

## Essays on convexity

### Minkowski on convexity

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Excerpts from Minkowski's paper on *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs* published only as item XXV in volume II of **Gesammelte Abhandlungen**, Leipzig, 1911. Reprinted by Chelsea, 1967.

#### §1. Definition of a convex body.

A point set will be called a **convex body** when (1) its intersection with an arbitrary line is either a bounded segment, a single point, or empty; (2) it is not contained in any single plane.

If  $\mathfrak{K}$  is a convex body, then it must contain four points not all in one plane, therefore the tetrahedron they span. It therefore contains an interior point. By translating, we may assume it to contain in its interior any given point.

#### §3. . . .

. . . Property (1) of a convex set may be broken down into three parts:

- (1a) Along with every two points the set contains all points on the segment between them;
- (1b) The set is bounded;
- (1c) The set is closed.

#### §4. Polyhedra. Supporting half-spaces.

If  $\mathfrak{p}$  is a point with coordinates  $x, y, z$  and  $s$  a constant, the point whose coordinates are  $sx, sy, sz$  will be written as  $s\mathfrak{p}$ . If  $\mathfrak{p}'(x', y', z')$  and  $\mathfrak{p}''(x'', y'', z'')$  are two points, then  $\mathfrak{p}' + \text{goth}\mathfrak{p}''$  will be the point with coordinates  $x' + x'', y' + y'', z' + z''$ . If  $\mathfrak{K}$  is a set of points and  $s$  a constant,  $s\mathfrak{K}$  will be the set of points  $s\mathfrak{p}$  with  $\mathfrak{p}$  in  $\mathfrak{K}$ .

If  $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$  is a *finite* set of points, the set of those points which can be expressed in the form

$$t_1\mathfrak{p}_1 + \dots + t_n\mathfrak{p}_n \tag{5}$$

where

$$t_1 \geq 0, t_2 \geq 0, \dots, t_n \geq 0, \quad t_1 + t_2 + \dots + t_n = 1 \tag{6}$$

forms a set with the three properties (1a), (1b), (1c) of a convex body, which we call the *convex hull* of the  $\mathfrak{p}_i$ . This set contains the points  $\mathfrak{p}_i$  and is completely contained in any convex set containing them.

If a point, say  $\mathfrak{p}_n$ , possesses a representation of the form (5) besides the obvious one with all  $t_j = 0$  for  $j \neq n$  and  $t_n = 1$ , in which case we must have all  $t_i < 1$ , then it already lies in the convex hull of  $\mathfrak{p}_1, \dots, \mathfrak{p}_{n-1}$ . The two convex hulls are the same. Continuing in this way, renumbering if necessary, we find points  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  with  $m \leq n$ , such that their convex hull is the same as that of  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ , and the

number of points with that property cannot be reduced. Thus every point  $t_1\mathfrak{p}_1 + \dots + t_n\mathfrak{p}_m$  has exactly one expression of this form (5). The points in this possibly smaller set are called the *extremal points* of the convex hull. On any line through an extremal point  $\mathfrak{p}_h$  no two  $\mathfrak{p}_i$  can lie on either side of  $\mathfrak{p}_h$ .

If the points  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  do not all lie in one plane, their convex hull is a convex body, and is called a (convex) *polyhedron*. In these circumstances, every point  $\mathfrak{p}$  that can be written in the form (6) with all  $t_i > 0$  is in the *interior* of this body. Because if  $\epsilon$  is any point in the hull other than  $\mathfrak{p}$  then all  $(1+t)\mathfrak{p} - t\epsilon$  also lies in the hull for small  $t$ , whereas if  $\mathfrak{p}$  lies on the border of the hull then there will exist such points outside it.

If the points  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  do all lie in one plane, but not on a single line, their convex hull is a (convex) polygon. Points of the form (6) with all  $t_i > 0$  lie in the interior of that polygon.

If the points  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  do all lie in one line, their hull is a segment, unless it is just a single point.

If

$$\alpha x + \beta y + \gamma z = d \quad (7)$$

with  $\alpha^2 + \beta^2 + \gamma^2 = 1$ , is the equation of a plane, the set of points  $x, y, z$  with

$$\alpha x + \beta y + \gamma z > d$$

is called the *side*  $(\alpha, \beta, \gamma)$  of the plane.

