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HEISUKE HIRONAKA

INTRODUCTION TO REAL-ANALYTIC SETS AND REAL-ANALYTIC MAPS

ISTITUTO MATEMATICO "L. TONELLI"
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Introduction

This note is made out of my lectures given at Istituto "Leonida Tonelli" in Pisa during the months of June, July 1973. My initial intention was to give a self-contained introduction to the theory of real-analytic spaces, real-analytic maps and the images of such maps. But the way it turned out was that it made full use of desingularization theorems, proven elsewhere, and proved generalizations to subanalytic sets (those generated by the images of proper real-analytic maps) for some basic properties known for real-analytic and semi--analytic sets.

One lecture was omitted from the note, afterward, in which I gave a proof of Whitney stratification of a subanalytic set. The reason for this is that the stratification theorem has been already published (See (*)) and the proof given in the lecture was essentially identical with the published one. As for the other lectures, essentially all contained in this note, the reader will find one new feature (especially in comparison with the previous work (*)), which is what me called the rectilinearization of subanalytic sets. This enable us to present any given subanalytic set as a finite union of images of quadrants in the standard real Euclidean spaces.

Many (but not all) of the important elementary properties of sub-analytic sets (semi-analytic sets as well) can be

(*) H.Hironaka - "Subanalytic sets" in number theory, Algebraic Geometry and Commutative Algebra. Volume in honour of Y. A.Kizuki, Kinokunya Pub. 1973.

deduced from the rectilinearization theorem. One of them is Lojasiewicz'inequalities, which is proven in the last section.

I like to mention that, during the course of discovering a proof of the rectilinearization theorem, much valuable suggestions were given to me by Monique Lejeune and Bernard Teissier.

I cannot refrain myself from expressing my gratitude to those italian mathematicians who organized the meeting and attended my lectures. To me, it was a pleasure of discovering a new world of friendship, in which a person, like myself, felt so happy in being a mathematician.

§ 1. Analytic spaces.

Let K be a fields, commutative. Then a K-**ringed space** to a pair $(|X|, \mathcal{O}_X)$ where $|X|$ is a topological space and $\mathcal{O}_X$ is a sheaf of commutative K-algebras on $|X|$.

Example (1.1) The complex number space of dimension n, often denoted by $\mathbb{C}^n$ for short, is the pair $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$ where $\mathcal{H}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions on the topological space $\mathbb{C}^n$, i.e., for each open subset U of $\mathbb{C}^n$, $\mathcal{H}_{\mathbb{C}^n}(U)$ is the $\mathbb{C}$-algebra of all holomorphic functions on U.

Example (1.2) Let U be an open subset of $\mathbb{C}^n$, and let $f=(f_1,\dots,f_m)$ be a finite system of holomorphic functions f_i on U. Then the data $(U \subset \mathbb{C}^n, f)$ defines a $\mathbb{C}$-ringed space $S = (|S|, \mathcal{H}_S)$ as follows. $|S| = \{z \in U \mid f_1(z)=\dots=f_m(z)=0\}$, and $\mathcal{H}_S$ is the restriction of $\mathcal{H}_U/(f)\mathcal{H}_U$ to the closed subset $|S|$ of U, where $\mathcal{H}_U$ denotes the restriction $\mathcal{H}_{\mathbb{C}^n}|U$. Such S will be called local model of complex-analytic space.

Example (1.3) The real number space of dimension n, often denoted by $\mathbb{R}^n$ for short, is the pair $(\mathbb{R}^n, \mathcal{A}_{\mathbb{R}^n})$ where $\mathcal{A}_{\mathbb{R}^n}$ is the sheaf of (real-valued) real-analytic functions in $\mathbb{R}^n$.

Example (1.4) Let U be an open subset of $\mathbb{R}^n$ and let $g = (g_1,\dots g_m)$ be a finite system of real analytic functions on U, i.e, $g_i \in \mathcal{A}_{\mathbb{R}^n}(U)$. Then the data $(U \subset \mathbb{R}^n, g)$ defines an $\mathbb{R}$-ringed space $S = (|S|, \mathcal{A}_S)$ where $|S| = \{x \in U \mid g_1(x)=\dots=g_m(x)=0\}$ and $\mathcal{A}_S$ is the restriction to $|S|$ of the sheaf of $\mathbb{R}$-algebras $\mathcal{A}_U/(g)\mathcal{A}_U$ where $\mathcal{A}_U = \mathcal{A}_{\mathbb{R}^n}|U$. Such S will be called local model of real-analytic space.

<u>Definition</u> (1.5) A <u>complex-analytic space</u> (or, <u>complex space</u>, for short) is a $\mathbb{C}$-ringed space $X = (|X|, \mathcal{H}_X)$ such that every point of $|X|$ admits an open neighborhood N for which the restriction $X|N = (N, \mathcal{H}_X|N)$ is isomorphic (as $\mathbb{C}$-ringed space) to a local model of complex-analytic space.

<u>Definition</u> (1.6) A <u>real-analytic space</u> (or, <u>real space</u>, for short) is an $\mathbb{R}$-ringed space $X=(|X|, \mathcal{A}_X)$ such that every point of $|X|$ admits an open neighborhood N for which the restriction $X|N$ is isomorphic (as $\mathbb{R}$-ringed space) to a local model of real-analytic space.

A <u>morphism</u> of K-ringed spaces, say $h: X \to X'$, is a pair $(|h|, \theta_h)$ where $|h|: |X| \to |X'|$ is a continuous map of topological spaces and $\theta_h: \mathcal{O}_{X'} \to \mathcal{O}_X$ is an $|h|$ - homomorphism of sheaves of K-algebras (which means a homomorphism of sheaves of K-algebras from $\mathcal{O}_{X'}$ to the direct image $h_* \mathcal{O}_X$ on the topological space $|X'|$). Note that θ_h induces a homomorphism of K-algebras $\theta_{h,x}: \mathcal{O}_{X',|h|(x)} \to \mathcal{O}_{X,x}$ for every $x \in |X|$, where $\mathcal{O}_{X,x}$ (resp. $\mathcal{O}_{X',|h|(x)}$ denotes the stalk of $\mathcal{O}_X$ at x (resp. $\mathcal{O}_{X'}$ at $|h|(x)$). In the case of complex-(or real-) analytic spaces, the stalks $\mathcal{O}_{X,x}$ are local rings and $\theta_{h,x}$ are necessarily local homomorphisms (i.e. homomorphisms which send maximal ideals into maximal ideals). This is easily checked by the fact that $f \in \mathcal{O}_{X,x}$ belongs to the maximal ideal, often denoted by $\max(\mathcal{O}_{X,x})$, if and only if $f + \xi$ has its inverse in $\mathcal{O}_{X,x}$ for all non-zero $\xi \in K$. ($\theta_{h,x}$ induces id_K).

To simplify notations, we shall write $x \in X$ for $x \in |X|$ and $h(x)$ for $|h|(x)$.

<u>Remark</u> (1.7) Let X and X' be complex-(resp. real.) analytic spaces. Let $x \in X$ and $x' \in X'$. Then, given any homomorphism of K-algebras, $\tau: \mathcal{O}_{X',x'} \to \mathcal{O}_{X,x}$ where $K = \mathbb{C}$(resp. $\mathbb{R}$), there

exists an open neighborhood N of x in X and a morphism $h: X|N \to X'$ such that $\theta_{h,x} = \tau$. Any two such h for a given τ coincide when restricted to a sufficiently small neighborhood of x in X.

<u>Proof</u>. We shall use the following fact from the theory of commutative algebras. "Let A and B be two noetherian local K-algebras (where K is a field) such that the natural homomorphisms from K into $A/\max(A)$ and $B/\max(B)$ are isomorphisms. Let $(\xi_1, \dots, \xi_m)$ be a system of generators of $\max(A)$. Then a K-homomorphism $\tau: A \to B$ is uniquely determined by the images $\tau(\xi_i)$, $1 \leq i \leq m$. (For a proof of this, extend τ to the completions of A and B and prove the uniqueness of the extended τ by means of the structure theorem for complete noetherian local rings).
Now, as for (1.7), since the question is local, we may assume that X and X' are local models of complex-(resp. real.) analytic spaces. Say X is defined by $(U \subset K^n, f_1, \dots f_m)$ and X' by $(U' \subset K^r, f'_1, \dots, f'_s)$. Let $(z_1, \dots, z_n)$ (resp. $(y_1, \dots, y_r)$) be the coordinate system of K^n (resp. K^r). We may assume that x and x' are the origins of K^n and K^r respectively. Then $\mathcal{O}_{X,x} = K\{z\}/(f)K\{z\}$ and $\mathcal{O}_{X',x'} = K\{y\}/(f')K\{y\}$, where $K\{\}$ denotes the convergent power series ring. Let ξ_i be the class of y_i in $\mathcal{O}_{X',x'}$. Let $h_i(z)$ be a representative in $K\{z\}$ for the class $\tau(\xi_i)$. Apply the fact quoted above to the composition of τ with the natural homomorphism $K\{y\} \to \mathcal{O}_{X',x'}$, and we find that $f'_j(h_1(z), \dots, h_r(z))$, $1 \leq j \leq s$, are all in the ideal $(f)K\{z\}$. Write

$$f'_j(h_1(z), \dots, h_r(z)) = \sum_{1 \leq k \leq m} a_{jk} f_k(z)$$

and let V be an open neighborhood of $x(=0)$ in K^n in which all the h_i and a_{jk} are convergent. Then the morphism $K^n|V \to K^r$ defined by $(h_1,\ldots,h_r)$ induces a morphism $h:X|N \to X'$ where $N=V\cap|X|$. Again, by the fact quoted above, we have $\theta_{h,x}=\tau$. Moreover, if $(h'_1,\ldots,h'_r)$ is another system having the same property as $(h_1,\ldots,h_r)$, then we can prove that $h'_i(z)-h_i(z)\in(f)K\{z\}$ for each i. (The proof of this is by the same argument as above). From this, it is easy to deduce the last uniqueness assertion of (1.7). Q.E.D.

<u>Remark</u> (1.8) Notations being the same as in (1.7), if τ is an isomorphism then there exists an open neighborhood N (resp. N') of x (resp. x') in X (resp. X') and an isomorphism $h:X|N \xrightarrow{\sim} X'|N'$ such that $\theta_{h,x}=\tau$.

<u>Proof</u>. Immediate from (1.7).

Let us denote by $\rho:\mathbb{C}\to\mathbb{C}$ the complex conjugation, i.e., $\rho(z)=\bar{z}$ for $z\in\mathbb{C}$.

<u>Proposition</u> (1.9) Let $X=(|X|,\mathcal{H}_X)$ be a complex space. By $\mathbb{C}\xrightarrow[\rho]{}\mathbb{C}\subset\mathcal{H}_X$ (i.e., $\mathbb{C}\xrightarrow[\rho]{}\mathbb{C}\subset\mathcal{H}_X(U)$ for every open subset U of X), we define a new $\mathbb{C}$-algebra structure in the same sheaf of rings $\mathcal{H}_X$. Let us denote by ${}^*\mathcal{H}_X$ the one with new $\mathbb{C}$-algebra structure. Then the $\mathbb{C}$-ringed space ${}^*X=(|X|,{}^*\mathcal{H}_X)$ is a complex space.

<u>Proof</u>. Take any $x\in X$. Then, by definition, there exists an open neighborhood N of x in $|X|$ and an isomorphism $h:X|N\xrightarrow{\sim}S=(|S|,\mathcal{H}_S)$ where S is a local model defined by (1.2). Since ρ^n is an involutive (i.e., $\rho^n\cdot\rho^n=\mathrm{id}$) automorphism of the topological space $\mathbb{C}^n$, it induces homeomorphisms in both way $U\leftrightarrow{}^*U=\rho^n(U)$. For each f_i of (1.2), define a function ${}^*f_i$ on ${}^*U$ by

$${}^*f_i(z)=\overline{f_i(\bar{z})}$$

It is easy to check that ${}^*f_i$ are holomorphic functions on ${}^*U\subset\mathbb{C}^n$. Let ${}^*S$ be the local model of complex space defined by the data $({}^*U\subset\mathbb{C}^n,{}^*f_1,\ldots,{}^*f_m)$. We then claim an isomorphism of $\mathbb{C}$-ringed spaces, ${}^*h: {}^*X|N \xrightarrow{\sim} {}^*S$. In fact, $|{}^*h| = \rho^n|h|$ and $\mathcal{O}_{{}^*h}$ is obtained from $\mathcal{O}_h$ together with the ρ^n-automorphism of $\mathcal{H}_{\mathbb{C}^n}$ defined by

$f \longmapsto {}^*f$, where ${}^*f(z) = \overline{f(\bar{z})}$ Q.E.D.

