\alpha(a)| \le 1 & \text{for } \alpha \in \Theta \end{array} \right\}.$$

This is a subset of $A_\emptyset^-$ stable under multiplication by A_Θ^-, and for any ϵ one sees that $A_\emptyset^-$ is the disjoint union of the ${}_\Theta A_\emptyset^-(\epsilon)$ as Θ ranges over all subsets of Δ.

Corollary 4.3.4. Let $v \in V$, $\widetilde{v} \in \widetilde{V}$ be given. For $\Theta \subseteq \Delta$ let u_Θ, $\widetilde{u}_\Theta$ be their images in V_{N_Θ}, $\widetilde{V}_{N_\Theta^-}$. There exists $\epsilon > 0$ such that for any $\Theta \subseteq \Delta$ and $a \in {}_\Theta A_\emptyset^-(\epsilon)$ one has

$$\langle \pi(a)v, \widetilde{v} \rangle = \langle \pi_{N_\Theta}(a)u_\Theta, \widetilde{u}_\Theta \rangle_{N_\Theta}.$$

Proof. Let ϵ be the minimum of the ϵ guaranteed by 4.3.3 as Θ ranges over all subsets of Δ. □

4.4. The next few results (4.4.1 through 4.4.3) are technical and elementary. I will not give proofs[8].

If X is a set on which a group H acts, then a function $F\colon X \longrightarrow \mathbf{C}$ is said to be H-finite if it and its H-translates span a finite-dimensional subspace of the space of all complex-valued functions on X. Suppose H is abelian. Then the characters associated to an H-finite F are the generalized eigencharacters of the representation of H on this finite-dimensional space. If $\chi_1, \dots, \chi_m$ are the characters associated to F then there exist $\ell_1, \dots, \ell_m \in \mathbf{N}$ such that $\prod (h - \chi_i(h))^{\ell_i} F = 0$ for all $h \in H$.

Lemma 4.4.1. Suppose that $F\colon \mathbf{Z}^n \longrightarrow \mathbf{C}$ is $\mathbf{Z}^n$-finite, and let $p > 0$. Then the restriction of $|F(x)|^p$ to $\mathbf{N}^n$ is summable if and only $|\chi(x)| < 1$ for all nonzero $x \in \mathbf{N}^n$ and χ associated to F.

Now let V be a vector space over $\mathbf{R}$, L a lattice in V, and $f_1, \dots, f_n$ elements of the dual lattice which form a basis for the linear forms on V. Let $I = \{1, 2, \dots, n\}$ and for each $J \subseteq I$, and each $c = (c_1, \dots, c_n) \in \mathbf{R}^n$ with all $c_i \ge 0$ let

$$\begin{array}{lcl} V_J & = & \left\{ v \in V \,\middle|\, f_i(v) = 0 \text{ for all } i \in J \right\}; \\ L_J & = & L \cap V_J; \\ V^+ & = & \left\{ v \in V \,\middle|\, f_i(v) \ge 0 \text{ for all } i \right\}; \\ {V_J}^+ & = & V_J \cap V^+; \\ M & = & \text{the sublattice spanned by the } L_J, \text{ as } J \\ & & \text{ranges over all subsets of } I \text{ with } \operatorname{card}(J) = n-1; \\ {}_J V(c) & = & \left\{ v \in V \,\middle|\, 0 \le f_i(v) < c_i \text{ for } i \in J \right\}; \\ {}_J V^+(c) & = & \left\{ v \in V \,\middle|\, \begin{array}{rl} f_i(v) \ge c_i & \text{for } i \in I \smallsetminus J \\ 0 \le f_i(v) < c_i & \text{for } i \in J \end{array} \right\}. \end{array}$$

[8] Reference?

Define ${}_J L^+(c)$ as ${}_J V^+(c)$, etc. The vector subspace V_J has dimension equal to $(n - \mathrm{card}(J))$, so that if $\mathrm{card}(J) = n-1$, L_J has rank one over $\mathbf{Z}$. The lattice M is of finite index in L, and each M_J^+ is isomorphic to $\mathbf{N}^{n-\mathrm{card}(J)}$. The vector space V itself is $V_\emptyset$, and for any c as above V^+ is equal to the disjoint union of the ${}_J V^+(c)$ as J ranges over all subsets of I. The group L_J acts on the set ${}_J V(c)$.

Lemma 4.4.2. For each $J \subseteq I$ and $c \in \mathbf{R}^n$ with all $c_i \geq 0$, there exists a finite set Λ such that ${}_J L^+(c) = \coprod_{\lambda \in \Lambda} (\lambda + {M_J}^+)$.

The above two results combine to give:

Lemma 4.4.3. Let $F\colon {}_J L^+(c) \longrightarrow \mathbf{C}$ be the restriction to ${}_J L^+(c)$ of an L_J-finite function on ${}_J L(c)$, $p > 0$. Then $|F|^p$ is summable on ${}_J L^+(c)$ if and only if for each nonzero $x \in {L_J}^+$ and χ associated to F, $|\chi(x)| < 1$.

For the rest of this section, I continue the notation of §4.3.

Proposition 4.4.4. Let Θ be a subset of Δ, $0 < \epsilon \leq 1$, $p > 0$. Let $F\colon {}_\Theta A_\emptyset^-(\epsilon) \longrightarrow \mathbf{C}$ be such that

- (a) F is the restriction to ${}_\Theta A_\emptyset^-(\epsilon)$ of an A_Θ-finite function;
- (b) there exists a unitary character $\omega\colon A_\Delta \longrightarrow \mathbf{C}^\times$ such that $R_a F = \omega(a) F$ for all $a \in A_\Delta$;
- (c) there exists an open subgroup $\mathfrak{A} \subseteq A_\emptyset(\mathcal{O})$ such that $R_a F = F$ for all $a \in \mathfrak{A}$.

Then $|F|^p$ is summable on ${}_\Theta A_\emptyset^-(\epsilon)/\mathfrak{A} A_\Delta$ if and only if $|\chi(a)| < 1$ for every $a \in A_\Theta^- \smallsetminus A_\emptyset(\mathcal{O}) A_\Delta$ and every character χ of A_Θ associated to F.

Proof. Let L be the lattice $A_\emptyset / A_\emptyset(\mathcal{O}) A_\Delta$, $V = L \otimes \mathbf{R}$, and define for each $\alpha \in \Delta$ a function $f_\alpha\colon A_\emptyset \longrightarrow \mathbf{Z}$ by $f_\alpha(a) = -\log_q |\alpha(a)|$, where q is the order of the residue field. To this situation one may apply the previous results—for example, each ${}_J L^+(c)$ may be identified with some ${}_\Theta A_\emptyset^-(\epsilon)/A_\emptyset(\mathcal{O}) A_\Delta$. Now the function $|F|$ may be considered as a function on ${}_\Theta A_\emptyset^-(\epsilon)/\mathfrak{A} A_\Delta$, which fibres naturally over ${}_\Theta A_\emptyset^-(\epsilon)/A_\emptyset(\mathcal{O}) A_\Delta$ with finite fibres. The proposition follows therefore from an immediate generalization of 4.4.3. $\square$

Corollary 4.4.5. Let Θ be a subset of Δ, $0 < \epsilon \leq 1$, $p > 0$, K a compact open subgroup of G. Let F be a function on $K \cdot {}_\Theta A_\emptyset^-(\epsilon) \cdot K$ whose restriction Φ to ${}_\Theta A_\emptyset^-(\epsilon)$ satisfies (a) and (b) of 4.4.4. Then $|F|^p$ is integrable mod Z_G if and only if for every character χ associated to Φ and $a \in A_\Theta^- \smallsetminus A_\emptyset(\mathcal{O}) A_\Delta$, $|\chi\delta^{-1/p}(a)| < 1$.

This follows from 4.4.4 together with 1.5.2. (Note that condition (c) of 4.4.4 automatically holds here[9].)

If P is a parabolic subgroup of G and (π, V) an admissible representation of G, then according to 3.3.1 the representation (π_N, V_N) is admissible as well. By 2.1.9 this representation is a direct sum of subrepresentations $(V_N)_{\chi,\infty}$, χ ranging over a

[9] In other words, F is automatically uniformly locally constant. Is this obvious?

set of characters of the center of M. I call the restriction to A of the characters χ such that $(V_N)_{\chi,\infty} \neq 0$ the *central* characters of π with respect to P. (These are what Harish-Chandra calls *exponents* in [20].) If π is an ω-representation for the character ω of Z_G, then $\chi|_{A\cap Z_G} = \omega|_{A\cap Z_G}$ for any central character χ. If $P_1 \subseteq P_2$ are two parabolics then $\pi_{N_1} \cong (\pi_{N_2})_{N_1\cap M_2}$, so that if χ_1 is a central character of π with respect to P_1 then $\chi_1|_{A_2}$ is a central character with respect to P_2.

Theorem 4.4.6. Let ω be a character of Z_G. If (π, V) is an admissible ω-representation of G, then it is square-integrable if and only if

(a) ω is unitary;
(b) for every $\Theta \subseteq \Delta$, if χ is a central character of π with respect to P_Θ then $|\chi\delta_\emptyset^{-1/2}(a)| < 1$ for all $a \in A_\Theta^- \smallsetminus A_\emptyset(\mathcal{O})A_\Delta$.

Proof. The condition (a) is trivially necessary. Suppose it then to hold.

Let Γ be a subgroup of G as in 1.4.6 such that

(i) Γ/A_Δ is compact;
(ii) $A_\emptyset(\mathcal{O}) \subseteq \Gamma$;
(iii) $G = \Gamma A_\emptyset^- \Gamma$.

Let $v \in V$, $\widetilde{v} \in \widetilde{V}$ be given, and consider $c_{v,\widetilde{v}}(g) = \langle \pi(g)v, \widetilde{v}\rangle$. Let K be a compact open subgroup of G, normal in Γ, such that v, $\widetilde{v}$ are fixed by K. If $\{\gamma_i\}$ is a set of representatives of Γ/KA_Δ, then $G = \coprod K\gamma_i A_\emptyset^- \gamma_j K$, so that $|c_{v,\widetilde{v}}|$ is square-integrable on G/Z if and only if it is on each $K\gamma_i A_\emptyset^- \gamma_j K/A_\Delta$. In order to prove that the conditions imply square-integrability, one may therefore replace v, $\widetilde{v}$ by the $\gamma_j v$, $\gamma_i^{-1}\widetilde{v}$ in turn and consider only $c_{v,\widetilde{v}}$ on $KA_\emptyset^- K/A_\Delta$. But to this one may apply 4.3.4 and 4.4.5. □

5. Absolutely cuspidal representations

5.1. An admissible representation (π, V) is said to be *absolutely cuspidal* (or, by Harish-Chandra in [20], *super-cuspidal*) if for any proper parabolic subgroup $P = MN$ in G, $V = V(N)$ and hence $V_N = 0$. If $P_1 \subseteq P_2$ are two parabolic subgroups, then $N_2 \subseteq N_1$ and $V(N_2) \subseteq V(N_1)$, so that in order to check this condition, it suffices to consider only maximal proper parabolics. If P_1 and P_2 are conjugate then $V(N_1)$ is conjugate to $V(N_2)$, so that it also suffices to consider only a fixed element of each conjugacy class.

Proposition 5.1.1. If (π, V) is an admissible representation of G, then it is absolutely cuspidal if and only if for any proper parabolic subgroup P and any admissible representation (σ, U) of M one has $\mathrm{Hom}_G(V, i_P^G\,\sigma) = 0$.

Proof. The necessity follows from 3.2.5. The sufficiency follows from 3.2.4 and 3.3.1, since one always has a G-morphism from V to $i_P^G\,(\pi_N \delta_P^{-1/2})$ which is non-trivial if $V_N \neq 0$. □

Theorem 5.1.2 (Jacquet). If (π, V) is an irreducible admissible representation of G, then there exists a parabolic subgroup P and an irreducible absolutely cuspidal representation (σ, U) of M such that (π, V) may be embedded into $i_P^G\,\sigma$.

Proof. Let r be the semisimple k-rank of G. We proceed by induction on r. For $r = 0$, G has no proper parabolic subgroups, so all admissible representations are absolutely cuspidal. (Note that when $r = 0$, G/Z is compact.)

Assume $r > 0$. If (π, V) is absolutely cuspidal, one is through. If not, then there exists some parabolic P with $V_N \neq 0$. The representation π_N is both admissible and finitely generated by 3.3.1, so that it possesses a non-trivial irreducible admissible quotient (ρ, W). By 3.2.4, there exists an embedding of π into $i_P^G\,(\rho\delta_P^{-1/2})$. Now the semisimple rank of M is less than r, so that by the induction hypothesis there exists a parabolic subgroup Q of M, with Levi decomposition $Q = M_Q N_Q$, and an irreducible admissible representation (σ, U) of M_Q such that $\rho\delta_P^{-1/2}$ may be embedded into $i_Q^M\,\sigma$. Apply 2.4.4 and 2.4.5 to prove the theorem. □

5.2. The following is also due to Jacquet:

Theorem 5.2.1. Suppose that (π, V) is an absolutely cuspidal representation of G. For $v \in V$, $\widetilde{v} \in \widetilde{V}$, the matrix coefficient $c_{v,\widetilde{v}}$ has compact support on G modulo Z.

Proof. Let $P_\emptyset$ be a minimal parabolic subgroup of G, Δ the set of positive simple roots corresponding to $P_\emptyset$, Γ the group given in 1.4.6. Note that by 2.1.3, any admissible representation of G is Γ-finite.