Suppose  $\mathfrak{K}$  to be a closed point set. If we find in (7) a plane which contains a point of  $\mathfrak{K}$ , but has no point of  $\mathfrak{K}$  lies on the side  $(\alpha, \beta, \gamma)$ , so that for all points  $x, y, z$  of  $\mathfrak{K}$

$$\alpha x + \beta y + \gamma z \leq d \quad (8)$$

we shall call this plane a *supporting half-plane* of  $\mathfrak{K}$ , with *normal*  $(\alpha, \beta, \gamma)$  and *condition* (8). Every point of  $\mathfrak{K}$  on this plane lies in the boundary of  $\mathfrak{K}$ .

We are now going to prove the following Theorem:

*Given any polyhedron  $\mathfrak{P}$ , one can always find a finite number of supporting half-planes whose intersection is exactly  $\mathfrak{P}$ .*

Let  $\mathfrak{P}$  be a polyhedron with extremal vertices  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ , and let  $\epsilon$  be an interior point of  $\mathfrak{P}$ . We consider for each pair of distinct points  $\mathfrak{p}_i, \mathfrak{p}_j$  the line through them and, whenever  $\epsilon$  does not lie on this line, the plane through  $\mathfrak{p}_i, \mathfrak{p}_j$ , and  $\epsilon$ . Let now  $\mathfrak{p}$  be a point of the boundary of  $\mathfrak{P}$  not in this plane and also not on any of the segments  $\mathfrak{p}_k\mathfrak{p}_\ell$ . The point  $\mathfrak{p}$  can be expressed as

$$\mathfrak{p} = t_i\mathfrak{p}_i + \dots + t_\ell\mathfrak{p}_\ell \quad (9)$$

with all these  $t_k > 0$  and their sum equal to 1. The number of  $\mathfrak{p}_k$  in this expression must be at least 3, since  $\mathfrak{p}$  has been chosen not to lie on a line through any pair of vertices. On the other hand, they must all lie on a plane, because otherwise  $\mathfrak{p}$  would be in the interior of their convex hull, hence in the interior of  $\mathfrak{P}$ . So the convex hull of these  $\mathfrak{p}_k$  is a polygon with  $\mathfrak{p}$  in its interior.

The plane of this polygon must be the boundary of a supporting half-plane of  $\mathfrak{P}$ . Because it doesn't contain  $\epsilon$ , there must exist among the  $\mathfrak{p}_k$  at least one, say  $\mathfrak{p}_g$  lying on the same side of this plane as  $\epsilon$ . If there exists a point  $\mathfrak{p}_h$  on the opposite side, then the pyramids  $\mathfrak{p}_g, \mathfrak{p}_i, \dots, \mathfrak{p}_\ell$  and  $\mathfrak{p}_h, \mathfrak{p}_i, \dots, \mathfrak{p}_\ell$  would be separated by this plane, and  $\mathfrak{p}$  would be an interior point of their union, hence also of  $\mathfrak{P}$ , a contradiction. We now come to the following result:

If we consider all the planes through non-planar triples of the vertices, among this finite collection are those which form supporting half-planes of the convex hull. Let these half-planes be

$$\alpha_i x + \beta_i y + \gamma_i z \leq d_i$$

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for  $i = 1, \dots, \nu$ . These planes pass through every point of the boundary of  $\mathfrak{F}$  that does not lie in some  $\epsilon p_i p_j$ . Since the number of planes  $\epsilon p_i p_j$  is finite, we can choose a direction  $\alpha_*, \beta_*, \gamma_*$  equal to one of their normals, which is a limit of  $\alpha, \beta, \gamma$  not among their normals. If  $p_*$  is on the boundary, it is a limit of  $p$  not on any  $\epsilon p_i p_j$ . . . . The planes (9) must therefore contain every point of the boundary of  $\mathfrak{F}$ .

The inequalities (9) are valid for every point of  $\mathfrak{F}$ . If  $q$  is a point outside  $\mathfrak{F}$ , the segment  $\epsilon q$  will contain a point  $p$  of the boundary. One of the planes (9) will therefore separate  $\epsilon$  from  $q$ . Therefore any point satisfying all the inequalities (9) must lie in  $\mathfrak{F}$ .