Definition (1.10) The complex space ${}^*X$ of (1.9) is called the conjugate complex space of X. The identity map $\rho_X : X \to {}^*X$, when there are viewed as $\mathbb{R}$-ringed spaces, will be called (complex) conjugation of X.

Note that X is the conjugate complex space of ${}^*X$ and $\rho_{*X}\rho_X = id_X$ (the identity of X as a complex space).

Definition (1.11) Let X and X' be complex spaces. Then a morphism $h: X \to X'$ viewed as $\mathbb{R}$-ringed spaces, is called a conjugate morphism of complex spaces if $\rho_{X'}h: X \to {}^*X'$ (or equivalently $h\rho_{*X}: {}^*X \to X'$) is a morphism of complex spaces.

Definition (1.2) An auto-conjugation of a complex space X is a morphism $h: X \to X$, viewed as an $\mathbb{R}$-ringed space, such that $h^2 = id_X$ (or h is involutive) and $\rho_X h: X \to {}^*X$ is an isomorphism of complex spaces.

Quite generally, let $X = (|X|, \mathcal{O}_X)$ be a K-ringed space as in the beginning of this section. Let σ be an automorphism of X. We then define the σ-invariance space of X as follows. Let $|F| = \{ x \in |X| \mid \sigma(x) = x \}$ with the topology induced from $|X|$. Let $j = |F| \subset |X|$ be the inclusion. Then $\mathcal{O}_\sigma$ induces an automorphism of the sheaf of K-algebras $j^*(\mathcal{O}_X)$ on $|F|$. Let $\mathcal{A}_F$ be the subsheaf of K-algebras of $j^*(\mathcal{O}_X)$ which consists of $\mathcal{O}_\sigma$-invariant sections. Then $(|F|, \mathcal{A}_F)$ is a K-ringed space which is called the σ-invariance space of X.

Proposition (1.13) Let σ_X be an autoconjugation of a complex space X. Then the σ_X-invariance space of X (which is naturally an $\mathbb{R}$-ringed space) is a real-analytic space.

The proof of this proposition will be given aften a few lemmas.

Lemma (1.13.1) The complex number space $\mathbb{C}^n$ (or, to be precise, $(\mathbb{C}^n, \mathcal{H}_{\mathbb{C}^n})$ of (1.1)) has a canonical auto-conjugation σ defined by

1) $\sigma(z)=\rho^n(z)=\bar{z}$ for $z \in \mathbb{C}^n$, and

2) for every open subset N of $\mathbb{C}^n$, $\theta_\sigma: \mathcal{H}_{\mathbb{C}^n}(N) \longrightarrow \mathcal{H}_{\mathbb{C}^n}(\rho^n(N))$

is by $\theta_\sigma(f)(z) = \overline{f(\bar{z})}$, $f \in \mathcal{H}_{\mathbb{C}^n}(N)$ and $z \in \rho^n(N)$.

Moreover, the σ-invariante space of $\mathbb{C}^n$ is the real number space $\mathbb{R}^n$ (or, to be precise, $(\mathbb{R}^n, \mathcal{Q}_{\mathbb{R}^n})$ of (1.3)).

Proof. For a point $\zeta \in \mathbb{C}^n$, if $f= \sum_{A \in \mathbb{Z}_0^n} c_A(z-\zeta)^A$ is the Taylor series of a holomorphic function f in some neighborhood of ζ in $\mathbb{C}^n$, then $\theta_\sigma(f)= \sum_{A \in \mathbb{Z}_0^n} \bar{c}_A(z-\bar{\zeta})^A$ is that of $\theta_\sigma(f)$ in a corresponding neighborhood of $\bar{\zeta}$ in $\mathbb{C}^n$. For $\zeta \in \mathbb{R}^n$, in particular, we have $\theta_\sigma(f)=f$ in some neighborhood of ζ in $\mathbb{C}^n$ if and only if $c_A \in \mathbb{R}$ for all $A \in \mathbb{Z}_0^n$. The assertions of (1.13.1) can be easily deduced from these facts.

Notation (1.13.2) The canonical auto-conjugation of $\mathbb{C}^n$ defined by (1.13.1) will be denoted by ρ^n.

Lemma (1.13.3) Let S be a local model of complex space defined by data $(U \subset \mathbb{C}^n, f_1, \ldots, f_m)$ as in (1.2).
Assume that $\rho^n(U) = U$ and every connected component of U meets $\mathbb{R}^n \subset \mathbb{C}^n$, and that the restriction of f_i to $U \cap \mathbb{R}^n$ belongs to $\mathcal{Q}_{\mathbb{R}^n}(U \cap \mathbb{R}^n)$ for every i, $1 \leq i \leq m$. Then the canonical auto-conjugation ρ^n of $\mathbb{C}^n$ induces an auto conjugation σ of S. Moreover

the σ-invariance space of S is the local model of real-analytic space defined by data $(U \cap \mathbb{R}^n \subset \mathbb{R}^n,\ f_1,\dots,f_m)$ as in (1.4).

<u>Proof</u>. The assumptions imply that $\theta_{\rho^n}(f_i)=f_i$ on U for all i. The first assertion is immediate from this. As for the second, $U \cap \mathbb{R}^n$ is clearly the fixed point set of σ. $\mathcal{H}_{\mathbb{C}^n}|\mathbb{R}^n$ is a free $\mathcal{O}_{\mathbb{R}^n}$-module of rank 2 generated by 1 and $\sqrt{-1}$. Hence $\mathcal{H}_U/(f)\mathcal{H}_U$, restricted to $U \cap \mathbb{R}^n$, is so as a module over $\mathcal{O}_V/(f)\mathcal{O}_V$ with $V=U \cap \mathbb{R}^n$. It is then clear that this last sheaf is the subsheaf of θ_{ρ^n}-invariants of the first.

<u>Lemma</u> (1.13.4) **Let the assumption be the same as in (1.13).** Let ξ be any point of X such that $\sigma_X(\xi)=\xi$. Then there exist an open neighborhood N of ξ in X and an isomorphism of complex spaces $h: S \xrightarrow{\sim} X|N$ such that

1) S is a local model of complex space defined by data satisfying the conditions of (1.13.3)., and

2) if σ denotes the auto-conjugation of S obtained in (1.13.3), then $(\sigma_X|N)h = h\sigma$. (In particular, $\sigma_X(N)=N$) .

<u>Proof.</u> For simplicity, we shall write $\mathcal{H}_\xi$ for the stalk $\mathcal{H}_{X,\xi}$ and θ for $\theta_{\sigma',\xi}$ with $\sigma'=\sigma_X$. So θ^ξ is an involutive automorphism of $\mathcal{H}_\xi$ such that $\theta|\mathbb{C} = \rho$. Let $\mathcal{O}_\xi$ be the ring of θ-invariants in $\mathcal{H}_\xi$. Every $f \in \mathcal{H}_\xi$ is uniquely written as $f_1+\sqrt{-1}\,f_2$ with $f_i \in \mathcal{O}_\xi$, $i=1.2$. In fact, let $f_1=\frac{1}{2}(f+\theta(f))$ and $f_2=\frac{1}{2\sqrt{-1}}(f-\theta(f))$, both of which are clearly θ-invariants. Moreover if $f_1+\sqrt{-1}\,f_2 = f_1'+\sqrt{-1}\,f_2'$ with $f_i, f_i' \in \mathcal{O}_\xi$, then $f_1-f_1'= {}$ $= \theta(f_1-f_1') = \theta(\sqrt{-1}(f_2'-f_2)) = -\sqrt{-1}(f_2'-f_2) = -(f_1-f_1')$ which

shows $f_1=f_1'$. It then follows that $f_2=f_2'$. In other words, $\mathcal{H}_\xi$ is a free $\mathcal{A}_\xi$-module of rank 2 generated by 1 and $\sqrt{-1}$. It follows that $\mathcal{A}_\xi$ is a noetherian local ring (as $\mathcal{H}_\xi$ is such) in which max $(\mathcal{A}_\xi) = \mathcal{A}_\xi \cap \max(\mathcal{H}_\xi)$. Clearly $\mathbb{R} \subset \mathcal{A}_\xi$. The canonical map $\mathbb{R} \longrightarrow \mathcal{A}_\xi/\max(\mathcal{A}_\xi)$ is an isomorphism. In fact, if $f=c+f' \in \mathcal{A}_\xi$ with $c \in \mathbb{C}$ and $f' \in \max(\mathcal{H}_\xi)$, then $\bar{c} + \theta(f') = \theta(f)=f=c+f'$. An automorphism of a local ring maps its maximal ideal into itself. So $\theta(f') \in \max(\mathcal{H}_\xi)$ and hence, by the isomorphism $\mathbb{C} \xrightarrow{\sim} \mathcal{H}_\xi/\max(\mathcal{H}_\xi)$, we must have $\bar{c}=c$, i.e., $c \in \mathbb{R}$. Hence $\theta(f')=f'$, i.e., $f' \in \mathcal{A}_\xi \cap \max(\mathcal{H}_\xi)=\max(\mathcal{A}_\xi)$. We thus conclude $\mathbb{R} \longrightarrow \mathcal{A}_\xi/\max(\mathcal{A}_\xi)$ is an isomorphism. This fact, combined with the freeness described above, implies that $\max(\mathcal{A}_\xi)\mathcal{H}_\xi = \max(\mathcal{H}_\xi)$. Let us now pick a finite system of generators $(\zeta_1,\ldots,\zeta_n)$ of $\max(\mathcal{A}_\xi)$. Then we get an epimorphism of $\mathbb{C}$-algebras $\mathbb{C}\{z\} \longrightarrow \mathcal{H}_\xi$, where $\mathbb{C}\{z\}$ is the convergent power series ring of n independent variables $z=(z_1,\ldots z_n)$ and each z_i is mapped to ζ_i, $1 \leq i \leq n$. The epimorphism clearly maps $\mathbb{C}[z]$ into $\mathcal{A}_\xi$. Since $\mathcal{A}_\xi$ is $\max(\mathcal{H}_\xi)$-adically closed (by the fact that $\max(\mathcal{H}_\xi)=\max(\mathcal{A}_\xi)\mathcal{H}_\xi$ and $\mathcal{H}_\xi$ is a finite $\mathcal{A}_\xi$-module), the epimorphism map $\mathbb{R}\{z\}$ into and hence onto $\mathcal{A}_\xi$. Therefore, it induces an isomorphism

$$\tau : \mathbb{C}\{z\}/(f)\,\mathbb{C}\{z\} \xrightarrow{\sim} \mathcal{H}_\xi$$

where $f=(f_1,\ldots f_m)$ with $f_i \in \mathbb{R}\{z\}$, $1 \leq i \leq m$. Pick any open neighborhood U of the origin in $\mathbb{C}^n$ such that $\rho^n(U)=U$ and all the f_i are convergent in U. Let S be the local model of complex space defined by $(U \subset \mathbb{C}^n, f_1,\ldots,f_m)$. Note that the source of τ is canonically identified with $\mathcal{H}_{S,0}$ with the origin 0 of $\mathbb{C}^n$. By (1.7), if we choose U small enough, then there exist an open neighborhood N of ξ in $|X|$ and an isomorphism of complex spaces $h:S \xrightarrow{\sim} X|N$ such that $\theta_{h,0} = \tau$. We may replace U (and N accordingly) by the union of its connected component that meet $\mathbb{R}^n$. Let σ be the auto-conjugation of S defined by (1.13.3). Then the two morphisms of complex spaces, $\rho_X(\sigma_X|N)h$ and $\rho_X h\sigma$, from S to $^*X$ induce the same isomorphism from $\mathcal{H}_{S,0}$ to $\mathcal{H}_{^*X,\xi}$. This follows from the definition of τ. (τ transmport the complex conjugation of $\mathbb{C}\{z\}$ over $\mathbb{R}\{z\}$ to $\mathcal{O}$). By the uniqueness assertion in (1.7), the two morphism coincide within a sufficiently small neighborhood of 0 in $\mathcal{S}$. Hence the same is true for $(\sigma_X|N)h$ and $h\sigma$. This means that the required properties of h in (1.13.4) can be realized by replacing U by a possibly smaller neeghborhood of 0 in $\mathbb{C}^n$.