Since the split component A_Δ of the center of G is equal to the connected component of $\bigcap_{\alpha\in\Delta}\ker(\alpha)$, the theorem will follow from this claim: There exists $\epsilon > 0$ such that for all $g \in \Gamma a \Gamma$ with $|\alpha(a)| < \epsilon$ for some $\alpha \in \Delta$, $c_{v,\widetilde{v}}(g) = 0$.

To prove this claim: Let α be an element of Δ, P the standard parabolic subgroup corresponding to $\Delta \setminus \{\alpha\}$. It is a maximal proper parabolic subgroup. Since (π, V) is absolutely cuspidal, $V = V(N)$. Fix $v \in V$, $\widetilde{v} \in \widetilde{V}$, and choose $N_1 \subseteq N_2$ compact open subgroups of N such that for every $\gamma \in \Gamma$, $\pi(\gamma)v \in V(N_2)$ and $\pi(\gamma)\widetilde{v} \in \widetilde{V}^{N_1}$. Choose $\epsilon_P > 0$ such that if $a \in A_\emptyset^-$ and $|\alpha(a)| < \epsilon_P$, then $aN_2a^{-1} \subseteq N_1$. If $g = \gamma_1 a \gamma_2$, with $a \in A_\emptyset^-(\epsilon_P)$, then one can apply Jacquet's Second Lemma (4.1.2) to see that[10]

$$\begin{aligned}\langle \pi(g)v, \widetilde{v}\rangle &= \langle \pi(\gamma_1 a \gamma_2)v, \widetilde{v}\rangle \\ &= \langle \pi(a)\pi(\gamma_2)v, \widetilde{\pi}(\gamma_1^{-1})\widetilde{v}\rangle \\ &= 0.\end{aligned}$$

(One could also apply 4.3.3 here.)

Let ϵ be the minimum of the ϵ_P thus chosen, as P ranges over the set of all standard maximal proper parabolic subgroups of G. The claim is clearly satisfied by ϵ. □

Lemma 5.2.2. There is no Lemma 5.2.2.

Corollary 5.2.3. Suppose that (π, V) is an irreducible absolutely cuspidal ω-representation of G, where ω is a unitary character of Z. Then (π, V) is unitary.

Proof. A matrix coefficient of compact support modulo Z is obviously square-integrable modulo Z. The result follows from 2.5.4. □

Proposition 5.2.4. If (π, V) is an irreducible absolutely cuspidal[11] representation of G, then there exists a real constant $d_\pi > 0$ such that for any $u, v \in V$, $\widetilde{u}, \widetilde{v} \in \widetilde{V}$, one has

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle \pi(g^{-1})v, \widetilde{v}\rangle \, dg = d_\pi^{-1} \langle u, \widetilde{v}\rangle \langle v, \widetilde{u}\rangle.$$

Proof. Note that the integral makes sense by 5.2.1.

For each $v_0 \in V$, $\widetilde{u}_0 \in \widetilde{V}$, the pairing which takes $u \in V$ and $\widetilde{v} \in \widetilde{V}$ to

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}_0\rangle \langle \pi(g^{-1})v_0, \widetilde{v}\rangle \, dg$$

is, as one can check, G-invariant, hence a multiple $c(v_0, \widetilde{u}_0)$ of the canonical pairing. Furthermore, the pairing taking $v \in V$ and $\widetilde{u} \in \widetilde{V}$ to $c(v, \widetilde{u})$ is also bilinear and G-invariant, hence some multiple of the canonical pairing. Thus, one has

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle \pi(g^{-1})v, \widetilde{v}\rangle \, dg = c_\pi \langle u, \widetilde{v}\rangle \langle v, \widetilde{u}\rangle$$

for some constant c_π.

[10] Perhaps this will be clearer if we cite 4.1.3.

[11] This result and its proof are valid for square-integrable representations.

Assume for the moment that ω is unitary. Then by 5.2.3, (π, V) is unitary as well, say with inner product $u \cdot v$. Fix $\widetilde{u}$, $\widetilde{v}$ for the moment and let $u_0, v_0 \in V$ be such that for all $x \in V$, $x \cdot u_0 = \langle x, \widetilde{u}\rangle$ and $x \cdot v_0 = \langle x, \widetilde{v}\rangle$. The formula already established gives

$$\begin{aligned} c_\pi (u \cdot v_0)(v \cdot u_0) &= \int_{G/Z} (\pi(g)u \cdot u_0)(\pi(g^{-1})v \cdot v_0)\, dg \\ &= \int_{G/Z} (\pi(g)u \cdot u_0)\overline{(\pi(g)v_0 \cdot v)}\, dg \end{aligned}$$

(which is the usual Schur orthogonality for unitary representations). One sees that $c_\pi > 0$ by setting $u = v_0$ and $v = u_0$.

Before completing the proof, we'll need the following result:

Lemma 5.2.5. If (π, V) is any smooth ω-representation of G, then there exists a unique positive real-valued character χ of G such that the restriction of $\pi \otimes \chi$ to Z is unitary.

Proof. What must be shown is that there exists a unique positive character χ of G such that $\omega \cdot (\chi|_Z)$ is a unitary character of Z. Now G has a maximum torus quotient T; the canonical morphism from G to T induces an isogeny of Z. If Z_0 and T_0 are the maximal compact subgroups of Z and T, then the groups Z/Z_0 and T/T_0 are free modules over $\mathbf{Z}$ of the same rank, and the canonical map from Z/Z_0 to T/T_0 is an injection with finite cokernel. The character $|\omega|$ on Z/Z_0 therefore extends uniquely to a positive character of T/T_0. Take χ to be its inverse. □

To complete the proof of 5.2.4, let (π, V) be given, and choose χ so $\pi \otimes \chi$ is unitary. The constant $c_{\pi\otimes\chi}$ will work for π as well as $\pi \otimes \chi$, and is positive. Let $d_\pi = c_{\pi\otimes\chi}^{-1}$. □

The constant d_π in 5.2.4 is called the *formal degree* of (π, V). It depends only on π and on the normalization of Haar measure. Harish-Chandra has recently[12] shown that if the characteristic of the field of definition of G is 0, then one may normalize the measure on G such that $d_\pi \in \mathbf{N}$ for every absolutely cuspidal π.

5.3. I give here some new characterizations of absolutely cuspidal representations.

Theorem 5.3.1. Let (π, V) be an admissible representation of G. The following are equivalent:

(a) (π, V) is absolutely cuspidal;
(b) $(\widetilde{\pi}, \widetilde{V})$ is absolutely cuspidal;
(c) for every $v \in V$, $\widetilde{v} \in \widetilde{V}$ the matrix coefficient $c_{v,\widetilde{v}}$ has compact support modulo Z.

[12] Not recently. PS will find a reference.

Proof. The equivalence of (a) and (b) is immediate from Theorem 4.2.3.

That (a) implies (c) is just 5.2.1. For the converse: Suppose that for every $v \in V$, $\widetilde{v} \in \widetilde{V}$, $c_{v,\widetilde{v}}$ has compact support modulo Z. Then for every $v \in V$, $\widetilde{v} \in \widetilde{V}$, and every parabolic subgroup P in G, there exists $\epsilon > 0$ such that $\langle \pi(a)v, \widetilde{v} \rangle = 0$ for $a \in A^-(\epsilon)$. Fix for the moment a compact open subgroup K_0 with an Iwahori factorization with respect to P. Then there exists $\epsilon > 0$ such that for every $v \in V^{K_0}$, $\widetilde{v} \in \widetilde{V}^{K_0}$, and $a \in A^-(\epsilon)$, we have $\langle \pi(a)v, \widetilde{v} \rangle = 0$, and this in turn implies that for every $v \in V^{K_0}$ and $a \in A^-(\epsilon)$, we have $\mathcal{P}_{K_0}(\pi(a)v) = 0$. This implies that $V_{A^-}^{K_0}$ (see §4) is zero. Thus, letting K_0 vary and applying 4.1.4, we see that $V_N = 0$. Since P was arbitrary, (π, V) is absolutely cuspidal. □

5.4. Recall from §2 that if ω is a character of Z, then the smooth ω-representations of G form an abelian category.

Theorem 5.4.1. If (π, V) is an absolutely cuspidal ω-representation, then it is projective and injective in this category.

Proof. I shall first prove projectivity, and start with the case where (π, V) is irreducible. Let (σ, U) be any smooth ω-representation, $F\colon U \longrightarrow V$ a G-surjection. I must show that there exists a G-morphism $\Phi\colon V \longrightarrow U$ splitting F.

Choose nonzero $\widetilde{v}_0 \in \widetilde{V}$ and $v_0 \in V$ such that $\langle v_0, \widetilde{v}_0 \rangle = d_\pi$ (see 5.2.4). For any $v \in V$, let Γ_v be the function $c_{v,\widetilde{v}_0}\colon g \longmapsto \langle \pi(g)v, \widetilde{v}_0 \rangle$. Because π is an ω-representation, it follows from 2.5.1 and 5.2.1 that $v \longmapsto \Gamma_v$ is a G-morphism from (π, V) to $(R, \mathcal{H}_{\omega^{-1}}(G))$. Identify V with its image in $\mathcal{H}_{\omega^{-1}}(G)$. Define the projection $P\colon \mathcal{H}_{\omega^{-1}} \longrightarrow V$ by the formula $Pf(y) = (\Gamma_{v_0} * f)(y)$. Then

$$\begin{aligned} Pf(y) &= \int_{G/Z} \Gamma_{v_0}(x) f(x^{-1}y)\, dx \\ &= \int_{G/Z} \Gamma_{v_0}(yx) f(x^{-1})\, dx \\ &= \int_{G/Z} f(x^{-1})(R_x \Gamma_{v_0})(y)\, dx. \end{aligned}$$

The last formula shows that $P(\mathcal{H}_{\omega^{-1}}) \subseteq V$; the first implies that P is a G-morphism: $P(R_g f) = R_g(Pf)$. Furthermore, if $f = \Gamma_v$ lies in V, then

$$\begin{aligned} Pf(y) &= \int_{G/Z} \langle \pi(x)v_0, \widetilde{v}_0 \rangle \langle \pi(x^{-1}y)v, \widetilde{v}_0 \rangle\, dx \\ &= d_\pi^{-1} \langle v_0, \widetilde{v}_0 \rangle \langle \pi(y)v, \widetilde{v}_0 \rangle \\ &= \langle \pi(y)v, \widetilde{v}_0 \rangle \\ &= \Gamma_v(y) \end{aligned}$$

by 5.2.4 and the choice of v_0, $\widetilde{v}_0$. In other words, $Pf = f$ for $f \in V$.

Recall (σ, U) and $F\colon U \longrightarrow V$. Choose $u_0 \in U$ with $F(u_0) = v_0$. Let Π be the map from $\mathcal{H}_{\omega^{-1}}$ to U, taking f to $\sigma(\check{f})u_0$, where for any $f \in \mathcal{H}_{\omega^{-1}}$ (or $C^\infty(G)$, for that matter) $\check{f}(x) = f(x^{-1})$. One can check easily that this is a G-morphism. And one can also check that the diagram

$$\begin{array}{ccc} & \mathcal{H}_{\omega^{-1}} & \\ {}^{\Pi}\swarrow & & \searrow^{P} \\ U & \xrightarrow{\quad F \quad} & V \end{array}$$

is commutative, which implies that the composition $\Phi = \Pi \circ \Gamma\colon V \longrightarrow U$ splits F.

Now assume (π, V) to be an arbitrary absolutely cuspidal ω-representation. Let K be some compact open subgroup of G, V_0 the G-stable subspace generated by V^K. This is finitely generated, hence has an irreducible (absolutely cuspidal) quotient, which is in fact isomorphic to a summand of V_0 by the case we have just dealt with. An induction argument implies that V_0 is a finite direct sum of irreducible absolutely cuspidal representations. If one applies Zorn's Lemma, letting K vary, one obtains:

Proposition 5.4.2. Any absolutely cuspidal ω-representation is a countable direct sum of irreducible absolutely cuspidal representations.

This clearly implies projectivity for all absolutely cuspidal ω-representations.

For injectivity: Let (σ, U) be any smooth ω-representation and $F\colon V \longrightarrow U$ a G-injection. I must construct $\Phi\colon U \longrightarrow V$ splitting F. But now one has a dual map $\widetilde{F}\colon \widetilde{U} \longrightarrow \widetilde{V}$, which splits by what I have just shown, say by a map $\widetilde{\Phi}$. But then one also has $\widetilde{\widetilde{\Phi}}\colon \widetilde{\widetilde{U}} \longrightarrow \widetilde{\widetilde{V}}$ splitting $\widetilde{\widetilde{F}}\colon \widetilde{\widetilde{V}} \longrightarrow \widetilde{\widetilde{U}}$. However, $\widetilde{\widetilde{V}} \cong V$ and one has a canonical embedding of U into $\widetilde{\widetilde{U}}$, so that one may define Φ as $\widetilde{\widetilde{\Phi}}|_U$. $\square$

Corollary 5.4.3. If P is a proper parabolic subgroup of G and (σ, U) is an irreducible admissible representation of M, then no irreducible composition factor of $i_P^G\,\sigma$ is absolutely cuspidal.

Proof. Let V_0 be a G-stable subspace of $i_P^G\,\sigma$, (π, V) an irreducible absolutely cuspidal representation of G, and $F\colon V_0 \longrightarrow V$ a G-morphism. If $F \neq 0$, then F splits by 5.4.1. But this implies that (π, V) is isomorphic to a subspace of $i_P^G\,\sigma$, a contradiction to 5.1.1. $\square$

Corollary 5.4.4. Assume π to be an admissible representation of G of finite length, whose composition series is multiplicity-free and whose irreducible composition factors are absolutely cuspidal. Then π is in fact isomorphic to the sum of its irreducible composition factors.