Proof of (1.13). Let $X^{\mathbb{R}}$ be the σ_X-invariance space of X. To prove that $X^{\mathbb{R}}$ is a real-analytic space, pick any point $\xi \in |X^{\mathbb{R}}|$. Then we have an isomorphism $h : \mathcal{S} \longrightarrow X|N$ Having the properties of (1.13.4). It then follows that $X^{\mathbb{R}}|N$ is isomorphic (as $\mathbb{R}$-ringed space) to the σ-invariance space of $\mathcal{S}$, which is a local model of real-anal tic space by (1.13.3).

Q.E.D.

Definition (1.14). Under the assumption of (1.13), the σ_X-invariance space of X will be called the real part of X with respect to the given auto-conjugation σ_X.

Definition (1.15). Let X be a real-analytic space. Then a complexification of X means a system $(\tilde{X},\sigma,h)$ where $\tilde{X}$ is a complex space, σ is an auto-conjugation of $\tilde{X}$ and h is an isomorphism of real-analytic spaces from X to the real part of $\tilde{X}$ with respect to σ. We shall often think of h as id_X. Also $\tilde{X}$ will be often called complexification of X (by above of terminology), and σ the canonical auto-conjugation of such $\tilde{X}$.

Remark (1.16). Let $\mathcal{S}$ be a local model of real-analytic space defined by data $(U \subset \mathbb{R}^n, g_1,\dots g_n)$ as in (1.4). Pick an open subset V_o of $\mathbb{C}^n$, containing U, to which all the g_i extend to holomorphic functions. Pick one such extension for each i and call it g_i again. Then we can always find an open neighborhood V of U in V_o

such that $\rho^n(V) = V$, $V \cap \mathbb{R}^n = \mathcal{V}$ and every connected component of V meet $\mathbb{R}^n$. Let $\tilde{S}$ be the local model of complex space defined by $(V \subset \mathbb{C}^n, g_1, \ldots, g_m)$ as in (1.2). Then, by (1.13.3) we have an auto-conjugation σ of $\tilde{S}$ induced by ρ^n on $\mathbb{C}^n$. According to the definition (1.15), (1.13.3) implies that $(\tilde{S}, \sigma, id_S)$ is a complexification of S. Thus every local model of real-analytic space admits a complexification, which is a local model of complex space.

Quite generally, if $g: X_1 \to X_2$ is a morphism of K-ringed spaces and if σ_i is an automorphism of X_i, $i=1,2$ such that $g\sigma_1 = \sigma_2 g$, then g induces a morphism of K-ringed spaces, from the σ_1- invariance space of X_1 to the σ_2-invariance space of X_2. In particular, take the case in which X_i is a complex space with an auto-conjugation σ_i, $i=1,2$ and g is a morphism of complex spaces. $(K=\mathbb{R})$. Let $X_i^{\mathbb{R}}$ denote the real part of X_i with respect to σ_i. If $g\sigma_1 = \sigma_2 g$ as above, then g induces a morphism of real-analytic spaces $X_1^{\mathbb{R}} \to X_2^{\mathbb{R}}$ which will be denoted by $g^{\mathbb{R}}$.

Let X_i be a real-analytic space and let $(\tilde{X}_i, \sigma_i, h_i)$ be a complexification of X_i, $i=1,2$. Then a <u>morphism</u> $g: (\tilde{X}_1, \sigma_1, h_1) \to (\tilde{X}_2, \sigma_2, h_2)$ will mean a morphism of complex spaces $g: \tilde{X}_1 \to \tilde{X}_2$ such that $g\sigma_1 = \sigma_2 g$. If this is so, $g^{\mathbb{R}}$ is well defined and there exists a unique morphism of real-analytic spaces $f: X_1 \to X_2$ such that $(g^{\mathbb{R}})h_1 = h_2 f$.

<u>Proposition</u> (1.17). Let $f: X_1 \to X_2$ be a morphism of real-analytic spaces. Let $(\tilde{X}_i, \sigma_i, h_i)$ be a complexification of X_i, $i=1,2$. Let be any point of X_1. Then we have:

1) There exists an open neighborhood V of ξ in $|\tilde{X}_1|$ such that $\sigma_1(V) = V$, and a morphism g from $(\tilde{X}_1, \sigma_1, h_1)|V$ to $(\tilde{X}_2, \sigma_2, h_2)$ such that $g^{\mathbb{R}}(h_1|V_1) = h_2(f|V_1)$ where $V_1 = h_1^{-1}(V)$.

2) If $g_\alpha : (\tilde{X}_1, \sigma_1, h_1)|V_\alpha \longrightarrow (\tilde{X}_2, \sigma_2, h_2)$ satisfy the condition of 1) for $\alpha = 1,2$, then there exists a neighborhood W of ξ in

$V_1 \cap V_2$ such that $g_1|W = g_2|W$.

<u>Proof</u>. Let us make identifications so as to have h_i=identity of X_i, $i=1,2$. Let $\mathcal{H}_i$ denote the local ring of $\tilde{X}_i$ at ξ_i, where $\xi = \xi_1$ and $\xi_2 = f(\xi_1)$. Let $\mathcal{Q}_i$ denote the local ring of X_i as ξ_i. Then $\mathcal{H}_i$ is a free $\mathcal{Q}_i$-module of rank 2 generated by 1 and $\sqrt{-1}$, i.e., there exists a canonical isomorphism of $\mathbb{C}$-algebras $\mathcal{Q}_i \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathcal{H}_i$. Note that the auto-morphism of $\mathcal{H}_i$ induced by σ_i corresponds to $\mathrm{id} \otimes_{\mathbb{R}} \rho$. Let $\tau : \mathcal{H}_2 — \mathcal{H}_1$ be the homomorphism of $\mathbb{C}$-algebras defined by $\theta \otimes_{\mathbb{R}} \mathrm{id}_{\mathbb{C}}$ where $\theta = \theta_{f,\xi} : \mathcal{Q}_2 \to \mathcal{Q}_1$. Now the existence and the uniqueness of (1.17) can be easily deduced by applying (1.7) to this τ (and to the composition of τ with the automorphisms induced by σ_i, $i=1,2$).

<u>Corollary</u> (1.17.1) Let $(\tilde{X}_i, \sigma_i h_i)$, $i=1,2$, be two complexifications of the same real-analytic space X. Then for every $\xi \in X$, we can find an open neighborhood V_i of $h_i(\xi)$ in $|\tilde{X}_i|$, $i=1,2$, and an isomorphism

$$g : (\tilde{X}_1, \sigma_1, h_1)|V_1 \xrightarrow{\sim} (\tilde{X}_2, \sigma_2, h_2)|V_2$$

such that $\sigma_i(V_i)=V_i$ for both $i=1,2$ and $g^{\mathbb{R}}(h_1|V)=h_2|V$, where $V=h_1^{-1}(V_1)$.

<u>Proof</u>. Immediate from the existence and the uniqueness of (1.17).

<u>Proposition</u> (1.18) Let $(\tilde{X}_i, \sigma_i, h_i)$, $i=1,2$, be two complexifications of the same real-analytic space X. Then there exists a third complexification $(\tilde{X}', \sigma', h')$ which admits morphisms

$$g_i : (\tilde{X}', \sigma', h') \to (\tilde{X}_i, \sigma_i, h_i)$$

such that $(g_i^{\mathbb{R}})\, h' = h_i$, for both $i=1,2$.

Proof. Let us assume $h_i = id_X$, $i=1,2$. In this case, we shall write $(\tilde{X}_i, \sigma_i)$ for (X_i, σ_i, h_i). For each $\xi \in |X|$, let us pick an isomorphism

$$f_\xi : (\tilde{X}_1, \sigma_1) | V_\xi \xrightarrow{\sim} (\tilde{X}_2, \sigma_2) | V'_\xi$$

where V_ξ (resp. V'_ξ) is an open neighborhood of ξ in $|\tilde{X}_1|$ (resp. $|\tilde{X}_2|$), such that $\sigma_1(V_\xi) = V_\xi$ and $f_\xi^{\mathbb{R}} = id_X$. Let $V_{\xi\eta}$ be the maximal open subset of $V_\xi \cap V_\eta$ such that $f_\xi | V_{\xi\eta} = f_\eta | V_{\xi\eta}$. Note that $V_{\xi\eta} \cap V_{\eta\zeta}$ is the maximal open subset of $V_\xi \cap V_\eta \cap V_\zeta$ in which f_ξ, f_η, and f_ζ induce the same morphism and hence independent of the ordering of ξ, η, ζ. This enables us to construct a complex space $\tilde{X}'$ by glueing $\{\tilde{X}_1 | V_\xi\}_{\xi \in |X|}$ exactly on $\tilde{X}_1 | V_{\xi\eta}$ for the pairs $(\xi, \eta) \in |X| \times |X|$ (and no more). We have an obvious morphism $g_1 : \tilde{X}' \to \tilde{X}_1$. Moreover, since $f_\xi | V_{\xi\eta} = f_\eta | V_{\xi\eta}$, we obtain a morphism $g_2 = \tilde{X}' \to \tilde{X}_2$ such that $g_2 | V_\xi = (f_\xi g_1) | V_\xi$ for all ξ. Since $\sigma_1(V_\xi) = V_\xi$, we get $\sigma_1(V_{\xi\eta}) = V_{\xi\eta}$ and hence σ_1 induces an auto-conjugation σ' of $\tilde{X}'$. As $V_{\xi\eta} \supset V_\xi \cap V_\eta \cap |X|$ by (1.17), we have the identity $\tilde{X}'^{\mathbb{R}} = \tilde{X}_1^{\mathbb{R}} = X$. Thus $(\tilde{X}', \sigma', h')$ with $h' = id_X$ is a complexification of X. It is easy to check that g_i, $i=1,2$ are morphisms satisfying the required conditions of (1.18).

Proposition (1.19) Let $g : (\tilde{X}', \sigma', h') \to (\tilde{X}, \sigma, h)$ be a morphism of two complexifications of the same real-analytic space X, such that $(g^{\mathbb{R}}) h' = h$. Assume that $\tilde{X}$ is Hausdorff and paracompact. Then there exist an open neighborhood V' (resp. V) of $h'(|X|)$ (resp. $h(|X|)$)

in $|\tilde{X}'|$ (resp. $|\tilde{X}|$) such that $\sigma'(V')=V'$ and g induces an isomorphism

$$(\tilde{X}',\sigma',h')|V' \xrightarrow{\sim} (\tilde{X},\sigma,h)|V \quad .$$

<u>Proof</u>. Again assume $h'=h=id_X$. For each $\xi \in |X|$, there exists an open neighborhood N'_ξ (resp. N_ξ) of ξ in $|\tilde{X}'|$ (resp. $|\tilde{X}|$) such that g induces an isomorphism $g_\xi : \tilde{X}'|N'_\xi \xrightarrow{\sim} \tilde{X}|N_\xi$. We may assume that N'_ξ (resp. N_ξ) is contained in a local model of $\tilde{X}'$ (resp. $\tilde{X}$). Since $\tilde{X}$ is Hausdorff, $|X|$ is closed in $|\tilde{X}|$ because it is the fixed point set of the continuous auto-morphism σ. Since $\tilde{X}$ is paracompact, we can find a locally finite refinement of the covering $\{N_\xi, |\tilde{X}|-|X|\}$, $\xi \in |X|$, of $|\tilde{X}|$. This implies that there exists a family of open subsets $\{M_\alpha\}_{\alpha\in\Delta}$ in $|\tilde{X}|$ together with a map $\xi : \Delta \longrightarrow |X|$ such that

1) $\{M_\alpha\}_{\alpha\in\Delta}$ is locally finite in $|\tilde{X}|$

2) $M_\alpha \subset N_{\xi(\alpha)}$ for all $\alpha \in \Delta$, and

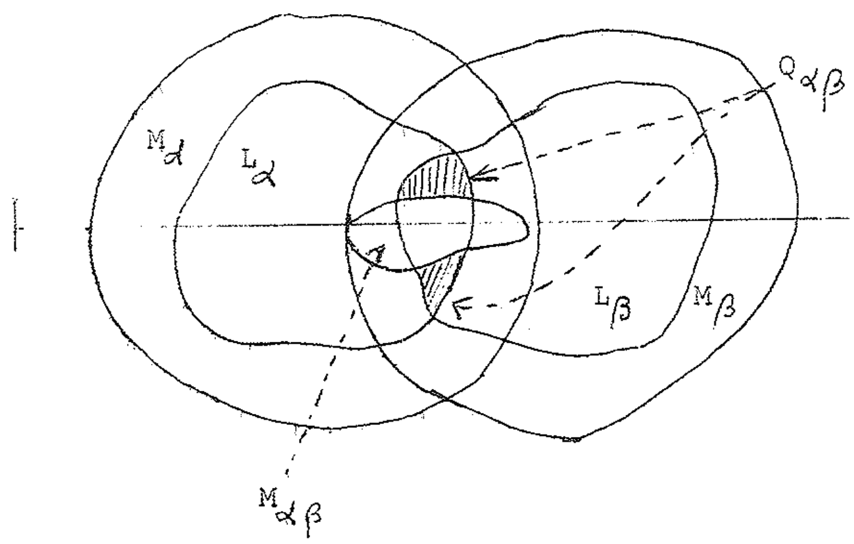
3) $\bigcup_{\alpha\in\Delta} M_\alpha \supset |X|$.

We shall write g_α for $g_{\xi(\alpha)}$. Since $\tilde{X}$ is paracompact and since M_α is locally compact and countable at infinity, we can find open subsets $\{L_\alpha\}_{\alpha\in\Delta}$ in $|\tilde{X}|$ such that

4) L_α is relatively compact in M_α for all $\alpha \in \Delta$,

5) $\bigcup_{\alpha\in\Delta} L_\alpha \supset |X|$.

Let $M_{\alpha\beta} = g_\alpha(g_\alpha^{-1}(M_\alpha)\cap g_\beta^{-1}(M_\beta))=g_\beta(g_\alpha^{-1}(M_\alpha)\cap g_\beta^{-1}(M_\beta))$ which is the maximal open subset of $M_\alpha \cap M_\beta$ such that $g_\alpha^{-1}|M_{\alpha\beta} = g_\beta^{-1}|M_{\alpha\beta}$. By (1.17), we have $|X|\cap M_\alpha \cap M_\beta = |X| \cap M_{\alpha\beta}$. Let $Q_{\alpha\beta} = \bar{L}_\alpha \cap \bar{L}_\beta - M_{\alpha\beta}$. This i a compact subset of $M_\alpha \cap M_\beta$ which does not meet $|X|$. ($Q_{\alpha\alpha} = \emptyset$ for all $\alpha \in \Delta$). Let $V_\alpha = L_\alpha - \bigcup_{\beta\in\Delta} Q_{\alpha\beta}$. This last union being finite by 1) and 4), V_α is open in $|\tilde{X}|$. Moreover $V_\alpha \cap |X| = L_\alpha \cap |X|$. As $V_\alpha \cap V_\beta \subset M_{\alpha\beta}$, $g_\alpha^{-1}|V_\alpha \cap V_\beta = g_\beta^{-1}|V_\alpha \cap V_\beta$. Let $V=(\bigcup_{\alpha\in\Delta} V_\alpha)\cap\sigma(\bigcup_{\alpha\in\Delta} V_\alpha)$,



which is an open neighborhood of $|X|$ in $|\tilde{X}|$ such that $\sigma(V)=V$. Moreover there exists a well-defined morphism $g' : \tilde{X}|V \to \tilde{X}'|V'$ with $V' = \bigcup_{\alpha\in\Delta} g_\alpha^{-1}(V_\alpha \cap V)$, such that $g'|V_\alpha = g_\alpha^{-1}|V_\alpha$ for all $\alpha\in\Delta$. Clearly $(g|V')g' = id_{\tilde{X}}|V$. Hence g induces an isomorphism

$$(\tilde{X}',\sigma',h')|V' \xrightarrow{\sim} (\tilde{X},\sigma,h)|V .$$

Corollary (1.19.1). If $(\tilde{X}_i,\sigma_i,h_i)$ are two complexifications of the same real-analytic space X, such that both $\tilde{X}_i$ are Hausdorff and paracompact, then there exist open neighborhoods V_i of $h_i(|X|)$ in $|\tilde{X}_i|$ and an isomorphism

$$g : (\tilde{X}_1, \sigma_1, h_1)|V_1 \xrightarrow{\sim} (\tilde{X}_2,\sigma_2,h_2)|V_2$$

where $\sigma_i(V_i)=V_i$, $i=1,2$, and $(g^{\mathbb{R}})(h_1|V) = h_2|V$ with $V = h_1^{-1}(V_1) = h_2^{-1}(V_2)$.

Proof. Take a third complexification $(\tilde{X}',\sigma',h')$ of X having g_i as in (1.18). Then apply (1.19) to each one of the g_i, $i=1,2$. Restricting the results to a suitable common domain, we obtain g by composition.

That every (not necessarily paracompact) real analytic space has a complexification way shown by A. Tognoli in: Introduzione alla teoria degli spazi analitici reali, Appunti redatti dalla dott. Dina Ghinelli Smit (contributi del Centro Linceo Interdisciplinare di Scienze Matematiche e loro applicazioni).

§. 2 Local blowing-ups

We have a natural map $p_\circ : \mathbb{C}^n - \{0\} \longrightarrow \mathbb{P}_{\mathbb{C}}^{n-1}$, where $\mathbb{P}_{\mathbb{C}}^{s}$ denotes the complex projective space of dimension s. The morphism $p_\circ$ is such that for every $\xi \in \mathbb{P}_{C}^{n-1}$, $p_\circ^{-1}(\xi) \cup \{0\}$ is a complex line through the origin in $\mathbb{C}^n$. By assigning to each $\xi \in \mathbb{P}_{\mathbb{C}}^{n-1}$ the complex line obtained in this way, we obtain a complex line bundle

$$p : L \longrightarrow \mathbb{P}_{\mathbb{C}}^{n-1}$$

The complement of the zero section of p in L is equal to $\mathbb{C}^n - \{0\}$, and we have a natural morphism

$$\pi : L \longrightarrow \mathbb{C}^n$$

which is isomorphic outside the zero section of p and the zero section is mapped to the origin 0.

With the coordinate system $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$, the complex space L can be constructed as follows:

$$L = \bigcup_{i=1}^{n} L_i \quad , \ L_i = \mathbb{C}^n, \quad 1 \leq i \leq n$$

where

(1) L_i has a coordinate system $t_{i1}, \ldots, t_{in}$ and $\pi | L_i$ is defined by

$$z_j \pi = \begin{cases} t_{ii} & \text{if } j = i \\ t_{ii} t_{ij} & \text{if } j \neq i \end{cases}$$

(2) $L_i \cap L_j$ in L_i is defined by $t_{ij} \neq 0$, and $t_{i\alpha}$ and $t_{j\beta}$ are related by

$$t_{ij} t_{ji} = 1, \ t_{ji} t_{i\alpha} = t_{j\alpha}, \ t_{ii} = t_{jj} t_{ji}, \ 1 \leq i,j \leq n .$$

Definition (2.1) Let X be a complex space and Y a closed complex subspace of X, i.e., there is a coherent ideal sheaf J in $\mathcal{H}_X$ such that $|Y| = \mathrm{supp}(\mathcal{H}_X/J)$ and $\mathcal{H}_Y$ is the restriction of $\mathcal{H}_X/J$ to $|Y|$. Then a morphism $\pi : X' \to X$ is said to be the blowing-up of X with center Y if the following conditions are satisfied:

1) $J\mathcal{H}_{X'}$ (= the ideal sheaf in $\mathcal{H}_{X'}$ generated by J with respect to θ_π) is invertible as $\mathcal{H}_{X'}$ - module, i.e., locally principal and generated by a non-zero divisor.

2) given any morphism of complex spaces $f : T \to X$ such that $J\mathcal{H}_T$ is invertible as $\mathcal{H}_T$ - module, there exists one and only one morphism $f' : T \to X'$ such that $\pi f' = f$.

Remark (2.2) The conditions 1) and 2) make a universal mapping property for π. So if $\pi_1 : X_1' \to X$ is any other morphism satisfying the two conditions, then there exists a unique isomorphism $h : X' \xrightarrow{\sim} X_1'$ such that $\pi_1 h = \pi$.

Lemma (2.3) The morphism $\pi : L \to \mathbb{C}^n$ defined above is the blowing-up with center O.

Proof. Let J be the ideal of O in $\mathcal{H}_{\mathbb{C}^n}$, i.e., $J = (z_1, \dots, z_n)\mathcal{H}_{\mathbb{C}^n}$. Then, by (1) on the coordinate system $(t_{i1}, \dots, t_{in})$ of L_i, $J\mathcal{H}_{L_i}$ is generated by $t_{ii} = z_i\pi$ and hence is invertible as $\mathcal{H}_{L_i}$ - module. This being true for all i, the condition 1) is verified. To prove the condition 2) of (2.1), take any $f : T \to \mathbb{C}^n$ such that $J\mathcal{H}_T$ is invertible as $\mathcal{H}_T$-module. Then we have an open covering $\{N_i\}_{1 \le i \le n}$ of $|T|$ such that $z_i f$ generate $J\mathcal{H}_T|N_i$ for

all i. (Here use the fact that in a local ring, as $\mathcal{H}_{T,t}$, if a principal ideal in it is generated by $u_1,\dots,u_m$ then in fact it is generated by one the u_i). It is then enough to prove the unique existence of the lifting of 2) of (2.1) for the restrictions of f to the N_i and to the $N_i \cap N_j$. In other words, it is enough to consider the case in which there exists an index i such that $z_i f$ generates $J\mathcal{H}_T$.
Since $J\mathcal{H}_T$ is invertible, there exist sections ξ_{ij} of $\mathcal{H}_T$ such that $z_j f = (z_i f)\xi_{ij}$ for each $j \neq i$. Let $\xi_{ii} = z_i f$. So if we define $h : T \to L_i \subset L$ by letting $\xi_{ij} = t_{ij} h$, $1 \leq j \leq n$, then we have $\pi h = f$. Now, for the uniqueness, if $h' : T \to L$ is any morphism with $\pi h' = f$, then first of all the image of h' must be in L_i. In fact, if not, there must be a point $t \in T$ and an index $j \neq i$ such that $(t_{ji} h')(t) = 0$, i.e., $t_{ji} h' \in \max(\mathcal{H}_{T,t})$. This means $z_i f \in (z_j f)\max(\mathcal{H}_{T,t})$, which contradicts the assumption that $z_i f$ generates $J\mathcal{H}_{T,t} = (z_1 f,\dots,z_n f)\mathcal{H}_{T,t}$. Now, knowing $\operatorname{Im}(h') \subset L_i$, $\pi h' = f$ implies that $z_j f = (z_j \pi) h'$ for all j, $1 \leq i \leq n$. This then implies that $\xi_{ij} = t_{ij} h'$, $1 \leq j \leq n$. Since $(t_{ij}, 1 \leq j \leq n)$ is a coordinate system for $L_i = \mathbb{C}^n$, it follows that $h' = h$.

<u>Lemma</u> (2.4) Let $\pi : X' \to X$ is the blowing-up with center Y as in (2.1). Let U be any open subset of $|X|$. Then $\pi \mid U : X' \mid \pi^{-1}(U) \to X \mid U$ is the blowing-up with center $Y|U$.

<u>Proof</u>. The properties 1)-2) of (2.1) for π imply the same for $\pi \mid U$.