Proof. Applying 2.1.9, one may assume that for some character ω of Z, each composition factor is an ω-representation. By an induction argument, one is reduced to the case where the length of π is two, i.e., one has an exact sequence

$$0 \longrightarrow \pi_1 \longrightarrow \pi \longrightarrow \pi_2 \longrightarrow 0$$

where π_1 and π_2 are both irreducible absolutely cuspidal ω-representations, and π_1 and π_2 are not isomorphic. Now for any $z \in Z$, the map $v \longmapsto \pi(z)v - \omega(z)v$ from the space of π into itself factors through π_1 and defines in fact a G-morphism from π_2 to π_1, which by assumption must be null. Therefore π itself is an ω-representation, and one can apply 5.4.1 to finish the proof. $\square$

6. Composition series and intertwining operators I

I shall show here among other things that every finitely generated admissible representation has finite length; this will fall out from an analysis of the Jacquet module for a representation V induced from a parabolic subgroup of G. This analysis also relates to G-morphisms between various induced representations and allows one a refinement of the earlier criterion for square-integrability.

6.1. I shall first prove some fairly technical results, related to some results of Bruhat in [11]. Let X be an analytic variety over the p-adic field k, Y an analytic subvariety of X. Suppose H is a k-analytic group acting on X and taking Y to itself, such that

(i) H acts freely and properly on X and
(ii) for each $x \in X$, the map $h \longmapsto h \cdot x$ is an immersion of H.

Then by [9, §6.2], the quotient $H\backslash X$ of X by H exists and is also an analytic variety, and contains the quotient $H\backslash Y$ as an analytic subvariety. The projection from X to $H\backslash X$ makes X into a principal bundle with H as fibre. Local analytic sections of this projection exist everywhere on $H\backslash X$.

Suppose (σ, U) is a smooth representation of H. Define $\mathrm{I}_c^\infty(\sigma|H, X)$ to be the space of locally constant sections, of compact support, of the vector bundle associated to the H-space U and the principal bundle X. Explicitly, this consists of functions $f\colon X \longrightarrow U$ which are locally constant and of compact support modulo H, such that $f(hx) = \sigma(h)f(x)$ for all $h \in H$ and $x \in X$. One may also define spaces $\mathrm{I}_c^\infty(\sigma|H, X \smallsetminus Y)$ and $\mathrm{I}_c^\infty(\sigma|H, Y)$.

Lemma 6.1.1. The sequence

$$0 \longrightarrow \mathrm{I}_c^\infty(\sigma|H, X \smallsetminus Y) \longrightarrow \mathrm{I}_c^\infty(\sigma|H, X) \longrightarrow \mathrm{I}_c^\infty(\sigma|H, Y) \longrightarrow 0$$

is exact.

Proof. Injectivity is clear.

Suppose $f \in \mathrm{I}_c^\infty(\sigma|H, X)$ and $f|Y = 0$. Since f has compact support on X modulo H, and is locally constant, there exists a finite set of compact open sets $X_i \subseteq X$ such that $\mathrm{supp}(f) \subseteq HX_i$ and $f|X_i$ is constant. Since $f|Y = 0$, one may assume $X_i \cap Y = \emptyset$ for each i. Thus, f lies in $\mathrm{I}_c^\infty(\sigma|H, X \smallsetminus Y)$.

Suppose $f \in \mathrm{I}_c^\infty(\sigma|H, Y)$. One has a finite disjoint set of compact open sets $Y_i \subseteq Y$ such that $\mathrm{supp}(f) \subseteq \bigcup HY_i$ and $f|Y_i$ is constant. The projection of each Y_i onto $H\backslash Y$ is open. Because local sections over $H\backslash Y$ exist, we may find disjoint compact sets $Z_j \subseteq Y$ such that

(i) the restriction of the projection from Y to $H\backslash Y$ is an isomorphism on each Z_j;
(ii) $\mathrm{supp}(f) \subseteq \bigcup HZ_j$;
(iii) f is constant on each Z_j.

One may then find disjoint compact open sets $X_j \subseteq X$ such that

(i) the restriction of the projection to each X_j is an isomorphism;
(ii) $X_j \cap Y = Z_j$;
(iii) for each $h \in H$, if $hX_i \cap X_j$ is nonempty, then so is $hY_i \cap Y_j$.

Extend f in the obvious way to X_j, again to HX_j, then to $\bigcup HX_j$. The function in $\mathrm{I}_c^\infty(\sigma|H, X)$ one obtains then has image f in $\mathrm{I}_c^\infty(\sigma|H, Y)$. □

6.2. In this section, let P be any locally compact group such that the compact open subgroups form a basis of neighborhoods of the identity, let Q be a closed subgroup, N a normal closed subgroup with arbitrarily large compact open subgroups, and assume QN closed. In particular, N is unimodular. Let (σ, U) be a smooth representation of Q. Then one has the representation $(\sigma_{Q\cap N}, U_{Q\cap N})$ of $Q/Q\cap N$, which may also be considered as a representation of $QN/N \subseteq P/N$ since $QN/N \cong Q/Q\cap N$.

Let δ be the modulus character of Q on $Q\cap N\backslash N$. It is trivial on $Q\cap N$.

Proposition 6.2.1. One has

$$(\text{c-Ind}_Q^P\,\sigma)_N \cong \text{c-Ind}_{QN/N}^{P/N}\,\sigma_{Q\cap N}\delta.$$

Proof. In several stages:

(i) For $u \in U$, let $\bar{u}$ be its image in $U_{Q\cap N}$. Since any $f \in \text{c-Ind}_Q^P\,\sigma$ has compact support modulo Q, for any $p \in P$ the function $R_p f|_N$ has compact support modulo $Q\cap N$. Also, for $q \in Q\cap N$, $n \in N$, and $p \in P$ one has $\overline{f(qnp)} = \overline{\sigma(q)f(np)} = \overline{f(np)}$. Thus the integral

$$\bar{f}(p) = \int_{Q\cap N\backslash N} \overline{f(np)}\,dn$$

is well-defined.

(ii) For $n \in N$, a change of variable in the integral shows that for all $p \in P$, $\bar{f}(np) = \bar{f}(p)$. Thus, $\bar{f}$ may be considered as a function on P/N.

(iii) For $q \in Q$,

$$\begin{aligned}\bar{f}(qp) &= \int_{Q\cap N\backslash N} \overline{f(nqp)}\,dn \\ &= \int_{Q\cap N\backslash N} \overline{f(q\cdot q^{-1}nq\cdot p)}\,dn \\ &= \sigma_{Q\cap N}(q)\delta(q)\bar{f}(p).\end{aligned}$$

(iv) The function $\bar{f}$ clearly has compact support modulo QN on P.

(v) From (ii)–(iv) one sees that $f \longrightarrow \bar{f}$ is a map from $\text{c-Ind}_Q^P\,\sigma$ to $\text{c-Ind}_{QN/N}^{P/N}\,\sigma_{Q\cap N}$. It is a P-morphism, clearly, and factors through a P/N-morphism from $(\text{c-Ind}_Q^P\,\sigma)_N$.

(vi) Now to see that $f \longmapsto \bar{f}$ is surjective.

Let $x \in P$ be given. Let K be a compact open subgroup of P and u an element of U fixed by $xKx^{-1} \cap Q$. Then $(xKx^{-1} \cap Q)(Q\cap N)$ is the inverse image of the image of $xKx^{-1} \cap Q$ in $Q/(N \cap Q)$; thus, if $K_0 \subseteq K$ is a compact open subgroup small enough so that the image of $(xK_0x^{-1})N \cap Q$ in $Q/(Q \cap N)$ is contained in that of $xKx^{-1} \cap Q$, one has $(xK_0x^{-1})N \cap Q \subseteq (xKx^{-1} \cap Q)(Q \cap N)$. Choose such a K_0. Define a function $\phi\colon P \longrightarrow U_{Q\cap N}$ to be 0 outside $QNxK_0$ and equal to $(\sigma_{Q\cap N}\delta)(q)\bar{u}$ at $qnxk \in QNxK_0$. The space $\text{c-Ind}_{Q/(Q\cap N)}^{P/N} \sigma_{Q\cap N}\delta$, with proper identifications, is spanned by such elements, so it suffices to find $f \in \text{c-Ind}_Q^P \sigma$ with $\bar{f} = \phi$. But now define f to be 0 off QxK_0 and equal to $\sigma(q)u$ at qxk. Then for $nx = qxk$ one has $q = nxk^{-1}x^{-1}$, hence, by assumption on K_0, $q \in (xKx^{-1} \cap Q)(Q \cap N)$, so that $\overline{f(nx)} = \overline{\sigma(q)u} = \bar{u}$. Thus $\bar{f}$ is equal to some scalar multiple of ϕ.

(vii) The map $f \longmapsto \bar{f}$ is injective. For this, we need a few preliminary results:

Lemma 6.2.2. If N_0 is any compact open subgroup of N and K is a compact subgroup of P, then there exists a compact open subgroup of N containing N_0 and normalized by K.

Proof. The set $\bigcup_{k\in K} kN_0k^{-1}$ is compact, hence by the assumption on N contained in some compact open subgroup N_1 of N. The group $\bigcap_{k\in K} kN_1k^{-1}$ contains N_0 and is thus open, but it is also clearly compact and K-stable. □

Corollary 6.2.3. The element $f \in \text{c-Ind}_Q^P \sigma$ has image 0 in $(\text{c-Ind}\,\sigma)_N$ (i.e., lies in $(\text{c-Ind}\,\sigma)(N)$) if and only if for every $p \in P$ there exists a compact open subgroup $N_p \subseteq N$ such that $\int_{N_p} f(np)\,dn = 0$.

Proof. The function f lies in $(\text{c-Ind}\,\sigma)(N)$ if and only if there exists a compact open subgroup $N_0 \subseteq N$ such that for every $p \in P$

$$\int_{N_0} f(pn)\,dn = 0.$$

Therefore, if $f \in (\text{c-Ind}\,\sigma)(N)$ one may choose $N_p = pN_0p^{-1}$.

Conversely, let f be given and assume that for every $p \in P$ there exists an N_p satisfying the conditions in 6.2.3. Choose K compact and open such that $f \in (\text{c-Ind}\,\sigma)^K$. For any $p \in P$, the Lemma implies that one may find a compact open subgroup $N_0 \subseteq N$ such that $\int_{N_0} f(pkn)\,dn = 0$ for every $k \in K$. If X is compact and f has support on QX, one may (again by the Lemma) find a compact open $N_1 \subseteq N$ such that $\int_{N_1} f(xn)\,dn = 0$ for all $x \in X$. But now consider the function $f_{N_1} = \int_{N_1} R_n f\,dn$. Since f has support on QX, f_{N_1} has support on QXN_1. But by the construction of N_1, for any $x \in X$ and $n_1 \in N_1$, $\int_{N_1} f(xn_1n)\,dn = 0$. Hence $f_{N_1} = 0$ and $f \in (\text{c-Ind}\,\sigma)(N_1)$. □

Now to attack the injectivity of $f \longmapsto \bar{f}$. Suppose $\bar{f} = 0$. In particular, $\bar{f}(1) = 0$, which means that $\int_{(Q\cap N)\backslash N} \overline{f(n)}\,dn = 0$. Choose a compact open subgroup N_0 of N

such that the support of $f|_N$ lies in $(Q \cap N)N_0$. Then

$$\int_{N_0} \overline{f(n)}\, dn = \int_{(N_0 \cap Q)\backslash N_0} \int_{N_0 \cap Q} \overline{f(n_1 n_2)}\, dn_1\, dn_2 = 0,$$

which means that $\int_{N_0} f(n)\, dn$ lies in $U(Q \cap N)$. Thus, $\int_{N_0} f(n)\, dn \in U(N_1)$ for some compact open subgroup $N_1 \subseteq Q \cap N$. If one chooses a compact open subgroup $N_0' \subseteq N$ containing $N_1 N_0$, then one has $\int_{N_0'} f(n)\, dn = 0$.

This argument applies to $R_p f$ as well, which guarantees that the condition of 6.2.3 holds, and hence $f \in (\text{c-Ind}\,\sigma)(N)$.

The proof of Proposition 6.2.1 is complete. □

Remark 6.2.4. The map $f \longmapsto \bar{f}$, and therefore the isomorphism of 6.2.1, depends on the choice of a measure on $(Q \cap N)\backslash N$. Otherwise, it is canonical.

6.3. Return to the notational conventions of §§3–5 (so that in particular G is the group of rational points of a reductive group defined over the p-adic field k). Fix a minimal parabolic $P_\emptyset$ and a maximal split torus $A_\emptyset \subseteq P_\emptyset$, and let Δ be the corresponding set of simple roots.

Fix also for a while (through the proof of 6.3.6) subsets $\Theta, \Omega \subseteq \Delta$. Given this choice, let $C(w)$ be the double coset $P_\Theta w P_\Omega$; recall that G is the disjoint union of the $C(w)$ as w ranges over $[W_\Theta\backslash W/W_\Omega]$ (as defined in §1.1). Define a partial ordering on $[W_\Theta\backslash W/W_\Omega]$ as follows: $x < y$ if $C(x)$ is contained in the closure of $C(y)$. Thus, 1 is minimal with respect to this ordering.

For each $w \in [W_\Theta\backslash W/W_\Omega]$, let G_w be the union of the $C(x)$ with $x > w$. It is open in G, and if $x > y$ then $G_x \subseteq G_y$. For any subset $R \subseteq [W_\Theta\backslash W/W_\Omega]$, let G_R be the open set $\bigcup_{w\in R} G_w$. It is also the union of all sets $C(x)$ where $x > w$ for some $w \in R$.