<u>Lemma</u> (2.5) If $\pi : X' \to X$ is the blowing-up with center Y and V is any complex space, then $\pi \times id_V : X' \times V \to X \times V$ is the

blowing-up with center $Y \times V$.

Proof. Let J be the ideal sheaf of Y in $\mathcal{H}_X$. Then $J\mathcal{H}_{X\times V}$ (with respect to the projection) is the ideal sheaf of $Y \times V$. Since $(J\mathcal{H}_{X\times V})\mathcal{H}_{X'\times V} = (J\mathcal{H}_{X'})\mathcal{H}_{X'\times V}$, 1) of (2.1) is evident for $\pi\times id_V$.
Denote the projections by $p_1 : X\times V \to X, p_2 : X\times V \to V$, $p_1' : X'\times V \to X'$ and $p_2' : X'\times V \to V$. Take any morphism $f : T \to X \times V$ such that $(J\mathcal{H}_{X\times V})\mathcal{H}_T$ is invertible as $\mathcal{H}_T$-module. Since this ideal is nothing but $J\mathcal{H}_T$ with respect to $p_1 f$, the definition of π implies there exists a unique $f_1' : T \to X'$ such that $\pi f_1' = p_1 f$. So there exists a unique $f' : T \to X'\times V$ such that $f_1' = p_1' f'$ and $p_2 f = p_2' f'$. Here the uniqueness is by the universal mapping property of product space. These conditions for f' is equivalent to the condition that $(\pi \times id_V)f' = f$, again by the universal mapping property of product space, this time for $X \times V$.

Example (2.6) Let $\mathbb{C}^r \hookrightarrow \mathbb{C}^m$ be a linear subspace. Then there exists the blowing-up $\pi' : L' \to \mathbb{C}^m$ with center $\mathbb{C}^r$. In fact, write $\mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^n$ so that $\mathbb{C}^r$ is identified with $\mathbb{C}^r \times 0$.
Then with $\pi : L \to \mathbb{C}^n$ of (2.3), let $\pi' = id_{\mathbb{C}^r} \times \pi$, which is the blowing-up with center $\mathbb{C}^r$ by (2.5). It is important to note that, by the definition (2.1), the blowing-up π' depends only upon the pairs $\mathbb{C}^r \subset \mathbb{C}^m$ as complex spaces and it is independent of the linear structures in them and of the choice of the decomposition $\mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^n$.

Proposition (2.7) Let $\pi : X' \to X$ be the blowing-up with center Y. Let Z be any locally closed (i.e., closed in some open subset) complex subspace of X. Then there exists the smallest (unique) closed complex subspace Z' of $\pi^{-1}(Z)$ such that $Z' - \pi^{-1}(Y) = \pi^{-1}(Z) - \pi^{-1}(Y)$.

Moreover, π induces the blowing-up $\pi' : Z' \longrightarrow Z$ with center $Z \cap Y$.

<u>Remark</u> (2.7.1) If Z is given as a closed complex subspace in an open restriction $X|U$, say defined by an ideal sheaf I in $\mathcal{H}_X|U$, then $\pi^{-1}(Z)$ is by definition the closed complex subspace of $X'|\pi^{-1}(U)$ defined by $I\mathcal{H}_{X'}|\pi^{-1}(U)$.
Let J be the ideal sheaf of Y in $\mathcal{H}_X$. Then $Z \cap Y$ means the closed complex subspace of Z defined by $J\mathcal{H}_Z$ (or a locally closed complex subspace of X defined by $(I+J)\mathcal{H}_X|U$).

The proof of (2.7) will be given after a few lemmas.

<u>Lemma</u> (2.7.2) for a coherent $\mathcal{H}_T$-module $\mathcal{F}$ on a complex space T, the ideal sheaf of annihilators of $\mathcal{F}$ in $\mathcal{H}_T$ is

$$\underline{\mathrm{Ann}}_{\mathcal{H}_T}(\mathcal{F}) = \underline{\mathrm{Ker}}(\mathcal{H}_T \xrightarrow{m} \underline{\mathrm{Hom}}_{\mathcal{H}_T}(\mathcal{F},\mathcal{F})) \quad \text{where } m \text{ is defined}$$

by the multiplication of $\mathcal{H}_T$ on $\mathcal{F}$. This equality shows that $\underline{\mathrm{Ann}}_{\mathcal{H}_T}(\mathcal{F})$ is again coherent.

(We omit the proof of this lemma in which the essential point is Oka's coherency theorem).

<u>Lemma</u> (2.7.3) Let $\mathcal{F}$ be a coherent $\mathcal{H}_T$-module on a complex space T. Let $\{\mathcal{F}_\alpha\}_{\alpha\in\Delta}$ be a family of coherent $\mathcal{H}_T$-submodules $\mathcal{F}_\alpha$ of $\mathcal{F}$, such that for every pair $\alpha, \beta \in \Delta$ there exists $\gamma \in \Delta$ with $\mathcal{F}_\gamma \supset \mathcal{F}_\alpha + \mathcal{F}_\beta$. Then $\bigcup_{\alpha\in\Delta}\mathcal{F}_\alpha$ is coherent (and hence, locally around any given point in T, isomorphic to some $\mathcal{F}_\alpha$).

(This is a theorem due to Cartan. A proof of this fact, which we omit here, invalues "Privileged Neighborhood Theorem" of Cartan and "Theorems A and B" of Cartan-Serre).

Lemma (2.7.4) Let S be a closed complex subspace of a complex space T, and let J be the ideal sheaf of S in $\mathcal{H}_T$. Assume that $\underline{\mathrm{Ann}}_{\mathcal{H}_T}(J) = (0)$. Then the pair (T,S) satisfies the minimality condition, i.e., if T' is any closed complex subspace of T such that $T'-S = T-S$, then $T' = T$.

(The proof of this lemma is based on the so-called Nullstellensatz in complex-analytic geometry. In fact, take any T' as in the Lemma and let N be the ideal sheaf of T' in $\mathcal{H}_T$. Then $\mathrm{Supp}(N) \subset |S|$. So, by Nullstellensatz, for every point $t \in S$ there exists an integer $n(t) > 0$ such that $J^{n(t)}N$ is zero in some neighborhood of t in T. Then the assumption of J implies that N is zero in some neighborhood of t in T. This being true for every $t \in S$, we conclude $T' = T$).

Proof of (2.7). Let J be the ideal sheaf of Y in $\mathcal{H}_X$. Let $T = \pi^{-1}(Z)$. Let us define

$$H = \bigcup_{m=1}^{\infty} \underline{\mathrm{Ann}}_{\mathcal{H}_T}(J^m \mathcal{H}_T)$$

Then this is a coherent ideal sheaf in $\mathcal{H}_T$ by Lemmas (2.7.2) and (2.7.3). Let Z' be the closed complex subspace of T defined by H. Since $J\mathcal{H}_T$ is the unit ideal in $T-\pi^{-1}(Y)$, we have $T-\pi^{-1}(Y) = Z'-\pi^{-1}(Y)$. The ideal sheaf of annihilators of $J\mathcal{H}_{Z'}$ in $\mathcal{H}_{Z'}$ is zero because its inverse image in $\mathcal{H}_T$ is necessarily contained in the union which defines H. On one hand, this implies that $(Z', Z' \cap \pi^{-1}(Y))$ satisfies the minimality condition by (1.7.4). On the other hand, it also implies that $J\mathcal{H}_{Z'}$ is invertible as $\mathcal{H}_{Z'}$-module because $J\mathcal{H}_{Z'}$ is locally principal as $J\mathcal{H}_{X'}$ is so. To prove that $\pi': Z' \longrightarrow Z$, induced by π, is the blowing-up of

Z with center $Y \cap Z$, let us take any morphism $f: F \to Z$ such that $J\mathcal{H}_F$ (which is the ideal generated by the ideal sheaf of $Y \cap Z$ in $\mathcal{H}_Z$) is invertible as $\mathcal{H}_F$-module. Viewing f as a morphism from F into X, we get a morphism $f': F \to X'$ such that $\pi f' = f$. Then, clearly, f' is also a morphism from F into $T = \pi^{-1}(Z)$. Since $\underline{\mathrm{Ann}}_{\mathcal{H}_F}(J^m\mathcal{H}_F) = (0)$ for all $m \geq 1$, $H\mathcal{H}_F = 0$. This means that f' is also a morphism into Z'.

<u>Definition</u> (2.8) The locally closed complex subspace Z' of (2.7) is called the strict transform of Z by the blowing-up π with center Y.

<u>Proposition</u> (2.9) For every complex space X and every closed complex subspace Y of X, there exists the blowing-up $\pi: X' \to X$ with center Y.
Moreover, π induces an isomorphism from $X' - \pi^{-1}(Y)$ to $X-Y$, π is proper and $\pi^{-1}(U)$ is Hausdorff for every Hausdorff open subset U of $|X|$.

<u>Proof</u>. Thanks to the uniqueness (2.2), the existence of blowing-up with a given center is a local question. Hence we may assume that X is given as a local model in some open subset V of $\mathbb{C}^r$ and that the ideal sheaf of Y in $\mathcal{H}_X$ is generated by a finite number of holomorphic functions $h_1, \ldots, h_n$ in V. Let us imbedd X into $\mathbb{C}^r \times \mathbb{C}^n$ by the inclusion $X \subset \mathbb{C}^r$ and by $(h_1, \ldots, h_n)$. In this way, we view X as being given as a locally closed complex subspace of $\mathbb{C}^{r+n}$ in such a way that Y is the intersection $X \cap \mathbb{C}^r$, where $\mathbb{C}^r$

denotes the linear subspace $\mathbb{C}^r \times 0$ of $\mathbb{C}^{r+n}$. Let $\pi' : L' \to \mathbb{C}^{n+r}$ be the blowing-up obtained in (2.6). Let X' be the strict transform of X by the blowing-up π' with center $\mathbb{C}^r$. (cf.(2.8)). Then, by (2.7), π' induces the blowing-up $\pi : X' \to X$ with center Y. This proves the existence of blowing-up. Now, by (2.4), π induces the blowing-up $X' - \pi^{-1}(Y) \to X - Y$ with the empty center. When the center is empty, the identity map has all the properties of blowing-up, given in (2.1). By the uniqueness, the induced morphism outside the center is an isomorphism. To prove that π is proper (i.e., the inverse image of any compact subset is compact), and to prove the assertion on Hausdorff property, the question is local in X. So, again, we may take the local situation as above. It is clear that $\pi' : L' \to \mathbb{C}^{n+r}$ is proper and hence so is π. Also clear is that L' is a Hausdorff space and hence so is X'.

<u>Proposition</u> (2.10) Let Y_α, $\alpha = 1,2$, be closed complex subspaces of a complex space X. Let I_α be the ideal sheaf of Y_α in $\mathcal{H}_X$. Then we have:

1) For every morphism $f : T \to X$, $I_1 I_2 \mathcal{H}_T$ is invertible as $\mathcal{H}_T$-module if and only if $I_\alpha \mathcal{H}_T$ is so for both $\alpha = 1,2$.

2) If Y_3 is the closed complex subspace of X defined by $I_1 I_2$ and $\pi_\beta : X_\beta \to X$ is the blowing-up with center Y_β, $\beta = 1,2,3$, then we have a commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "\pi_1"] & \\ X_3 \arrow[ur, "q_1"] \arrow[rr, "\pi_3"] \arrow[dr, "q_2"'] & & X \\ & X_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Proof. The existence of q_1 and q_2 in 2) follows immediately from 1). So it is enough to prove 1). The assertion in 1) is easily reduced to the corresponding question in the local rings $\mathcal{H}_{T,t}$, $t \in T$. So all we need is to prove the following

Lemma (2.10.1). Let A be a local ring and let I_α, $\alpha = 1,2$, be two ideals in A. If $I_1 I_2$ is principal and generated by a non-zero divisor in A, then the same is true for each I_α, $\alpha = 1,2$.

Proof. Pick any system of generators $\{g_{\alpha,j}\}_{1\leq j\leq m}$ of I_α. Then $\{g_{1j}g_{2k}\}$ generate $I_1 I_2$. Since A is local, there exists a pair (j,k) such that $I_1 I_2$ is generated by $g_{1j}g_{2k}$. Let $g_1 = g_{1j}$ and $g_2 = g_{2k}$ with this (j,k). Then $g_1 g_2$ is not a zero-divisor in A and hence none of the two g_α is. $I_1 I_2 \supset g_1 I_2 \supset g_1 g_2 A = I_1 I_2$. So $g_1 I_2 = g_1 g_2 A$. Hence $I_2 = g_2 A$. Similarly $I_1 = g_1 A$.

Definition (2.11) A morphism of complex spaces $f : V \to W$ is said to be strict if there exists a closed complex subspace F of V such that

1) f is locally isomorphic in $V-F$, and
2) (V,F) satisfies the minimality condition. (cf. (2.7.4)).

Lemma (2.12) Let Y be a closed complex subspace of a complex space X. Let J be the ideal sheaf of Y in $\mathcal{H}_X$. Let $\pi : X' \to X$ be the blowing-up with center Y. Let $f : V \to X$ be any strict morphism. Then there exists at most one morphism $f' : V \to X'$ such that $\pi f' = f$. Moreover, if such f' exists, then $J\mathcal{H}_V$ is invertible as $\mathcal{H}_V$-module and f' is also a strict morphism.

Proof. Assume that there exists a morphism f' with $\pi f'=f$. We shall prove that $J\mathcal{H}_V$ (with respect to this f') is invertible as $\mathcal{H}_V$-module. Note that if this is proven then the uniqueness of f' follows from the universal mapping property of the blowing-up π. Pick a closed complex subspace F of V which has the two properties of (2.11). Pick any point $v \in V - F$. If $f(v) \notin Y$, then f' is locally isomorphic in some neighborhood of v because π is isomorphic on $X-Y$. Assume that $f(v) \in Y$. In any case, we have an open neighborhood N of v in V such that f induces a locally isomorphic open imbedding of $V|N$ to X. So f' induces a locally closed imbedding of $V|N$ into X', i.e., there exists an open subset M of X' and a closed complex subspace X'' of $X'|M$ such that f' induces an isomorphism from $V|N$ to X''. Then by what was seen above, $X'' - \pi^{-1}(Y) = X'|M - \pi^{-1}(Y)$. But, since the ideal sheaf of $\pi^{-1}(Y)$ is invertible as $\mathcal{H}_{X'}$-module, it follows by (2.7.4) that $X''=X'|M$. We have thus proven that f' is isomorphic at every point $v \in V-F$. This proves that f' is also strict and that $J\mathcal{H}_V$ is invertible as $\mathcal{H}_V$-module at least on $V-F$. Clearly $J\mathcal{H}_V$ is locally principal everywhere on V due to the existence of f'. If K is the ideal sheaf of annihilators of $J\mathcal{H}_V$ in $\mathcal{H}_V$, then K defines a closed complex subspace V' of V such that $V'-F = V-F$. By the minimality assumption on (V,F), we must have $V'=V$ and hence $K=(0)$. Therefore $J\mathcal{H}_V$ is invertible as $\mathcal{H}_V$-module.

Corollary (2.12.1) Let $f : V \rightarrow X$ be a strict morphism of complex spaces. Let J be any coherent ideal sheaf in $\mathcal{H}_X$, which is invertible as $\mathcal{H}_X$-module. Then $J\mathcal{H}_V$ is also invertible as $\mathcal{H}_V$-module.

Proof. In this case, id_X is the blowing-up with center Y, where Y is the closed complex subspace of X defined by J. Now (2.12.1) follows immediately from (2.12) with $\pi = id_X$.

Definition (2.13) A local blowing-up over a complex space W is a triple (U,E,π) where

1) U is open subset of W,
2) E is a closed complex subspace of $W|U$, and
3) $\pi: W' \to W$ is the composition of the blowing-up $W' \to W|U$ with center E and the inclusion $W|U \subset W$.

Proposition (2.14). Every morphism $\pi : W' \to W$ obtained by a finite sequence of local blowing-up is strict. In fact, there is a closed complex subspace F of W', defined by an ideal sheaf invertible as $\mathcal{H}_{W'}$-module and such that the condition of (2.11) are satisfied by π and F. If π is such and $f : V \to W$ is any strict morphism, then there exists at most one $h: V \to W'$ such that $f = \pi h$. Such h is again strict.

Proof. Let us first prove that if π is obtained by a single local blowing-up, say (U,E,π), then π is strict. In fact, let J be the ideal sheaf of E in $\mathcal{H}_W|U$. Then $J\mathcal{H}_{W'}$ is invertible as $\mathcal{H}_{W'}$-module. So, by (2.7.4), $(W', \pi^{-1}(E))$ satisfies the minimality condition. As π is locally isomorphic outside $\pi^{-1}(E)$, π is strict. Now let us consider the general case in which π is obtained by a finite sequence $\{(U_i, E_i, \pi_i)\}_{0 \leq i < m}$. Let J_i be the ideal sheaf of E_i in $\mathcal{H}_{W_i}|U_i$, where $\pi_i: W_{i+1} \to W_i$, $W_0 = W$, $W_m = W'$ and $\pi = \pi_0 \pi_1 \cdots \pi_{m-1}$. Let F be the closed complex subspace of W' defined by $J = \prod_{i=0}^{m-1} J_i \mathcal{H}_{W'}$. It is clear that π is locally isomorphic on $W'-F$. Since $J_i \mathcal{H}_{W_{i+1}}$ is invertible as $\mathcal{H}_{W_{i+1}}$-module and since π_j are all strict by the above result, it follows from (2.12.1) that all the $J_i \mathcal{H}_{W'}$ and hence J are invertible as $\mathcal{H}_{W'}$-module. Hence, by (2.7.4), π is strict. To prove the second assertion of (2.14), assume that h exists. Then we get morphisms $h_i : V \to W_i$, $0 \leq i \leq m$, by composing h with the π_j, $m-1 \geq j \geq i$ for each i. As h_1 exists, f is a morphism $V \to W|U_o$ and so (2.12) is applicable to this f. By (2.12), h_1 is unique and strict. Now, replacing f by h_1, the same reasoning shows that h_2 is unique and strict. Repeating this reasoning $m-1$ times, we obtain the uniqueness of h and the fact that h is strict if it exists.

<u>Corollary</u> (2.14.1) Let $\pi_\alpha : W_\alpha \to W$, $\alpha = 1, 2$, be two morphisms of complex spaces into the same W, each of which is obtained by a finite sequence of local blowing-ups over W. Then there exists at most one morphism $h : W_1 \to W_2$ such that $\pi_1 h = \pi_2$.

<u>Proof</u>. Immediate from (2.14).

<u>Notation</u> (2.15) Let us denote by $\mathcal{E}(W)$ for a complex space W, the category in which an object is a morphism $\pi : W' \to W$ which is obtained by a finite sequence of local blowing-ups, and in which a morphism $g : \pi_1 \to \pi_2$ for $\pi_i : W_i \to W$ in $\mathcal{E}(W)$ is a morphism of complex spaces $g : W_1 \to W_2$ such that $\pi_2 g = \pi_1$. The set of morphism $\pi_1 \to \pi_2$ in $\mathcal{E}(W)$ is denoted by $\mathrm{Hom}(\pi_1, \pi_2)$, which consists of at most one element by (2.14.1).

<u>Proposition</u> (2.16) Given any two $\pi_i : W_i \to W$ in $\mathcal{E}(W)$, there exists a third $\pi_3 : W_3 \to W$ in $\mathcal{E}(W)$ such that

1) there exists $q_i \in \mathrm{Hom}(\pi_3, \pi_i)$ for both $i = 1,2$, and

2) for every strict morphism $f : V \to W$, if there exist morphisms $h_i : V \to W_i$ with $f = \pi_i h_i$ for both $i=1,2$, then there exist morphisms $h_i : V \to W_i$ with $f = \pi_i h_i$ for both $i=1,2$, then there exists a unique $h_3 : V \to W_3$ with $q_i h_3 = h_i$, $i=1,2$.
Moreover, q_i belongs to $\mathcal{E}(W_i)$ for both $i=1,2$.

<u>Proof</u>. Let us first consider the case in which π_i is obtained by a single local blowing-up (U_i, E_i, π_i) for both $i=1,2$. In this case, we claim

<u>Lemma</u> (2.16.1) Let $U_3 = U_1 \cap U_2$ and let E_3 be the closed complex subspace of $W|U_3$ which is defined by $J_1 J_2 \mathcal{H}_W | U_3$ where J_i denotes the ideal sheaf of E_i in $\mathcal{H}_W | U_i$, $i=1,2$. Let (U_3, E_3, π_3) be a local blowing-up over W. Then $\pi_3 : W_3 \to W$ has the properties 1) and 2) of (2.16).

In fact, by applying (2.10) to the restrictions to U_3, we find morphisms $q_i : W_3 \to W_i$ with $\pi_i q_i = \pi_3$ for $i=1,2$. Namely 1) of (2.16). Given any f of 2) of (2.16), the existence of the h_i, $i=1,2$, implies that f is actually a morphism $V \to W|U_3$. By (2.12), $J_i \mathcal{H}_V$ is invertible as $\mathcal{H}_V$-module for $i=1,2$. So $J_1 J_2 \mathcal{H}_V$ is invertible as $\mathcal{H}_V$-module. This implies the unique existence of $h_3 : V \to W_3$ with the property 2) of (2.16) by the universal mapping property of the blowing-up $\pi_3 : W_3 \to W|U_3$. Namely 2) of (2.16).
We have proven (2.16.1).

<u>Lemma</u> (2.16.2) In the situation of (2.16.1), we have a local blowing-up $(\pi_i^{-1}(U_j), \pi_i^{-1}(E_j), q_i)$ with $q_i \in \mathrm{Hom}(\pi_3, \pi_i)$, where $(i,j) = (1,2)$ or $(2,1)$. Namely $q_i \in \mathcal{E}(W_i)$, $i=1,2$.

In fact, take the case of $(i,j) = (1,2)$. Let $W' = W_1 | \pi_1^{-1}(U_2)$. $J_2\mathcal{H}_{W'}$ is the ideal sheaf of $\pi_1^{-1}(E_2)$ in W'. Call it J'. Since $J'\mathcal{H}_{W_3}$ (with respect to q_1) is equal to $J_2\mathcal{H}_{W_3}$ it is invertible as $\mathcal{H}_{W_3}$-module. So if $q': W'' \to W'$ is the blowing-up with center $\pi_1^{-1}(E_2)$, then there exists a unique morphism $g: W_3 \to W''$ such that $q'g = q_1$. Since q' is strict by (2.14), $J_1\mathcal{H}_{W''}$ (with respect to $\pi_1 q'$) is invertible as $\mathcal{H}_{W''}$-module. Also $J_2\mathcal{H}_{W''} = J'\mathcal{H}_{W''}$ is invertible as $\mathcal{H}_{W''}$-module by the definition of q'. Therefore $J_1J_2\mathcal{H}_{W''}$ is invertible as $\mathcal{H}_{W''}$-module. This implies that there exists a morphism $g': W'' \to W_3$ such that $\pi_1 q' = \pi_3 g'$. We shall prove that g and g' are inverse of each other. We have $\pi_3(g'g) = (\pi_3 g')g = (\pi_1 q')g = \pi_1(q'g) = \pi_1 q_1 = \pi_3$. By (2.14.1), $g'g$ is the identity of W_3. Also we have $(\pi_1 q')(gg') = \pi_1(q'g)g' = \pi_1 q_1 g' = \pi_3 g' = \pi_1 q'$. Again by (2.14.1), gg' is the identity of W''. We have proven that g is an isomorphism, and hence $(\pi_1^{-1}(U_2), \pi_1^{-1}(E_2), q_1)$ is a local blowing-up over W_1. We have proven (2.16.2).