If x is a minimal element of R, then the intersection of G_R with the closure of $C(x)$ is just $C(x)$ itself, which is therefore closed in G_R. A subset $S \subseteq R$ is said to be *minimal* if all its elements are minimal; in this circumstance, the intersection of G_R with the union of the closures of the $C(x)$ $(x \in S)$ is simply the union of the $C(x)$ $(x \in S)$, which is again closed in G_R.

For each $w \in [W_\Theta\backslash W/W_\Omega]$, let $d(w)$ be the dimension of the algebraic variety $P_\Theta\backslash P_\Theta w P_\Omega$ over k. As a particular case of the above definitions, for $n \geq 0$ one may take R to be the set of all w with $d(w) \geq n$. In this case we write G_n for G_R. Of course, $G_{n+1} \subseteq G_n$, and $G = G_0$.

Now let (σ, U) be an admissible representation of M_Θ, and let $I = I(\sigma)$ be $i_{P_\Theta}^G\,\sigma$. For any $w \in [W_\Theta\backslash W/W_\Omega]$ define I_w to be the subspace of $f \in I$ with support on G_w, and similarly define I_R for $R \subseteq [W_\Theta\backslash W/W_\Omega]$. Each I_R is stable under P_Ω. The subspaces I_w define on I a decreasing filtration of P_Ω-spaces indexed by the partially ordered set $[W_\Theta\backslash W/W_\Omega]$. It is difficult to describe the whole space I as a P_Ω-module, which is unfortunately useful in many applications, but it is not so difficult to determine the graded modules associated to the filtration.

First of all, for each $w \in [W_\Theta\backslash W/W_\Omega]$ define J_w to be (in the notation of §6.1) $\mathrm{I}_c^\infty(\sigma\delta_\Theta^{1/2} | P_\Theta, P_\Theta w P_\Omega)$. (Of course this definition depends in reality on the double coset $P_\Theta w P_\Omega$.) Then letting $H = P_\Theta$ one may apply 6.1.1 and the above remarks immediately to obtain:

Proposition 6.3.1. Let R be any subset of $[W_\Theta\backslash W/W_\Omega]$, S a minimal subset of R, and let $R' = \left\{ x \in [W_\Theta\backslash W/W_\Omega] \smallsetminus S \,\middle|\, x > w \text{ for some } w \in R \right\}$. Then the sequence

$$0 \longrightarrow I_{R'} \longrightarrow I_R \longrightarrow \bigoplus_{w\in S} J_w \longrightarrow 0$$

is exact.

In particular one may take $R = S = \{w\}$, and then one obtains the exact sequence

$$0 \longrightarrow \bigoplus_{\substack{x>w \\ x\neq w}} J_x \longrightarrow I_w \longrightarrow J_w \longrightarrow 0.$$

As another special case one may let $R = \left\{ w \,\middle|\, d(w) \geq n \right\}$ and $S = \left\{ w \,\middle|\, d(w) = n \right\}$. In this case I write I_R as I_n. The proposition then gives a decreasing filtration $\{I_n\}$ indexed by $\mathbf{N}$, with

$$I_n/I_{n+1} \cong \bigoplus_{d(w)=n} J_w.$$

Second, the spaces J_w may be described more explicitly. For each $x \in G$, let $x^{-1}(\sigma\delta_\Theta^{1/2})$ be the representation of $x^{-1}P_\Theta x$ on the same space as that of σ which takes p to $\sigma\delta_\Theta^{1/2}(xpx^{-1})$. The isomorphism class of this representation depends only on the coset $P_\Theta x$, but the representation itself will in general depend on the particular x, and it is important to keep this in mind.

Define for each $x \in G$ the representation J_x of P_Ω:

$$J_x = \text{c-Ind}_{x^{-1}P_\Theta x \cap P_\Omega}^{P_\Omega} x^{-1}(\sigma\delta_\Theta^{1/2}).$$

There is a possibility of confusion with previous notation, but it is not too serious:

Proposition 6.3.2. Let w be an element of $[W_\Theta\backslash W/W_\Omega]$, x any element of $P_\Theta w P_\Omega$. The map taking f to ϕ_f, where

$$\phi_f(p) = f(xp),$$

induces an isomorphism of J_w with J_x.

The proof of this is straightforward.

Propositions 6.3.1 and 6.3.2 together provide the description I have already mentioned of the graded P_Ω-module associated to the filtration of $I(\sigma)$ by $[W_\Theta\backslash W/W_\Omega]$. The next step is to determine the corresponding Jacquet modules.

Proposition 6.3.3. Let w be an element of $[W_\Theta\backslash W/W_\Omega]$, $x \in N(A_\emptyset)$ representing w. Then

$$(J_x)_{N_\Omega} \cong \text{c-Ind}_{x^{-1}P_\Theta x\cap M_\Omega}^{M_\Omega} \sigma_{M_\Theta \cap xN_\Omega x^{-1}}\delta^{1/2},$$

where δ is the modulus of the unique rational character of $x^{-1}P_\Theta x \cap M_\Omega$ which restricts to

$$\prod_{\substack{\alpha\in\Sigma^+\setminus\Sigma_\Omega^+ \\ w\alpha\in\Sigma^-\setminus\Sigma_\Theta^-}} \alpha^{m(\alpha)} \qquad \prod_{\substack{\alpha\notin\Sigma^+\setminus\Sigma_\Omega^+ \\ w\alpha\in\Sigma^-\setminus\Sigma_\Theta^-}} \alpha^{-m(\alpha)}$$

on $A_\emptyset$, where $m(\alpha)$ is as defined in §1.6.

I shall first interpret this. The group $P_\emptyset \cap M_\Omega$ is a minimal parabolic of M_Ω and the corresponding set of simple roots is Ω. The group $x^{-1}P_\Theta x \cap M_\Omega$ is the standard parabolic corresponding to $w^{-1}\Theta \cap \Omega$, with radical $x^{-1}N_\Theta x \cap M_\Omega$ and reductive component $x^{-1}M_\Theta x \cap M_\Omega$ (see 1.3.3). Similarly, the group $xP_\Omega x^{-1} \cap M_\Theta$ is a standard parabolic in M_Θ, corresponding to $\Theta \cap w\Omega$, with radical $xN_\Omega x^{-1} \cap M_\Theta$ and reductive component $xM_\Omega x^{-1} \cap M_\Theta$. The representation $\sigma_{M_\Theta\cap xN_\Omega x^{-1}}$ may be considered as a representation of the latter, hence also of $xP_\Omega x^{-1} \cap M_\Theta$. Thus $x^{-1}(\sigma_{M_\Theta\cap xN_\Omega x^{-1}})$ is to be considered as a representation of $x^{-1}M_\Theta x \cap M_\Omega$ or of $x^{-1}P_\Theta x \cap M_\Omega$, and $\text{c-Ind}_{x^{-1}P_\Theta x\cap M_\Omega}^{M_\Omega} x^{-1}(\sigma_{M_\Theta\cap xN_\Omega x^{-1}})\delta^{1/2}$ is at least of a familiar sort if the δ factor makes sense, which I shall show in a moment. In particular, Ind and c-Ind are the same.

The proof is not much longer than the statement. Let $P = P_\Omega$, $Q = x^{-1}P_\Theta x \cap P_\Omega$, $N = N_\Omega$ in 6.2.1. Note that $QN/N \cong Q/(Q\cap N) \cong x^{-1}P_\Theta x \cap M_\Omega$ and that $P/N \cong M_\Omega$. One obtains quickly that

$$(J_x)_{N_\Omega} \cong \text{Ind}_{x^{-1}P_\Theta x\cap M_\Omega}^{M_\Omega} x^{-1}(\sigma_{M_\Theta\cap xN_\Omega x^{-1}})(w^{-1}\delta_\Theta^{1/2})\gamma$$

where γ is the modulus character of $x^{-1}P_\Theta x \cap P_\Omega$ acting on $N_\Omega/(x^{-1}P_\Theta x \cap N_\Omega)$.

It remains only to examine the character $(w^{-1}\delta_\Theta^{1/2})\gamma$ more closely.

The character γ is the norm of the rational character

$$\det \text{Ad}_{\mathfrak{n}_\Omega} / \det \text{Ad}_{\text{Ad}(x^{-1})\mathfrak{p}_\Theta\cap\mathfrak{n}_\Omega}$$

where $\mathfrak{p}_\Theta$ is the Lie algebra of P_Θ, $\mathfrak{n}_\Theta$ that of N_Θ. The character $(w^{-1}\delta_\Theta^{1/2})\gamma$ is thus the square root of the norm of the rational character equal to

$$\prod_{\alpha\in\Sigma^+\setminus\Sigma_\Theta^+} w^{-1}\alpha^{m(\alpha)} \qquad \prod_{\substack{\alpha\in\Sigma^+\setminus\Sigma_\Omega^+ \\ w\alpha\in\Sigma^-\setminus\Sigma_\Theta^-}} \alpha^{2m(\alpha)}$$

on $A_\emptyset$ (and by 1.6.1 this determines it as a rational character of $x^{-1}P_\Theta x \cap M_\Omega$). This in turn is equal to

$$\prod_{\alpha\in\Sigma^-\setminus\Sigma^-_\Theta} \alpha^{-m(\alpha)} \prod_{\substack{\alpha\in\Sigma^+\setminus\Sigma^+_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{2m(\alpha)}$$

$$= \prod_{\substack{\alpha\in\Sigma^+\setminus\Sigma^+_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\notin\Sigma^+\setminus\Sigma^+_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{-m(\alpha)}$$

which is the formula in the proposition. □

Corollary 6.3.4. Suppose σ is absolutely cuspidal, $w \in [W_\Theta\backslash W/W_\Omega]$, and x is an element of $N(A_\emptyset)$ representing w. Then

(a) one has $(J_x)_{N_\Omega} = 0$ unless $w^{-1}\Theta \subseteq \Omega$;
(b) when $w^{-1}\Theta \subseteq \Omega$, one has

$$(J_x)_{N_\Omega} \cong (i^{M_\Omega}_{x^{-1}P_\Theta x\cap M_\Omega}\, x^{-1}\sigma)\delta_\Omega^{1/2}.$$

Proof. Part (a) is easy, since if σ is absolutely cuspidal then $\sigma_{M_\Theta\cap xN_\Omega x^{-1}} = 0$ unless $M_\Theta \cap xN_\Omega x^{-1}$ is trivial, and this is equivalent to the inclusions $M_\Theta \subseteq xM_\Omega x^{-1}$ or $w^{-1}\Theta \subseteq \Omega$.

Part (b) is a matter of computing the δ-factor correctly. By the definition of normalized induction, what must be shown is that

$$(J_x)_{N_\Omega} \cong \mathrm{Ind}^{M_\Omega}_{x^{-1}P_\Theta x\cap M_\Omega}(x^{-1}\sigma)\delta^{1/2},$$

where δ is now the norm of the rational character which restricts to

$$\prod_{\alpha\in\Sigma^+\setminus w^{-1}\Sigma^+_\Theta} \alpha^{m(\alpha)}$$

on $A_\emptyset$. Now if $w^{-1}\Theta \subseteq \Omega$ then $w^{-1}\Sigma^+_\Theta \subseteq \Sigma^+_\Omega$ and $w^{-1}\Sigma^-_\Theta \subseteq \Sigma^-_\Omega$. Hence for $\alpha \in \Sigma^+ \setminus \Sigma^+_\Omega$, $w\alpha \in \Sigma^+ \setminus \Sigma^+_\Theta$ or $\Sigma^- \setminus \Sigma^-_\Theta$, and according to 6.3.3 the proper δ-factor is the norm of

$$\prod_{\substack{\alpha\in\Sigma^+\setminus\Sigma^+_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\in\Sigma^-\setminus\Sigma^-_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{-m(\alpha)} \prod_{\substack{\alpha\in\Sigma_\Omega \\ w\alpha\in\Sigma^-\setminus\Sigma^-_\Theta}} \alpha^{-m(\alpha)}$$

(note that there is no $\alpha \in \Sigma^+_\Omega$ such that $w\alpha \in \Sigma^- \setminus \Sigma^-_\Theta$)

$$= \prod_{\alpha\in\Sigma^+\setminus\Sigma^+_\Omega} \alpha^{m(\alpha)} \prod_{\substack{\alpha\in\Sigma^+_\Omega \\ w\alpha\in\Sigma^+\setminus\Sigma^+_\Theta}} \alpha^{m(\alpha)}$$

$$= \prod_{\alpha\in\Sigma^+\setminus\Sigma^+_\Omega} \alpha^{m(\alpha)} \prod_{\alpha\in\Sigma^+_\Omega\setminus w^{-1}\Sigma^+_\Theta} \alpha^{m(\alpha)}. \quad \square$$

The next result summarizes what the results so far say about the space I_{N_Ω} when σ is absolutely cuspidal.

Theorem 6.3.5. Let Θ, Ω be subsets of Δ, and let σ be an absolutely cuspidal representation of M_Θ, $I = i_{P_\Theta}^G \sigma$. There exists a filtration

$$0 \subseteq I_{n_\ell} \subseteq \cdots \subseteq I_0 = I$$

by P_Ω-stable subspaces such that $(I_n/I_{n+1})_{N_\Omega} \cong (I_n)_{N_\Omega}/(I_{n+1})_{N_\Omega}$ is isomorphic to the direct sum $\bigoplus (J_w)_{N_\Omega}$, the sum ranging over $w \in [W_\Theta \backslash W / W_\Omega]$ with $d(w) = n$. Furthermore:

$$(J_w)_{N_\Omega} \cong \begin{cases} 0 & \text{if } w^{-1}\Theta \not\subseteq \Omega \\ i_{w^{-1}P_\Theta w \cap M_\Omega}^{M_\Omega} w^{-1}\sigma & \text{if } w^{-1}\Theta \subseteq \Omega. \end{cases}$$

The use of elements of W rather than representing elements in $N(A_\emptyset)$ is justified because it is only a matter of isomorphism class.