Now, going back to the proof of (2.16) in the general case, let us assume that π_1 is obtained by a sequence of local blowing-ups

$$\{(U_{i0}, E_{i0}, \pi_{i0})\}_{0 \le i < m}$$

and π_2 by

$$\{(U'_{0j}, E'_{0j}, \pi'_{0j})\}_{0 \le j < n}$$

where $W = W_{oo}$, $W_1 = W_{mo}$, $W_2 = W_2 = W_{on}$, $\pi_1 = \pi_{oo}\pi_{1o}\cdots\pi_{m-1\,o}$ and $\pi_2 = \pi'_{oo}\pi'_{o1}\cdots\pi'_{o\,n-1}$. We can then construct a net of local blowing-ups

$$\left\{(U_{ij}, E_{ij}, \pi_{ij})\right\}_{o\leq i < m,\, o\leq j \leq n}$$

and $$\left\{(U'_{ij}, E'_{ij}, \pi'_{ij})\right\}_{o\leq j < n,\, o\leq i \leq m}$$

by induction as follows. Suppose we have already obtained $(U_{ij}, E_{ij}, \pi_{ij})$ and $(U'_{ij}, E'_{ij}, \pi'_{ij})$ for a pair (i,j) with $0 \leq i < m$, $0 \leq j < n$. Then let $\tilde{U}_{ij} = U_{ij} \cap U'_{ij}$ and let $\tilde{E}_{ij}$ be the closed complex subspace of $W_{ij}|\tilde{U}_{ij}$ defined by the product of the ideal sheaves of E_{ij} and E'_{ij}. Let $(\tilde{U}_{ij}, \tilde{E}_{ij}, \tilde{\pi}_{ij})$ be the local blowing-up. Let $\tilde{\pi}_{ij}: W_{i+1,j+1} \longrightarrow W_{ij}$. Then, by (2.16.1), there exist morphisms $\pi_{ij+1}: W_{i+1\,j+1} \longrightarrow W_{ij+1}$ and $\pi'_{i+1\,j}: W_{i+1\,j+1} \longrightarrow W_{i+1\,j}$ which make commutative diagrams with $\tilde{\pi}_{ij}$, π_{ij} and π'_{ij}.

If $U_{ij+1} = (\pi'_{ij})^{-1}(U_{ij})$ and $E_{ij+1} = (\pi'_{ij})^{-1}(E_{ij})$, then by (2.16.2), $(U_{ij+1}, E_{ij+1}, \pi_{ij+1})$ is a local blowing-up. By simmetry, we define $U'_{i+1\,j}$ and $E'_{i+1\,j}$ so as to have a local blowing-up. By simmetry, we define $U'_{i+1\,j}$ and $E'_{i+1\,j}$ so as to have a local blowing-up $(U'_{i+1\,j}, E'_{i+1\,j}, \pi'_{i+1\,j})$. By continuing this way, we obtain the net of local blowing-ups.

$$\begin{tikzcd}
 & W_{i+1\,j+1} \arrow[dl, "\pi'_{i+1\,j}"'] \arrow[dr, "\pi_{i\,j+1}"] \arrow[dd, "\tilde{\pi}_{ij}"] & \\
W_{i+1\,j} \arrow[dr, "\pi_{ij}"'] & & W_{i\,j+1} \arrow[dl, "\pi'_{ij}"] \\
 & W_{ij} &
\end{tikzcd}$$

Let us then define W_3 to be W_{mn} and $\pi_3 : W_3 \longrightarrow W$ to be the composition of maps π_{ij} and π'_{ij} along the commutative net. Let $q_1 = \pi'_{mo}\pi'_{m1}\cdots\pi'_{mn-1}$ and $q_2 = \pi_{on}\pi_{1n}\cdots\pi_{m-1n}$. Then 1) of (2.16) is clear. As for 2) of (2.16), if f with h_i, $i=1,2$, is given, then it induces a commutative system of morphisms $h_{ij} : V \longrightarrow W_{ij}$, $o \leq i \leq m$ and $o \leq j \leq n$, into the commutative net of $\{\pi_{ij}, \pi'_{ij}\}$, where $f = h_{oo}, h_1 = h_{mo}$ and $h_2 = h_{on}$. To verify this, we apply (2.16.1) to each of the small squares in the above constructed net. Here we need the uniqueness and the strictness assertions of (2.14).
By letting $h_3 = h_{mn}$, we get 2) of (2.16). It is clear that $q_i \in \mathcal{E}(W_i)$ for $i=1,2$.

<u>Definition</u> (2.17) Given $\pi_i \in \mathcal{E}(W)$, $i=1,2$, $\pi_3 \in \mathcal{E}(W)$ having the properties of (2.16) is called the join of π_1 and π_2. We then write $\pi_3 = \pi_1 \wedge \pi_2$. It should be noted that the properties 1) and 2) of (2.16) say that $\pi_1 \wedge \pi_2$ is the "product" of π_1 and π_2 in the category $\mathcal{E}(W)$. Clearly $\pi_1 \wedge \pi_2$ is uniquely determined by π_1 and π_2 up to a (canonical) isomorphism in $\mathcal{E}(W)$.

Remark (2.17.1) It is immediate from the definition (2.17) (or, by (2.16)) that for $\pi_\alpha \in \mathcal{E}(W), \alpha = 1,2,3$, we have $(\pi_1 \wedge \pi_2) \wedge \pi_3 = \pi_1 \wedge (\pi_2 \wedge \pi_3)$. Moreover if $\mathrm{Hom}(\pi_1, \pi_2) \neq \emptyset$ then $\pi_1 = \pi_1 \wedge \pi_2$.

Proposition (2.18) Let $\pi_\alpha : W_\alpha \to W, \in \mathcal{E}(W)$, for $\alpha = 1,2$. Let $\pi_1 \wedge \pi_2 : W_3 \to W$. Let $q_\alpha \in \mathrm{Hom}(\pi_3, \pi_\alpha), \alpha = 1,2$. Then the morphism $W_3 \to W_1 \times_W W_2$ (the fibre product in the usual sense) defined by (q_1, q_2) is a closed imbedding.

Proof. By (2.14), there exists a closed complex subspace F_α of W such that F_α is defined by an ideal sheaf J_α invertible as $\mathcal{H}_{W_\alpha}$-module and (π_α, F_α) satisfies the strictness conditions 1) and 2) of (2.11). Let V be the smallest closed complex subspace of $W_1 \times_W W_2$ which coincides with $W_1 \times_W W_2$ outside $F = F_1 \times_W W_2 \cup W_1 \times_W F_2$. (cf. the proof of (2.7).) As a point set, F is defined by the product of the ideal sheaves generated by I_α, $\alpha = 1,2$, in the structure sheaf of $W_1 \times_W W_2$. The existence of the q_α implies that $J_\alpha \mathcal{H}_{W_3}$ and hence its product for $\alpha = 1,2$, are all invertible as $\mathcal{H}_{W_3}$-module by (2.14) and (2.12.1). Therefore the canonical map $W_3 \to W_1 \times_W W_2$ induces a morphism $g : W_3 \to V$. On the other hand, by (2.16), there exists a morphism $g' : V \to W_3$ such that $q_\alpha g'$ are induced by the projections of $W_1 \times_W W_2$, because the canonical morphism $V \to W$ is strict. (This satisfies the conditions of (2.11) with F defined above). The endomorphism $g'g$ clearly commutes with $\pi_1 \wedge \pi_2$ and hence $g'g = id_{W_3}$ by the uniqueness of morphisms (2.14.1). It follows that g is closed imbedding and hence so is $W_3 \to W_1 \times_W W_2$.

§ 3. La voûte étoilée $\mathcal{E}_W$

Throughout this section, all the complex spaces will be assumed to be Hausdorff without any explicit mentioning. This is done so mainly for the sake of simplicity; Hausdorff case is at any rate enough for the applications we are interested in what follows. It should be noted, however, that if W is an Hausdorff complex space then all the complex spaces which appear in the category $\mathcal{E}(W)$ are necessarily Hausdorff by virtue of (2.9).

Definition (3.1) An étoile **e** over a complex space W is a maximal one (with respect to inclusion) among those subcategories of $\mathcal{E}(W)$ satisfying the following condition:

(3.1.1) For every pair of $\pi_\alpha : W_\alpha \to W, \in e$, with $\alpha = 1,2$, there exists $\pi_3 : W_3 \to W, \in e$, such that

1) there exists $q_\alpha \in \mathrm{Hom}(\pi_3, \pi_\alpha)$ for both $\alpha = 1,2$, so that we have a commutative diagram:

$$\begin{tikzcd} & W_1 \arrow[dr, "\pi_1"] & \\ W_3 \arrow[ur, "q_1"] \arrow[rr, "\pi_3"] \arrow[dr, "q_2"'] & & W \\ & W_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

2) $W_3 \neq \emptyset$ and $q_\alpha(W_3)$ is relatively compact in W_α for both $\alpha = 1,2$.

Remark (3.2) Given any subcategory e_0 of $\mathcal{E}(W)$ satisfying the condition (3.1.1), there exists at least one étoile e over W which contains e_0: this can be proven by applying Zorn's lemma to the partially ordered set of all the subcategories of $\mathcal{E}(W)$ which satisfy (3.1.1).

<u>Remark</u> (3.3) Let e be an étoile over W. Then for any $\pi_\alpha \in \mathcal{E}(W)$, $\alpha = 1,2$, we have $\pi_1 \wedge \pi_2 \in e$ if and only if $\pi_\alpha \in e$ for both $\alpha = 1,2$.

<u>Proof</u>. If $\pi_\alpha \in e$ for $\alpha = 1,2$, then there exists $\pi_3 \in e$ such that $\mathrm{Hom}(\pi_3, \pi_\alpha) \neq \emptyset$ for $\alpha = 1,2$.
Also, in any case, $\mathrm{Hom}(\pi_1 \wedge \pi_2, \pi_\alpha) = \emptyset$ for $\alpha = 1,2$. For this reason, it is enough to prove that if $\pi' \in e$ and $\mathrm{Hom}(\pi', \pi) \neq \emptyset$ with $\pi \in \mathcal{E}(W)$ then $\pi \in e$. Let e' be the subcategory of $\mathcal{E}(W)$ obtained from e by adding those $\pi \in \mathcal{E}(W)$ for which there exist $\pi' \in e$ with $\mathrm{Hom}(\pi', \pi) \neq \emptyset$.
It is easy to see that e' satisfies (3.1.1). Hence by the maximality of e, we get $e' = e$.

<u>Remark</u> (3.4) Let S be a set of object, in $\mathcal{E}(W)$ such that if $\pi_i \in S$, $i=1,2$, then $\pi_1 \wedge \pi_2 \in S$. Let e be an étoile over W. Then $S \subset e$ if and only if the following condition is satisfied.

(3.4.1) For every pair $(\pi_\alpha, \pi_i) \in e \times S$, there exists at least one $(\pi'_\alpha, \pi'_i) \in e \times S$ such that $\mathrm{Hom}(\pi'_\alpha, \pi_\alpha) \neq \emptyset \neq \mathrm{Hom}(\pi'_i, \pi_i)$ and $\mathrm{Im}(q)$ with $q \in \mathrm{Hom}(\pi'_\alpha \wedge \pi'_i, \pi_\alpha \wedge \pi_i)$ is relatively compact and non-empty.

<u>Proof</u>. Let e' be the subcategory of $\mathcal{E}(W)$ obtained from e by adding all the $\pi_\alpha \wedge \pi_i$ with $\pi_\alpha \in e$ and $\pi_i \in S$. Then e' contains S because e contains id_W. So it is enough to prove that e' satisfies (3.1.1). Take any $(\pi_\alpha, \pi_i) \in e \times S$ and $(\pi_\beta, \pi_j) \in e \times S$. Then by (3.4.1), there exists $(\pi_\gamma, \pi_k) \in e \times S$ such that $\mathrm{Hom}(\pi_\gamma, \pi_\alpha \wedge \pi_\beta) \neq \emptyset \neq$

$\mathrm{Hom}(\pi_k, \pi_i \wedge \pi_j)$ and $q \in \mathrm{Hom}(\pi_\gamma \wedge \pi_k, (\pi_\alpha \wedge \pi_\beta) \wedge (\pi_i \wedge \pi_j))$ has a non-empty relatively compact image. If we let $\pi_3 = \pi_\gamma \wedge \pi_k$, then there exist $q_1 \in \mathrm{Hom}(\pi_3, \pi_\alpha \wedge \pi_i)$ and $q_2 \in \mathrm{Hom}(\pi_3, \pi_\beta \wedge \pi_j)$ such that $\mathrm{Im}(q_p)$ is non-empty and relatively compact for both $p=1,2$.
The condition (3.4.1) for e' is now immediate.