The only new ingredient here is that a filtration of I gives one of I_{N_Ω} as well, and in such a way that $\mathrm{Gr}(I_{N_\Omega}) \cong \mathrm{Gr}(I)_{N_\Omega}$. This follows from the exactness of the functor (3.2.3).

Theorem 6.3.6. Let Θ, Ω be subsets of Δ, and let σ be an irreducible absolutely cuspidal representation of M_Θ. If π is an irreducible composition factor of $i_{P_\Theta}^G \sigma$ and ρ an irreducible absolutely cuspidal representation of M_Ω such that π may be embedded into $i_{P_\Omega}^G \rho$, then there exists $w \in W(\Theta, \Omega)$ (i.e., $w \in N(A_\emptyset)$ such that $w^{-1}M_\Theta w = M_\Omega$) with $w^{-1}\sigma \cong \rho$.

Proof. The assumption on π implies, by Frobenius reciprocity (3.2.4), that there exists a non-trivial M_Ω-morphism from π_{N_Ω} to $\rho\delta_\Omega^{1/2}$. Therefore, $\rho\delta_\Omega^{1/2}$ occurs as a composition factor of I_{N_Ω} (where $I = i_{P_\Theta}^G \sigma$), hence of some $(J_w)_{N_\Omega}$ with $w^{-1}\Theta \subseteq \Omega$ (6.3.4). But $(J_w)_{N_\Omega}$ is isomorphic to $i_{w^{-1}P_\Theta w \cap M_\Theta}^{M_\Theta} w^{-1}\sigma\delta_\Omega^{1/2}$, so that by 5.4.3 if $\rho\delta_\Omega^{1/2}$ is a composition factor then $w^{-1}\Theta = \Omega$ and $\rho \cong w^{-1}\sigma$. $\square$

Corollary 6.3.7. Let Θ be a subset of Δ, and let σ be an irreducible, absolutely cuspidal representation of M_Θ. Then $i_{P_\Theta}^G \sigma$ has finite length. If π is an irreducible composition factor then there exist $\Omega \subseteq \Delta$ and $w \in W(\Theta, \Omega)$ such that π has an embedding into $i_{P_\Omega}^G w^{-1}\sigma$.

Proof. If π is an irreducible composition factor, then according to 5.1.2 there exist $\Omega \subseteq \Delta$ and an irreducible, absolutely cuspidal representation ρ of M_Ω with $\pi \subseteq i_{P_\Omega}^G \rho$. Apply 6.3.6 to conclude the proof of the second claim.

To see that $i\,(\sigma)$ has finite length: If $U \subsetneq V$ are G-stable subspaces of $I = i\,(\sigma)$, then there exists a finitely generated non-trivial subspace of V/U, and by Zorn's Lemma an irreducible quotient of that. Therefore if one is given an ascending chain $I_1 \subseteq I_2 \ldots$ one can, if necessary, replace it by another $I_1 \subseteq J_1 \subseteq I_2 \subseteq J_2 \subseteq \ldots$ where

J_i/I_i is irreducible. For any $\Omega \subseteq \Delta$ one has a corresponding chain of Jacquet modules $(I_i)_{N_\Omega} \subseteq (J_i)_{N_\Omega}$, and $(J_i)_{N_\Omega}/(I_i)_{N_\Omega} \cong (J_i/I_i)_{N_\Omega}$ by 3.2.3. By 6.3.5ff, this chain is of finite length. If the original chain were infinite, then by 5.4.3 there would be some $\Omega \subseteq \Delta$ for which the chain of associated Jacquet modules is of infinite length, and this would be a contradiction. A similar argument shows that the descending chain condition holds, so that I has a Jordan-Hölder composition series. □

Corollary of the Corollary 6.3.8. Let P be any parabolic subgroup of G. If σ is any admissible representation of M of finite length, then $i_P^G\,\sigma$ has finite length.

Proof. See 2.4.4 and 2.4.5. □

When σ is irreducible and absolutely cuspidal, one can obtain easily from the proof of 6.3.7 that the length of $i_{P_\Theta}^G\,\sigma$ is bounded by the product of the number of associates of Θ and the order of $W(\Theta,\Theta)$, but a stronger bound will be given later. When $\Theta = \emptyset$, however, the bound here is already the best possible one that can be obtained by these techniques:

Corollary 6.3.9. If σ is an irreducible, admissible (hence, finite-dimensional) representation of $M_\emptyset$, then

(a) the length of $i_{P_\emptyset}^G\,\sigma$ is at most the order of the Weyl group;
(b) if π is an irreducible composition factor of $i_{P_\emptyset}^G\,\sigma$ then there exists $w \in W$ and an embedding of π into $i_{P_\emptyset}^G\,w\sigma$.

This was proven independently by Matsumoto [27] and Silberger [30] when σ is unramified. Other results of this section have been proved independently by Bernstein and Zelevinskii [2].

Theorem 6.3.10. Any finitely generated admissible representation of G has finite length.

Proof. Let π be the given representation. By 2.1.9, there exists a finite composition series for π, each factor of which is an ω-representation for some character ω of Z_G. Thus, one may assume π itself to be an ω-representation.

Proceed by induction on the rank of G^{der}. If this is 0, then G^{der} is compact and any finitely generated admissible representation is finite-dimensional. For arbitrary G, let $P_1, \ldots, P_n$ be representatives of all the conjugacy classes of maximal proper parabolics of G, and $\sigma_i = \pi_{N_i}$. By 3.3.1, each of these is finitely generated and admissible, hence by the induction assumption each is of finite M_i-length. But then by 6.3.8 each $I_i = i_{P_i}^G\,\sigma_i$ has finite length. By Frobenius reciprocity there exists a canonical G-morphism from π into each I_i, hence one into $\bigoplus I_i$. It remains to show that the kernel of this map has finite length. This kernel, however, is absolutely cuspidal, by remarks made in §3.2, and by 5.4.1 a summand of π. Therefore it is finitely generated, and by 5.4.2 a finite direct sum of irreducible absolutely cuspidal representations. □

To conclude this section, I mention a sort of converse to 6.3.7 (due, I imagine[13], to Harish-Chandra):

Theorem 6.3.11. Suppose $\Theta, \Omega \subseteq \Delta$ to be associate, σ a finitely generated absolutely cuspidal representation of M_Θ, $w \in W(\Theta, \Omega)$. Then the irreducible composition factors of $i_{P_\Theta}^G \sigma$ and $i_{P_\Theta}^G w^{-1}\sigma$ are the same.

Proof. There are many possible proofs, but the briefest is to observe that the characters are the same (see 2.3.3, [22, Theorem 9.2(iii)], and [19]). □

6.4. Let the notation be as in §6.3. Let (σ, U) be an irreducible absolutely cuspidal representation of M_Θ, $I = i_{P_\Theta}^G \sigma$. Because of Frobenius reciprocity, the structure of I_{N_Ω} is related to G-morphisms from I to representations induced from P_Ω. I want to make this more explicit.

Define $W(\sigma)$ to be $\left\{ w \in W(\Theta, \Theta) \,\middle|\, w\sigma \cong \sigma \right\}$ and call σ *regular* if $W(\sigma) = \{1\}$. (Indeed, for this definition σ need not be absolutely cuspidal.)

Proposition 6.4.1. Assume that σ is regular and that Ω is an associate of Θ. Then I_{N_Ω} is isomorphic to the direct sum

$$\bigoplus_{w \in W(\Theta, \Omega)} (w^{-1}\sigma)\delta_\Omega^{1/2}.$$

It is allowable to use elements of W rather than representatives in G because it is only a matter of isomorphism.

Proof. The proposition follows from 6.3.5, 6.3.6, and 5.4.4. □

In other words, the filtration of I_{N_Ω} by the $(I_n)_{N_\Omega}$ splits, and I_{N_Ω} is isomorphic to the associated graded representation, when σ is regular. (It is not necessary that σ be regular in order for 6.4.1 to hold, but there are certainly examples where σ is not regular and 6.4.1 does not hold. See §9.)

For the remainder of §6.4, assume σ to be regular.

Let me be more precise and restate slightly results from §§6.1–6.3. Let Ω be an associate of Θ. Recall from §6.3 that G_w is then the union $\bigcup_{x > w} P_\Theta x P_\Omega$, that I_w is the subspace of all $f \in I$ with support in G_w, and that $J_w = \mathrm{I}_c^\infty(\sigma\delta_\Theta^{1/2} | P_\Theta, P_\Theta w P_\Omega)$. Restriction determines a canonical P_Ω-morphism from I_w to J_w, which is surjective. Its kernel is $\sum J_x$ ($x > w$, $x \neq w$). This in turn determines an M_Ω-morphism from $(I_w)_{N_\Omega}$ to $(J_w)_{N_\Omega}$. Recall that $(J_w)_{N_\Omega} = 0$ unless $w \in W(\Theta, \Omega)$, and that if $w \in W(\Theta, \Omega)$ then there exists an isomorphism of $(J_w)_{N_\Omega}$ with $(w^{-1}\sigma)\delta_\Omega^{1/2}$. I want

[13] Can we find out?

to describe this isomorphism explicitly because there are two choices to be made in obtaining it which I want to keep track of.

The first choice is of an element $x \in N(A_\emptyset)$ representing w. This gives the isomorphism of J_w with J_x given by $f \longmapsto \phi_f$. The second choice is of a measure dn on $(x^{-1}N_\Theta x \cap N_\Omega)\backslash N_\Omega$, which is isomorphic to the group $N_{w^{-1}}$ (see §1.3). This determines an isomorphism of $(J_x)_{N_\Omega}$ with $(x^{-1}\sigma)\delta_\Omega^{1/2}$: it takes $\phi \in J_x$ to

$$\int_{(x^{-1}N_\Theta x\cap N_\Omega)\backslash N_\Omega} R_n\phi\, dn.$$

Let $\Lambda_{x^{-1},dn}\colon (I_w)_{N_\Omega} \to (J_x)_{N_\Omega} \longrightarrow (x^{-1}\sigma)\delta_\Omega^{1/2}$ be the composition of the various maps. Proposition 6.4.1 then says two things:

(1) the morphism $\Lambda_{x^{-1},dn}$ splits uniquely, so that $(x^{-1}\sigma)\delta_\Omega^{1/2}$ occurs as a subrepresentation of $(I_w)_{N_\Omega}$;
(2) the inclusion of $(x^{-1}\sigma)\delta_\Omega^{1/2} \subseteq (I_w)_{N_\Omega}$ into I_{N_Ω} also splits, so that there exists a unique extension of $\Lambda_{x^{-1},dn}$ to all of $(I)_{N_\Omega}$. This, and the corresponding map from I to $(x^{-1}\sigma)\delta_\Omega^{1/2}$, I shall also call $\Lambda_{x^{-1},dn}$.

There is a relationship with results of Bruhat in [11]. Let $C_c^\infty(G,U)$ be the space of all locally constant functions from G to U of compact support. There is an almost canonical G-surjection from $C_c^\infty(G,U)$ to $i_{P_\Theta}^G\,\sigma$

$$\mathcal{P}_\sigma f(g) = \int_{P_\Theta} \sigma^{-1}\delta^{1/2}(p) f(pg)\, dp.$$

The composition

$$C_c^\infty(G,U) \longrightarrow I \longrightarrow I_{N_\Omega} \longrightarrow (x^{-1}\sigma)\delta_\Omega^{1/2}$$

corresponds to a distribution on G with values in $\mathrm{Hom}_{\mathbf{C}}(U,U)$ satisfying certain equations concerning left and right multiplication by elements of P_Θ and P_Ω. Viewed in these terms, the content of 6.4.1 is that a certain distribution defined initially only on G_w actually extends covariantly to all of G.

Fix now measures on each N_α ($\alpha \in \Sigma$) and hence on each product $\prod N_\alpha$. I will drop the subscripts referring to measure from now on.

For any $x \in N(A_\emptyset)$ with image $w \in W(\Theta,\Omega)$ define $T_{x^{-1}}(\sigma)$ (often just $T_{x^{-1}}$) to be the G-morphism from $i_{P_\Theta}^G\,\sigma = I(\sigma)$ to $i_{P_\Omega}^G\, x^{-1}\sigma$ corresponding to the M_Ω-morphism $\Lambda_{x^{-1}}\colon I(\sigma)_{N_\Omega} \longrightarrow (x^{-1}\sigma)\delta_\Omega^{1/2}$ (recall that σ is assumed regular). On I_w, $\Lambda_{x^{-1}}$ is defined by the formula

$$\Lambda_{x^{-1}}(f) = \int_{N_{w^{-1}}} f(xn)\, dn,$$

and in fact it may be defined by this formula on a somwhat larger space. For each $w \in [W_\Theta\backslash W/W_\Omega]$ let G_w^{**} be the complement in G of the closure of $P_\Theta w P_\Omega$ and let G_w^* be the union of this with $P_\Theta w P_\Omega$ itself. Define I_w^{**} and I_w^* to be the subspaces of $I(\sigma)$ consisting of functions with support on G_w^{**} and G_w^*, respectively. Because $P_\Theta w P_\Omega$ is closed in G_w^*, the restriction of any $f \in I_w^*$ to $P_\Theta w P_\Omega$ has compact support modulo P_Θ. This remark together with 6.3.1 implies:

Lemma 6.4.2. For any $f \in I_w^*$,

$$\Lambda_{x^{-1}}(f) = \int_{N_{w^{-1}}} f(xn)\, dn.$$

The map $\Lambda_{x^{-1}}$ induces an isomorphism of I_w^*/I_w^{**} with J_x.

For the next results, recall first of all from 1.3.2 that if u and v are elements of W with $\ell(uv) = \ell(u) + \ell(v)$ and $x \in N(A_\emptyset)$ represents u, then the map

$$(N_u, N_v) \longrightarrow xN_v x^{-1} N_u$$

is a bijection of $N_u \times N_v$ with N_{uv}. Recall, second, from §1.2 the notion of *height* and its connection with length in W. For any $x \in N(A_\emptyset)$ representing $w \in W$, let $\delta_\alpha(x)$ be the Radon derivative or modulus factor of the transformation $\mathrm{Ad}(x)\colon N_\alpha \longrightarrow N_{w\alpha}$, so that $d(xnx^{-1}) = \delta_\alpha(x)\, dn$.

Lemma 6.4.3. Assume that Θ, Ω, and Ξ are associates in Δ, $u \in W(\Theta, \Omega)$, $v \in W(\Omega, \Xi)$ with $\mathrm{ht}(uv) = \mathrm{ht}(u) + \mathrm{ht}(v)$, and $x \in N(A_\emptyset)$ representing u. Then $T_{x^{-1}}$ takes I_{uv}^* and I_{uv}^{**} to I_v^* and I_v^{**}, respectively.

Proof. By 1.3.5, $P_\Theta uv P_\Xi = P_\Theta u P_\Omega \cdot P_\Omega v P_\Xi$, and similarly for their closures. For f to be in I_{uv}^{**} means that $f = 0$ on $\overline{P_\Theta uv P_\Xi}$. Hence for any $g \in \overline{P_\Omega v P_\Xi}$, $R_g f = 0$ on $\overline{P_\Theta u P_\Omega}$, so that $R_g f$ lies in I_u^{**} and

$$T_{x^{-1}} f(g) = \Lambda_{x^{-1}}(R_g f) = \int_{N_{u^{-1}}} f(xng)\, dn.$$

However, the restriction of $R_g f$ to $N_{u^{-1}} \cap P_\Theta u P_\Omega$ is zero and so is this integral, so that $T_{x^{-1}} f \in I_v^{**}$. Something similar works for I^*. $\square$

Theorem 6.4.4. Assume that Θ, Ω, and Ξ are associates in Δ, let $u \in W(\Theta, \Omega)$ and $v \in W(\Omega, \Xi)$ be such that $\mathrm{ht}(uv) = \mathrm{ht}(u) + \mathrm{ht}(v)$, and let $x, y \in N(A_\emptyset)$ represent u and v. Then

$$T_{y^{-1}} T_{x^{-1}} = \prod_{\substack{\alpha > 0 \\ u\alpha < 0}} \delta_\alpha(y^{-1})^{-1} T_{y^{-1}x^{-1}}.$$

Proof. Since all G-morphisms from $i_P^G\, \sigma$ to $i_P^G\,(y^{-1}x^{-1}\sigma)$ are scalar multiples of $T_{y^{-1}x^{-1}}$, it suffices to prove that for all $f \in i_P^G\, \sigma$

$$T_{y^{-1}} T_{x^{-1}} f(1) = \prod_{\substack{\alpha > 0 \\ u\alpha < 0}} \delta_\alpha(y^{-1})^{-1} T_{y^{-1}x^{-1}} f(1),$$

or that

$$\Lambda_{y^{-1}}(T_{x^{-1}}f) = \prod_{\substack{\alpha>0\\ u\alpha<0}} \delta_\alpha(y^{-1})^{-1}\Lambda_{y^{-1}x^{-1}}(f).$$

Furthermore, it suffices to prove this only for $f \in I^*_{uv}$. But for these (applying 6.4.3):

$$\begin{aligned}\Lambda_{y^{-1}}(T_{x^{-1}}f) &= \int_{N_{v^{-1}}} (T_{x^{-1}}f)(yn_1)\,dn_1\\ &= \int_{N_{v^{-1}}} \Lambda_{x^{-1}}(R_{yn_1}f)\,dn_1\\ &= \int_{N_{v^{-1}}} dn_1 \int_{N_{u^{-1}}} f(xn_2yn_1)\,dn_2\end{aligned}$$

which by 1.3.2 and the definition of the δ_α is

$$\begin{aligned}&\prod_{\substack{\alpha>0\\ u\alpha<0}} \delta_\alpha(y^{-1})^{-1}\int_{N_{v^{-1}u^{-1}}} f(xyn)\,dn\\ &= \prod_{\substack{\alpha>0\\ u\alpha<0}} \delta_\alpha(y^{-1})^{-1}\Lambda_{y^{-1}x^{-1}}(f). \quad \square\end{aligned}$$

Remark 6.4.5. The operators $T_{x^{-1}}(\sigma)$ depend analytically on σ. Recall first of all that the set of unramified characters $X_{\rm nr}(M)$ has a complex analytic structure (§1.6) and hence so does the set X_σ of representations of M of the form $\sigma\chi$, $\chi \in X_{\rm nr}(M)$. The group $W(\Theta,\Theta)$ acts analytically on this, and the set of regular representations $X_\sigma^{\rm reg}$ in this family is the complement of an analytic subset. The family $i_{P_\Theta}^G(\sigma\chi)$ is an analytic family over X_σ (§3.4) and so is the restriction to $X_\sigma^{\rm reg}$. Furthermore, for $\Omega \subseteq \Delta$, the family $i\,(\sigma\chi)_{N_\Omega}$ is an analytic family of admissible representations of M_Ω over X_σ, and all the filtrations of $I_w(\sigma\chi)$ are also analytic. Over $X_\sigma^{\rm reg}$ one has an analytic splitting of $I(\sigma\chi)_{N_\Omega}$ as a direct sum of families isomorphic to $w^{-1}(\sigma\chi)\delta_\Omega^{1/2}$ ($w \in W(\Theta,\Theta)$) and hence the projections $\Lambda_{x^{-1}}$ are analytic as well. In more down-to-earth terms: for a fixed $f \in C_c^\infty(G,U)$ the image of f under the composition

$$C_c^\infty(G,U) \xrightarrow{P_{\sigma\chi}} I(\sigma\chi) \longrightarrow I(\sigma\chi)_{N_\Omega} \xrightarrow{\Lambda_{x^{-1}}} U$$

varies holomorphically with χ.

6.5. One consequence of the results so far in §6 is a refinement of the earlier criterion for square-integrability. Continue the previous notation.

Theorem 6.5.1. Suppose that π is irreducible and embedded in $i_{P_\Theta}^G\,\sigma$, where σ is an absolutely cuspidal representation of M_Θ. Then in order for π to be square-integrable mod Z_G it is necessary and sufficient that

(a) $\pi|_{A_\Delta}$ is unitary and
(b) for every $\Omega \subseteq \Delta$ associate to Θ and every central character χ of π with respect to P_Ω, $|\chi\delta^{-1/2}(a)| < 1$ for all $a \in A_\Omega^- \smallsetminus A_\emptyset(\mathcal{O})A_\Delta$.

Proof. According to 6.3.5, if $\Omega \subseteq \Delta$ does not contain an associate of Θ then $\pi_{N_\Omega} = 0$. If Ω does contain an associate of Θ, say Ξ, then 6.3.5 implies that any central character of π with respect to P_Ω is also one for P_Ξ. Apply 4.4.6. $\square$

Note that the case $\Theta = \emptyset$ is particularly simple.

6.6. I include here some relatively elementary consequences about irreducibility of representations induced from parabolic subgroups. We retain the earlier notation.

The first result is due to Bruhat [11].

Theorem 6.6.1. If σ is an irreducible, unitary, regular admissible representation of M_Θ (not necessarily absolutely cuspidal) then $i_{P_\Theta}^G\,\sigma$ is irreducible.

Proof. Because σ is unitary, so is $I = i\,(\sigma)$. Any G-subspace is therefore a summand, and it has only to be shown that $\mathrm{End}_G(I) \cong \mathbf{C}$, or, by Frobenius reciprocity, that I_{N_Θ} contains $\sigma\delta_\Theta^{1/2}$ exactly once in its composition series. Now Proposition 6.3.3 gives a composition series for I_{N_Θ} whose factors are indexed by $[W_\Theta\backslash W/W_\Omega]$ but are not necessarily irreducible: to $w \in [W_\Theta\backslash W/W_\Omega]$, one associates the factor

$$(J_w)_{N_\Theta} \cong \mathrm{Ind}_{w^{-1}P_\Theta w\cap M_\Theta}^{M_\Theta}\,(w^{-1}\sigma_{M_\Theta\cap wN_\Theta w^{-1}})\delta^{1/2}$$

where δ is the modulus of the rational character γ of $w^{-1}M_\Theta w \cap M_\Theta$ which restricts to

$$\prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma^-\smallsetminus\Sigma_\Theta^-}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\notin\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma^-\smallsetminus\Sigma_\Theta^-}} \alpha^{-m(\alpha)}$$

on $A_\emptyset$.

If $\sigma\delta_\Theta^{1/2}$ is to occur as a composition factor, then since $\sigma|_{A_\Theta}$ is unitary, this rational character must be γ_Θ itself. Therefore:

$$\begin{aligned}\prod_{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+} \alpha^{m(\alpha)} &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma^-\smallsetminus\Sigma_\Theta^-}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\notin\Sigma^-\smallsetminus\Sigma_\Theta^-\\ w\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+}} \alpha^{m(\alpha)} \\ &= \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma^-\smallsetminus\Sigma_\Theta^-}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+}} \alpha^{m(\alpha)} \prod_{\substack{\alpha\in\Sigma_\Theta\\ w\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+}} \alpha^{m(\alpha)}\end{aligned}$$

or

$$\prod_{\substack{\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+\\ w\alpha\in\Sigma_\Theta}} \alpha^{m(\alpha)} = \prod_{\substack{\alpha\in\Sigma_\Theta\\ w\alpha\in\Sigma^+\smallsetminus\Sigma_\Theta^+}} \alpha^{m(\alpha)}.$$

But since roots in Σ_Θ are trivial on A_Θ, the right-hand side is also. The left-hand side must therefore be trivial, which happens if and only if the set $\left\{ \alpha \in \Sigma^+ \setminus \Sigma_\Theta^+ \,\middle|\, w\alpha \in \Sigma_\Theta \right\}$ is empty. Thus $w^{-1}\Sigma_\Theta \subseteq \Sigma_\Theta$, and since $w^{-1}\Theta > 0$, $w^{-1}\Theta = \Theta$.

The assumption of regularity now implies that $w = 1$, and this concludes the proof. □

Assume now that σ is an irreducible, absolutely cuspidal, regular admissible representation of M_Θ (not necessarily unitary). If $w = w_\ell w_{\ell,\Theta}$, then $w\Theta = \bar{\Theta}$, the conjugate of Θ in Δ; if $x \in N(A_\emptyset)$ represents w, then $T_{x^{-1}}(x\sigma)T_x(\sigma)$ is a G-morphism from $i\,(\sigma)$ to itself, hence a scalar multiple, say $\gamma(\sigma)$, of the identity.

Theorem 6.6.2. The representation $i\,(\sigma)$ is irreducible if and only if $\gamma(\sigma) \neq 0$.

Proof. Note that no T_x is trivial, since it is constructed by Frobenius reciprocity from a non-trivial M_Θ-morphism.

If $\gamma(\sigma) = 0$, then $T_x(\sigma)$ cannot be an isomorphism, since otherwise $T_{x^{-1}}(x\sigma)$ would be trivial. Therefore either the kernel of $T_x(\sigma)$ is a non-trivial subspace of $i\,(\sigma)$, in which case of course $i\,(\sigma)$ is reducible, or its image is a non-trivial subspace of $i\,(x\sigma)$, in which case the latter is reducible. We then apply 6.3.11 to see that $i\,(\sigma)$ is also reducible.

If $i\,(\sigma)$ is reducible, then it has a non-trivial irreducible quotient, say π, which by 6.3.7 may be embedded in some $i\,(y^{-1}\sigma)$, where $y \in N(A_\emptyset)$ represents an element u of some $W(\Theta, \Omega)$ for some $\Omega \subseteq \Delta$. The G-morphism

$$i\,(\sigma)\colon \pi \longrightarrow i\,(y^{-1}\sigma)$$

must be a scalar multiple of $T_{y^{-1}}$, and is neither trivial nor an isomorphism. Now according to 1.2.9, $\mathrm{ht}(w) = \mathrm{ht}(wu) + \mathrm{ht}(u^{-1})$, so that by 6.4.4, T_x and $T_{xy}T_{y^{-1}}$ agree up to a nonzero scalar. Therefore T_x is likewise neither trivial nor an isomorphism. But then $T_{x^{-1}}(x\sigma)T_x(\sigma)$ cannot be an isomorphism, so $\gamma(\sigma) = 0$. □

Note that by 6.4.5 the condition $\gamma(\sigma) = 0$ is analytic. Furthermore, since by 6.6.1 $\gamma(\sigma) \neq 0$ for unitary σ and since there will exist $\chi \in X_{\mathrm{nr}}(M_\Theta)$ such that $\sigma\chi$ is unitary, the representations $i\,(\sigma)$ are generically irreducible.

7. Composition series and intertwining operators II

In this section I give refinements of the main theorems of §6. Fix a minimal parabolic $P_\emptyset$.

The results of §§7.1–7.2 are entirely due to Harish-Chandra (correspondence, 1972).

7.1. Let P be a maximal proper parabolic, say corresponding to $\Theta = \Delta \smallsetminus \{\alpha\}$. Let $\bar{P}$ be the unique standard parabolic conjugate to P^-, the opposite of P. It will correspond to the subset $\bar{\Theta} = -w_\ell\Theta$, where w_ℓ is the longest element in W. If $w_{\ell,\Theta}$ is the longest element in W_Θ, then this subset is also $w_\ell w_{\ell,\Theta}\Theta$. The element $w = (w_\ell w_{\ell,\Theta})^{-1}$ lies in $W(\Theta, \bar{\Theta})$. There are two cases which must be considered: either $P = \bar{P}$, in which case P is said to be self-dual, or $P \neq \bar{P}$. In either case, $\bar{P}$ is associate to P by w, and $\bar{P}$ is the only associate of P by 1.2.3. In the first case, $W(\Theta, \Theta) = \{1, w\}$ while in the second $W(\Theta, \bar{\Theta}) = \{w\}$.

The split torus A/A_Δ is one-dimensional.

Let σ be an irreducible absolutely cuspidal representation of M. The restriction of σ to A will be a scalar character. Let $I = i_P^G\,\sigma$.

Lemma 7.1.1. (a) If $P = \bar{P}$, then I_N fits into an exact sequence

$$0 \longrightarrow (w^{-1}\sigma)\delta_P^{1/2} \longrightarrow I_N \longrightarrow \sigma\delta_P^{1/2} \longrightarrow 0;$$

(b) If $P \neq \bar{P}$, then $I_N \cong \sigma\delta_P^{1/2}$ and $I_{\bar{N}} \cong (w^{-1}\sigma)\delta_{\bar{P}}^{1/2}$.

Proof. This follows from 6.3.5. □

Corollary 7.1.2. The length of I is at most 2.

Proof. Suppose one has a composition series $0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq I_3 = I$. According to 6.3.7 and 1.2.3, if U is I_1, I_2/I_1, or I_3/I_2, then by 3.2.4 either U_N or $U_{\bar{N}}$ is nonzero. This is a contradiction, by 7.1.1 and 3.2.3. □

Proposition 7.1.3. Suppose that $\sigma|_A$ is unitary and that $|\sigma(a)| < 1$ for $a \in A^- \smallsetminus A_\emptyset(\mathcal{O})A_\Delta$. Then any proper subrepresentation $\pi \subseteq I$ is square-integrable mod Z_G.

Proof. First suppose $P = \bar{P}$. Since the quotient I/π must have $(I/\pi)_N \neq 0$ by 6.3.7 and 1.2.3, and there exists a non-trivial map from π_N to $\sigma\delta_P^{1/2}$ by Frobenius reciprocity, one has $\pi_N = \sigma\delta_P^{1/2}$ by 7.1.1(a). The condition of 6.5.1 is thus satisfied.

Suppose $P \neq \bar{P}$. Here, similar reasoning shows that $\pi_N \cong \sigma\delta_P^{1/2}$ and $\pi_{\bar{N}} = 0$, so once again the condition of 6.5.1 holds. □

Theorem 7.1.4. Suppose $P \neq \bar{P}$. Then I is irreducible.

Proof. According to 5.2.5, one may as well assume that $I|_{A_\Delta}$, hence $\sigma|_{A_\Delta}$, is unitary. Since I is irreducible if and only if $\widetilde{I}$ is, one may also assume that $|\sigma(a)| < 1$ for $a \in A^- \setminus A_\emptyset(\mathcal{O})A_\Delta$. Then by 7.1.3 and its proof, if π is an irreducible subspace of I, one has $\pi_N \neq 0$ and $\pi_{\bar{N}} = 0$, and π is square-integrable, hence unitary. Thus π is isomorphic to the conjugate of $\widetilde{\pi}$. By 4.2.5, however, $\widetilde{\pi}_{N^-}$ is isomorphic to the contragredient of π_N, hence is nonzero. But since N^- is conjugate to $\bar{N}$, this implies that $\widetilde{\pi}_{\bar{N}} \neq 0$, hence the conjugate of $\widetilde{\pi}_{\bar{N}}$ is nonzero, and finally $\pi_{\bar{N}} \neq 0$, a contradiction. □

Corollary 7.1.5. If $P = \bar{P}$, then one has $i_P^G\,\sigma \cong i_{\bar{P}}^G\,w^{-1}\sigma$ for all irreducible absolutely cuspidal σ.

Proof. Since $I_{\bar{N}} \cong (w^{-1}\sigma)\delta_{\bar{P}}^{1/2}$, there exists a non-trivial G-morphism from $i\,(\sigma)$ to $i\,(w^{-1}\sigma)$. By 7.1.4, it has no kernel and is surjective. □

7.2. Suppose P_Θ to be an arbitrary standard parabolic, corresponding to $\Theta \subseteq \Delta$, and (σ, U) an irreducible absolutely cuspidal representation of M_Θ.

Theorem 7.2.1. For Ω associate to Θ and w a primitive element of $W(\Theta, \Omega)$, one has $i_{P_\Theta}^G\,\sigma \cong i_{P_\Omega}^G\,w^{-1}\sigma$.

Proof. Applying 1.2.6, we can use 7.1.5. □

Corollary 7.2.2. If π is an irreducible composition factor of $i_{P_\Theta}^G\,\sigma$, then there exists $w \in W(\Theta, \Theta)$ such that π may be embedded into $i_{P_\Theta}^G\,w^{-1}\sigma$.

Proof. According to 6.3.7, there exist P_Ω and $w \in W(\Theta, \Omega)$ such that π may be embedding to $i_{P_\Omega}^G\,w^{-1}\sigma$. Apply 7.2.1. □

This is a refinement of 6.3.7. The promised refinement of 6.3.8 is:

Corollary 7.2.3. The length of $i_{P_\Theta}^G\,\sigma$ is at most the order of $W(\Theta, \Theta)$.

Theorem 7.2.4. Let $K_0 = N_0^- M_0 N_0$ be an Iwahori factorization with respect to P_Θ of a compact open subgroup of G, and assume that $U^{M_0} \neq 0$. If V is any composition factor of $i_{P_\Theta}^G\,\sigma$, then $V^{K_0} \neq 0$.

Proof. This follows from 3.3.6 and 7.2.2. □

7.3. For each $\Theta \subseteq \Delta$, let $\{\Theta\}$ be the equivalence class of subsets associate to Θ.

If π is an irreducible admissible representation, it is said to be of *type* $\{\Theta\}$ if there exists $\Omega \in \{\Theta\}$ and an irreducible absolutely cuspidal representation σ of M_Ω such that π has an embedding into $i_{P_\Omega}^G \sigma$. According to 7.2.1, this holds for all $\Omega \in \{\Theta\}$ if it holds for one. According to 6.3.6, the type of π is uniquely determined.

An arbitrary admissible representation of finite length is said to be of *type* $\{\Theta\}$ if every irreducible composition factor is.

Theorem 7.3.1. If π is a finitely generated admissible representation, then there exists a unique set of representations $\left\{\pi_{\{\Theta\}}\right\}_{\Theta\subseteq\Delta}$ such that

(a) $\pi \cong \bigoplus \pi_{\{\Theta\}}$ and
(b) each $\pi_{\{\Theta\}}$ is of type $\{\Theta\}$.

Proof. The argument is the same as that of Theorem 6.3.10; apply the above remarks as well. □

If P is the parabolic corresponding to Θ and σ is an irreducible representation of M, let $\{\sigma\}$ be the equivalence class of representations $\{w\sigma\}_{w\in W(\Theta,\Theta)}$. If π is an irreducible admissible representation, I say that π is of *type* $\{\sigma\}$ if π_N has $\rho\delta_P^{1/2}$ as a composition factor for some $\rho \in \{\sigma\}$. According to 6.3.6, then, all irreducible composition factors of π_N are of this form.

If π is an arbitrary admissible representation of finite length, it is said to be of *type* $\{\sigma\}$ if each irreducible composition factor is.

Theorem 7.3.2. If π is a finitely generated admissible representation of type $\{\Theta\}$, then there exists a unique finite set of types $\{\sigma\}$ and a unique set of representations $\left\{\pi_{\{\sigma\}}\right\}$ such that

(a) $\pi \cong \bigoplus \pi_{\{\sigma\}}$ and
(b) each $\pi_{\{\sigma\}}$ is of type $\{\sigma\}$.

Proof. Apply the following result. □

Lemma 7.3.3. Suppose π_1, π_2, and π_3 are all finitely generated admissible representations of type $\{\Theta\}$ fitting into an exact sequence

$$0 \longrightarrow \pi_1 \longrightarrow \pi_2 \longrightarrow \pi_3 \longrightarrow 0.$$

Suppose that π_1 is of type $\{\sigma\}$ but that no factor of π_3 is. Then the sequence splits.

Proof. Arguing as in 5.4.4, we conclude that the sequence

$$0 \longrightarrow (\pi_1)_N \longrightarrow (\pi_2)_N \longrightarrow (\pi_3)_N \longrightarrow 0$$

splits, and in particular there exists a projection from $(\pi_2)_N$ to $(\pi_1)_N$. Corresponding to this is a morphism from π_2 to $i_P^G(\pi_1)_N\delta_P^{1/2}$. One can check that the image is isomorphic to π_1 and the kernel to π_3. □

8. An example: the Steinberg representation

Here I will justify the announcements in [14].

8.1. Fix a minimal parabolic $P_\emptyset$. For each $\Theta \subseteq \Delta$, define the G-representation π_Θ to be

$$C_c^\infty(P_\Theta, G) \cong i_{P_\Theta}^G \, \delta_\Theta^{-1/2}.$$

For $\Theta \subseteq \Omega$ one has a canonical inclusion $\pi_\Omega \subseteq \pi_\Theta$. In particular, π_Δ is the trivial representation of G, contained in all other π_Θ.

Lemma 8.1.1. One has

$$(\pi_\Theta)_{N_\emptyset} \cong \bigoplus_{w \in [W/W_\Theta]} (w^{-1}\delta_\emptyset^{-1/2})\delta_\emptyset^{1/2}.$$

Proof. From 6.3.5, one obtains a filtration of $(\pi_\Theta)_{N_\emptyset}$ indexed by $[W/W_\Theta]$, with the factor associated to w isomorphic to $(w^{-1}\delta_\emptyset^{-1/2})\delta_\emptyset^{1/2}$. This filtration splits since all these characters of $M_\emptyset$ are distinct. □

Note that this result is stated incorrectly in [14] (Proposition 2).

Define the *Steinberg representation* to be $\sigma = \pi_\emptyset / \sum_{\Theta \neq \emptyset} \pi_\Theta$.

Lemma 8.1.2. The Jacquet module $\sigma_{N_\emptyset}$ is isomorphic to $\delta_\emptyset$.

Proof. From the injections of each π_Θ into $\pi_\emptyset$, one has corresponding injections of each $(\pi_\Theta)_{N_\emptyset}$ into $(\pi_\emptyset)_{N_\emptyset}$, and $\sigma_{N_\emptyset}$ is the quotient of $(\pi_\emptyset)_{N_\emptyset}$ by the sum of the images for $\Theta \neq \emptyset$. Since the only $w \in W$ such that $w\alpha < 0$ for all $\alpha \in \Delta$ is w_ℓ, 8.1.1 implies the lemma. □

Theorem 8.1.3. The representation σ is irreducible and square-integrable mod Z_G.

Proof. It has no proper quotient by 6.3.7 and 8.1.2. It is square-integrable mod Z_G by 6.5.1. □

9. Another example: the unramified principal series of SL_2

9.1. Let η be a generator of the prime ideal of $\mathcal{O}$, and let q be the order of the residue field. Let

$$\begin{aligned}
G &= \mathrm{SL}_2(k);\\
A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in k^\times \right\};\\
A^- &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, 0 < |a| \le 1 \right\};\\
N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in k \right\};\\
P &= AN;\\
W &= N(A)/A = \{w, 1\} \quad \text{where } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\\
K &= \mathbf{G}(\mathcal{O}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, a, b, c, d \in \mathcal{O} \right\};\\
B &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \,\middle|\, c \in \wp \right\};\\
P(\mathcal{O}) \text{ etc.} &= P \cap \mathbf{G}(\mathcal{O}) \text{ etc.}
\end{aligned}$$

The effect of w on A is to take $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ to $\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$.

One also has various decompositions:

$$\begin{array}{rcll}
G &=& NAK & \text{(Iwasawa)}\\
G &=& KA^-K & \text{(Cartan)}\\
G &=& PwP \cup P & \text{(Bruhat)}\\
K &=& BwB \cup B. &
\end{array}$$

The subgroup B is called the *Iwahori subgroup* of G, and has an Iwahori factorization with respect to P.

I shall actually need an explicit form of the Bruhat decomposition:

Lemma 9.1.1. If $c \ne 0$, then (with $ad - bc = 1$):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -c^{-1} & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix}.$$

This gives not only the Bruhat decomposition for G, but also the decomposition $K = BwB \cup B$ as well, and one step further:

Corollary 9.1.2. One has $K = P(\mathcal{O})wN(\mathcal{O}) \cup B$.

Since $A \cong k^\times$, the characters of A may (and will) be identified with the characters of $k^\times$. In particular one has the modulus $\nu\colon x \longmapsto |x|$, which turns out to be the modulus $\delta_P^{1/2}$ as well, and its complex powers $\nu^s\colon x \longmapsto |x|^s$. These complex powers are precisely the unramified characters $X_{\rm nr}(A)$. Since $|x| = q^{-n}$ if $x = \eta^n$, the true parameter space of $X_{\rm nr}(A)$ is $\mathbf{C}/(2\pi i/\log q)\mathbf{Z}$. The unitary characters correspond to purely imaginary s. The element $w \in W$ acts on $X_{\rm nr}(A)$ by taking s to $-s$, and has as fixed points $s = 0$ and $s = \pi i/\log q$. Corresponding to $s = 0$ is the trivial character of A, and to $s = \pi i/\log q$ the quadratic character $\mathrm{sgn}_{\rm nr}\colon \eta^n \longmapsto (-1)^n$ (which by class field theory corresponds to the unramified quadratic extension of k).

9.2. I want to discuss in §9 the representations $I_s = i_P^G\,\nu^s$. The first remark is that they all have isomorphic restrictions to K:

Proposition 9.2.1. The restriction of I_s to K is isomorphic to the space of all locally constant functions $f\colon K \longrightarrow \mathbf{C}$ such that $f(pk) = f(k)$ for all $p \in P(\mathcal{O})$, $k \in K$.

The K-morphism from one to the other is simply the restriction of functions in I_s to K, which is an isomorphism because of the Iwasawa decomposition. (This result is a special case of 3.1.1.)

Corollary 9.2.2. For any s, $I_s^K \cong \mathbf{C}$ and $I_s^B \cong \mathbf{C}^2$.

The second isomorphism follows from the fact that $K = P(\mathcal{O})wN(\mathcal{O}) \cup B = P(\mathcal{O})wB \cup B$. Explicitly, one has the function $\phi_K = \phi_{K,s}$:

$$\phi_K(nak) = \nu^{s+1}(a) \quad \text{for } n \in N,\ a \in A\,, k \in K$$

as a basis for I_s^K, and the functions ϕ_1, ϕ_w:

$$\phi_1(nak) = \begin{cases} \nu^{s+1}(a) & k \in B \\ 0 & k \notin B \end{cases}$$

$$\phi_w(nak) = \begin{cases} 0 & k \notin BwB \\ \nu^{s+1}(a) & k \in BwB, \end{cases}$$

as a basis for I_s^B. Another way to describe these elements is as follows: one has a projection $\mathcal{P}_s$ from $C_c^\infty(G)$ to I_s:

$$\mathcal{P}_s f(x) = \int_P \nu^{-s+1}(a) f(px)\,dp$$

and

$$\begin{aligned}\phi_K &= \mathcal{P}_s(\mathrm{ch}_K)\\ \phi_1 &= \mathcal{P}_s(\mathrm{ch}_B)\\ \phi_w &= \mathcal{P}_s(\mathrm{ch}_{BwB}).\end{aligned}$$

Proposition 9.2.3. If (π, V) is any admissible representation of G, then the canonical projection is an isomorphism of V^B with $V_N^{A(\mathcal{O})}$.

Proof. Surjectivity follows from 3.3.3. For injectivity, it suffices to prove that, in the terminology of §4.1, $V^B = V^B_{A^-}$, or that for any $a \in A^-$ the operator $\pi(BaB)$ is invertible. However, as one can easily check, in the Hecke algebra $\mathcal{H}(G, B)$ one has the identity

$$\mathrm{ch}_{BaB}^2 - (q-1)\,\mathrm{ch}_{BaB} - q\,\mathrm{ch}_B = 0$$

(assuming $\mathrm{meas}(B) = 1$ for the moment), and this does it, since ch_B is the identity element of the Hecke algebra. □

Corollary 9.2.4. If (π, V) is an irreducible admissible representation of G, then $V^B \neq 0$ if and only if π embeds into some I_s.

Proof. This follows from 9.2.3 and Frobenius reciprocity. □

Incidentally, the proof of injectivity in 9.2.3 (which works for more general groups) is due to Borel.

9.3. A first result on the structure of $(I_s)_N$ is an immediate corollary of the results of §6.3:

Proposition 9.3.1. There is an exact sequence:

$$0 \longrightarrow \nu^{-s+1} \longrightarrow (I_s)_N \longrightarrow \nu^{s+1} \longrightarrow 0.$$

As an exercise, one might try to prove this directly. I should mention that the map from $(I_s)_N$ to ν^{s+1} is induced by $f \longmapsto f(1)$ and is therefore canonical, but that the injection of ν^{-s+1} into $(I_s)_N$ depends on several choices I have made. To be precise, let $I_{w,s}$ be the subspace of I_s of functions with support on PwP; the map

$$\Lambda_{w,s}(f) = \int_N f(wn)\,dn$$

then induces the isomorphism of $(I_{w,s})_N$ with the A-representation ν^{-s+1}. I will need later the observation that if $D_{w,s}$ is the composition

$$C_c^\infty(PwP) \xrightarrow{\mathcal{P}_s} I_{w,s} \longrightarrow \nu^{-s+1}$$

then

$$D_{w,s}(f) = \int_{PwP} \Phi(x) f(x)\,dx,$$

where Φ is the function on $PwP = PwN$ defined by

$$\Phi(nawn_1) = \nu^{-s+1}(a)$$

and the measure on PwP is the unique multiple of the measure induced by the Haar measure on G (PwP is open in G) such that $\mathrm{meas}(P(\mathcal{O})wN(\mathcal{O})) = 1$. The multiple is therefore $(q+1)/q$. In particular, the measure of B is now taken to be $1/q$.

If ν^s is regular, the exact sequence in 9.3.1 splits, and one has therefore

(1) an extension of $\Lambda_{w,s}$ to all of I_s;
(2) an extension of $D_{w,s}$ to all of $C_c^\infty(G)$;
(3) a G-morphism $T_{w,s}$ from I_s to I_{-s} such that

$$T_{w,s}f(1) = \Lambda_{w,s}(f) \qquad (f \in I_s)$$

or

$$T_{w,s}(\mathcal{P}_s f)(1) = D_{w,s}(f) \qquad (f \in C_c^\infty(G)).$$

The operator $T_{w,s}$ is a G-morphism, in particular a K-morphism, so it takes $\phi_{K,s}$ to a scalar multiple of $\phi_{K,-s}$.

Proposition 9.3.2. For s regular, one has

$$T_{w,s}(\phi_K) = c(s)\phi_K$$

with

$$c(s) = \frac{1-q^{-1-s}}{1-q^{-s}}.$$

Proof. Since T_w takes ϕ_K to a multiple of ϕ_K, and $\phi_K(1) = 1$, one only has to evaluate

$$\Lambda_{w,s}(\phi_K) = D_{w,s}(\mathrm{ch}_K).$$

Furthermore, since $\Lambda_{w,s}$ depends holomorphically on s, one only has to evaluate this for an open subset of complex s.

Now what happens is that the integral above defining $D_{w,s}$ on $C_c^\infty(PwP)$ actually converges for all $f \in C_c^\infty(G)$ as long as $\mathrm{Re}(s) > 1$, and that this integral defines the extension to all of $C_c^\infty(G)$ giving rise to $T_{w,s}$. I shall not prove this explicitly, because it will actually fall out of the calculation below:

$$D_{w,s}(\mathrm{ch}_K) = \int_{BwB\cap PwP} \Phi(x)\,dx + \int_{B\cap PwP} \Phi(x)\,dx.$$

Because (9.1.1) $BwB = P(\mathcal{O})wN(\mathcal{O}) \subseteq PwP$ and $\Phi \equiv 1$ on $P(\mathcal{O})wN(\mathcal{O})$ (9.1.1 again), the first integral is just $\mathrm{meas}(BwB) = 1$. For the second, express B as the disjoint union of the sets $B_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B \,\middle|\, c \in \wp^n \setminus \wp^{n+1} \right\}_{n\geq 1}$ and $P(\mathcal{O})$. Then $B \cap PwP$ is exactly the union of the B_n, so one must evaluate

$$\sum_{n=1}^{\infty} \int_{B_n} \Phi(x)\, dx.$$

Step (1): According to 9.1.1 once more, $\Phi = |\eta^{-n}|^{-s+1} = q^{n(-s+1)}$ on B_n. Step (2): The set $\bigcup_{n\geq m} B_n$ is a subgroup of B of index q^{m-1}, so that

$$\begin{aligned} \operatorname{meas} B_n &= (q^{-(n-1)} - q^{-n})\operatorname{meas} B \\ &= (q^{-(n-1)} - q^{-n})(1/q) \\ &= (q-1)/q^{n+1}. \end{aligned}$$

Therefore, the above sum is

$$\begin{aligned} \sum_{n=1}^{\infty}((q-1)/q^{n+1})(q^n)^{-s+1} &= \frac{q-1}{q} \cdot q^{-s} \sum_{n=0}^{\infty} q^{-ns} \\ &= \frac{(1-q^{-1})q^{-s}}{1-q^{-s}} \end{aligned}$$

and

$$c(s) = 1 + \frac{(1-q^{-1})q^{-s}}{1-q^{-s}} = \frac{1-q^{-1-s}}{1-q^{-s}}. \quad \square$$

Corollary 9.3.3. If s is regular, then I_s is reducible only when $s = \pm 1$.

This follows from 6.6.2, because $\gamma(s) = c(s)c(-s)$ is 0 only when $s = \pm 1$.

What happens at $s = -1$ has been discussed in §8. The representation I_{-1} contains the trivial representation and the quotient is the Steinberg representation. Since I_1 is the contragredient of I_{-1}, it contains the Steinberg representation and has the trivial representation as quotient. Incidentally, the sequence

$$0 \longrightarrow \mathbf{C} \longrightarrow I_{-1} \longrightarrow \mathrm{St} \longrightarrow 0$$

does not split, as one can prove easily.

9.4. The case of s irregular is more delicate.

Let me continue to assume that s is regular at first. Since $I_s^B \cong (I_s)_N \cong \nu^{-s+1} \oplus \nu^{s+1}$, there exist two elements $f_{1,s}$ and $f_{w,s}$ in I_s^B dual to the $\Lambda_{x,s}$—i.e., such that $\Lambda_x(f_y) = \delta_{xy}$. The element $f_{1,s}$ will depend on s, but it turns out that $f_{w,s}$ is in some sense independent of s.

Lemma 9.4.1. For any regular s, $f_{w,s} = \phi_{w,s}$.

Proof. One has

$$\Lambda_1(\phi_w) = \phi_w(1) = 0$$
$$\Lambda_w(\phi_w) = \int_N \phi_w(wn)\,dn = \int_{N(\mathcal{O})} dn = 1. \quad \square$$

Lemma 9.4.2. For any regular s, one has

$$\phi_1 = f_1 + (c(s)-1)f_w$$
$$\phi_w = f_w.$$

Proof. The second equation is in 9.4.1. For the first, clearly $\Lambda_1(\phi_1) = 1$, while to find $\Lambda_w(\phi_1)$ one merely notices that $\phi_1 = \phi_K - \phi_w$ and applies 9.3.2. $\square$

Lemma 9.4.3. For any regular s and any $a \in A^-$, one has

$$\pi_s(BaB)f_1 = \nu^{s+1}(a)f_1$$
$$\pi_s(BaB)f_w = \nu^{-s+1}(a)f_w$$

Proof. This follows from 4.1.1. $\square$

Proposition 9.4.4. For any s at all and $a \in A^-$, one has

$$\pi_s(BaB)\phi_1 = \nu^{s+1}(a)\phi_1 + (c(s)-1)(\nu^{-s+1}(a) - \nu^{s+1}(a))\phi_w$$
$$\pi_s(BaB)\phi_w = \nu^{-s+1}(a)\phi_w.$$

Proof. Because the action of $\mathcal{H}(G,B)$ on I_s^B varies holomorphically with s, it suffices to prove this for regular s. But for regular s, one has

$$\phi_1 = f_1 + (c(s)-1)f_w$$
$$\phi_1 = \qquad\qquad f_w \quad (9.4.2)$$
$$\pi_s(BaB)\phi_1 = \nu^{s+1}(a)f_1 + (c(s)-1)\nu^{-s+1}(a)f_w$$
$$\pi_s(BaB)\phi_w = \qquad\qquad \nu^{-s+1}(a)f_w$$

and then uses 9.4.2 to express the f_x in terms of the ϕ_x again. $\square$

Let ψ_1, ψ_w be the images in $(I_s)_N$ of ϕ_1, ϕ_w. Because of 4.1.1, Proposition 9.4.4 tells what $a \in A^-$ does to ϕ_1, ϕ_w. If one then specializes to the irregular s, one gets:

Proposition 9.4.5. With respect to the basis ψ_1, ψ_w of $(I_s)_N$ one has:

(a) When $s = 0$,

$$\pi_N(a) = \begin{pmatrix} |a| & 0 \\ 2(1-q^{-1})|a|\log_q|a| & |a| \end{pmatrix};$$

(b) When $s = (\pi i / \log q)$,

$$\pi_N(a) = \begin{pmatrix} |a| \operatorname{sgn}_{\mathrm{nr}}(a) & 0 \\ 0 & |a| \operatorname{sgn}_{\mathrm{nr}}(a) \end{pmatrix}.$$

Here $\operatorname{sgn}_{\mathrm{nr}}$ is the character corresponding to $(\pi i / \log q)$.

Corollary 9.4.6. The representation I_s is

(a) irreducible when $s = 0$;
(b) reducible, and in fact a direct sum of two irreducibles, when $s = (\pi i / \log q)$.

Proof. Since I_s is unitary in each case, it suffices to see whether or not $\mathrm{Hom}_G(I_s, I_s) = \mathbf{C}$ in each case. By Frobenius reciprocity, one only has to decide whether or not $(I_s)_N$ is a semi-simple representation of A, and 9.4.5 answers this. □

List of symbols

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Note: In earlier versions of this paper, [26] and [27] were cited in preprint form.