<u>Definition</u> (3.5) For a complex space W, we denote by $\mathcal{E}_W$ the set of all the étoiles over W. Then we introduce the topology in $\mathcal{E}_W$ by taking a basis of open subsets to be

$$\mathcal{E}_\pi = \left\{ e \in \mathcal{E}_W \mid \pi \in e \right\}, \quad \pi \in \mathcal{E}(W) .$$

<u>Remark</u> (3.6) Let $\pi : W' \to W, \in \mathcal{E}(W)$. Then $\mathcal{E}_W$ and $\mathcal{E}_{W'}$ are related to each other by the following correspondence:

(3.6.1) Given $e' \in \mathcal{E}_{W'}$, there exists one and only one $e \in \mathcal{E}_W$ such that e contains all the $\pi\pi'$ with $\pi' \in e'$.

(3.6.2) Given $e \in \mathcal{E}_W$, let e'' be the subcategory (full) of $\mathcal{E}(W')$ consisting of exactly those $q \in \mathrm{Hom}(\pi \wedge \pi_\alpha, \pi)$ with $\pi_\alpha \in e$. Then e'' has the property (3.1.1) if and only if $e \subset \mathcal{E}_\pi$.

(3.6.3) Given $e \in \mathcal{E}_\pi$, the e'' of (3.6.2) is an étoile over W' which consists of exactly those $q \in \mathcal{E}(W')$ such that $\pi q \in e$.

<u>Proof</u>. Let us first prove (3.6.2). If e'' satisfies (3.1.1), then applying (3.4) with $S = \{\pi\}$, we get $\pi \in e$, i.e., $e \in \mathcal{E}_\pi$. The converse is obvious. To prove (3.6.1) and (3.6.3), take any étoile $e' \in \mathcal{E}_{W'}$. Let e_o be the subcategory of $\mathcal{E}(W)$ which consists of exactly those $\pi\pi'$ with $\pi' \in e'$. Then clearly e_o satisfies (3.1.1). So there exists $e \in \mathcal{E}_W$ which contains e_o by (3.2). As $\pi \in e$, e'' defined by (3.6.2) satisfies (3.1.1). Since $e'' \supset e'$, $e'' = e'$. Also pick any $e \in \mathcal{E}_W$ and define e'' as in (3.6.2). Since e'' satisfies (3.1.1), there exists $e' \in \mathcal{E}_{W'}$ which contains e''. Define e_o with this e' as above. Let e_1 be any étoile $\in \mathcal{E}_W$ containing e_o. Then for every $\pi_\alpha \in e$, there exists $\pi_\beta \in e_1$ such that $\mathrm{Hom}(\pi_\beta, \pi_\alpha \wedge \pi) \neq \emptyset$.
This implies $\pi_\alpha \in e_1$ by (3.3). Namely $e \subset e_1$. Hence

$e = e_1$. There two results, dual to each other, imply (3.6.3) and that the map $e \mapsto e''$ defined by (3.6.3) is a bijection from $\mathcal{E}_\pi \to \mathcal{E}_{V'}$. Hence (3.6.1) follows.

<u>Proposition</u> (3.7) Given $\pi : W' \to W, \in \mathcal{E}(W)$, there exists a unique map $j_\pi : \mathcal{E}_{W'} \to \mathcal{E}_W$ such that for every $\pi' \in e' \in \mathcal{E}_{W'}$ we have $\pi\pi' \in j_\pi(e')$. Moreover j_π induces an isomorphism of topological spaces $\mathcal{E}_{W'} \xrightarrow{\sim} \mathcal{E}_\pi$.

<u>Proof</u>. The existence of j_π follows (3.6.1). By (3.6.2) and (3.6.3), it induces a bijective map $\mathcal{E}_{W'} \to \mathcal{E}_\pi$. If $\pi' \in \mathcal{E}_{W'}$, then j_π maps $\mathcal{E}_{\pi'}$ to $\mathcal{E}_{\pi\pi'}$. This proves the bicontinuity of $\mathcal{E}_{W'} \to \mathcal{E}_\pi$ by j_π.

<u>Proposition</u> (3.8) Take any $\pi : W' \to W, \in e \subset \mathcal{E}_W$. Then the intersection of $\mathrm{Im}(\pi')$ for all $\pi' \in \mathcal{E}(W')$ such that $\pi\pi' \in e$, is a single point of W'. Moreover $\pi/U' \in e$ for every open neighborhood of this point in W'.

<u>Proof</u>. By means of j_π, the proof is easily reduced to the case $\pi = id_W$. Let $S = \{ id_W/U \}$ where U ranges through the set of all open neighborhood of any fixed point $y \in \bigcap_{\pi_\alpha \in e} \overline{\mathrm{Im}(\pi_\alpha)}$, provided this intersection is not empty. Then it is easy to check that S and e satisfy the condition (3.4.1). So $S \subset e$. This means $\{y\} = \bigcap_{\pi_\alpha \in e} \overline{\mathrm{Im}(\pi_\alpha)}$. But, for each $\pi_\alpha \in e$ there exists $\pi_\beta \in e$ such that $\mathrm{Im}(q)$ with $q \in \mathrm{Hom}(\pi_\beta, \pi_\alpha) \neq \emptyset$ is relatively compact, so that $\overline{\mathrm{Im}(\pi_\beta)} \subset \mathrm{Im}(\pi_\alpha)$. So, we yet $\{y\} = \bigcap_{\pi_\alpha \in e} \mathrm{Im}(\pi_\alpha)$. Thus all

that remains to be proven is that $\bigcap_{\pi_\alpha \in e} \overline{\mathrm{Im}(\pi_\alpha)}$ is not empty.

From (3.1.1), it follows that $\overline{\mathrm{Im}(\pi_\alpha)}$ is compact for some $\pi_\alpha \in e$ and that $\{\overline{\mathrm{Im}(\pi_\alpha)}\}$ has the finite intersection property. That intersection is hence non-empty.

<u>Definition</u> (3.9) We denote $p_W : \mathcal{E}_W \to W$ to be the map defined by

$$\{p_W(e)\} = \bigcap_{\pi \in e} \mathrm{Im}(\pi)$$

<u>Proposition</u> (3.10) $p_W : \mathcal{E}_W \to W$ is continuous and surjective. For $\pi : W' \to W, \in \mathcal{E}(W)$, we have $p_W j_\pi = \pi\, p_{W'}$.

<u>Proof</u>. For every open subset U of W, we have that $p_W(e) \in U$ if and only if $id_W / U \in e$. (A proof of this is included in the proof of (3.8).) This means that $p_W^{-1}(U) = \mathcal{E}_\pi$ with $\pi = id_W / U$. Hence p_W is continuous. Next, for every $y \in W$, take $e_o = \{id_W / U\}$ where U ranges through all the open neighborhoods of y in W. Then there exists an étoile $e \in \mathcal{E}_W$ which contains e_o. It is clear that $p_W(e) = y$. Thus p_W is surjective. The rest of the proposition is clear.

<u>Notation</u> (3.11) We denote by $p_\pi : \mathcal{E}_\pi \to W'$ the map defined by $p_\pi j_\pi = p_{W'}$ where $\pi : W' \to W$ is any object in $\mathcal{E}(W)$. (cf.(3.7) and (3.10).)

Lemma (3.12) Let $\pi : W' — W$, (W), and let e' $_{W}$. If (U', E', π') is a local blowing-up over W' such that $y' = p_{W'}(e')$ U' and E' is nowhere dense in some neighborhood of y' in U', then $\pi\pi'$ $j_\pi(e')$.

<u>Proof</u>. For every neighborhood U'' of y' in U', we have $\pi/U'' \in e$. Hence we may assume that $U' = |W'|$ and E' is nowhere dense in U'. Moreover, as the assertion is equivalent to $\pi' \in e'$, we may assume from the beginning that $\pi = id_W$. Thus the problem is reduced to prove:

(3.12.1) Let $e \in \mathcal{E}_W$ and let $\pi : W' \to W$ be the blowing-up whose center E is a closed nowhere dense complex subspace of W. Then $\pi \in e$.

To prove (3.12.1), we shall prove that if $S = \{\pi\}$ then S and e satisfy the condition (3.4.1). Pick any $\pi_\alpha \in e$. We shall first prove that the source of $\pi_\alpha \wedge \pi$ is not empty. Let the notation as follows:

$$\begin{tikzcd} W'_\alpha \arrow[r] \arrow[d, "r"'] \arrow[dr, "\pi_\alpha \wedge \pi"] & W' \arrow[d, "\pi"] \\ W_\alpha \arrow[r, "\pi_\alpha"'] & W \supset E \end{tikzcd} \qquad \text{(commutative)}$$

since π_α is strict by (2.14) and a strict morphism is locally isomorphic in an open dense subset of it, source, $\pi_\alpha^{-1}(E)$ is nowhere dense in W_α. Then, by (2.6.1)-(2.6.2) (applied to each of the local blowing-ups which yield π_α), r is the blowing-up with center $\pi_\alpha^{-1}(E)$. Therefore r is surjective. Hence W'_α is not empty. Now we have $\pi_\beta \in e$, depending upon any given $\pi_\alpha \in e$, such that $Hom(\pi_\beta, \pi_\alpha) \neq \emptyset$ and if $q \in Hom(\pi_\beta, \pi_\alpha)$ then $Im(q)$ is relatively compact in W_α. Set the notation as follows:

$$\begin{tikzcd} W'_\beta \arrow[r, "q'"] \arrow[d, "\pi'_\beta"] \arrow[dr, "\pi_\alpha \wedge q"] & W'_\alpha \arrow[d, "\pi'"] & \\ W_\beta \arrow[r, "q"] & W_\alpha \arrow[r, "\pi_\alpha"] & W \end{tikzcd}$$ (commutative)

The morphism $W'_\beta \to W$ obtained by these diagrams is nothing but $\pi_\beta \wedge \pi$. Hence, once again, $W'_\beta \neq \emptyset$. Since π'_α is proper, the relative compactness of $\mathrm{Im}(q)$ implies the same of $\mathrm{Im}(q')$. Thus we have proven (3.4.1) for our S and e.

<u>Lemma</u> (3.13) Let $\pi : W' \to W$, $\in e \in \mathcal{E}_{W'}$ and let (U', E', π') be a local blowing-up over W' such that $\pi\pi' \in e$. Let $y' = p_\pi(e)$. Let U'' be any open neighborhood of y' in U' and let E'' be any closed complex subspace of E'/U''. If (U'', E'', π'') is the local blowing-up over W', then $\pi\pi' \in e$.

<u>Proof</u>. We have $e' \in \mathcal{E}_{W'}$ such that $j_\pi(e') = e$. (cf.(3.7)). The assumption is that $\pi' \in e'$, and the assertion is equivalent to saying that $\pi'' \in e'$. Let us take the following notation:

$$\begin{tikzcd} & W'_2 \arrow[dr, "\pi''"] & & \\ W'_3 \arrow[ur, "q_2"] \arrow[rr, "\pi' \wedge \pi''"] \arrow[dr, "q_1"'] & & W' \arrow[r, "\pi"] & W \\ & W'_1 \arrow[ur, "\pi'"'] & & \end{tikzcd}$$ (commutative)

Let $y'_1 = p_{\pi'}(e') (= p_{\pi\pi'}(e))$ so that $\pi'(y'_1) = y'$, clearly $(\pi')^{-1}(U'') \ni y'_1$. We know that $((\pi')^{-1}(U''), (\pi')^{-1}(E''), q_1)$ is a local blowing-up over W_1'. (cf.(2.16.1)-(2.16.2).) Now $(\pi')^{-1}(E')$ is nowhere dense in $(\pi')^{-1}(U') = |W_1'|$ because it is defined by an ideal sheaf which is invertible as $\mathcal{H}_{W_1'}$-module. Hence $(\pi')^{-1}(E'')$ is nowhere dense in $(\pi')^{-1}(U'')$. Hence by (3.12). $\pi' \wedge \pi'' = \pi' q_1 \in e'$. Hence $\pi'' \in e'$.

<u>Lemma</u> (3.14) Let (U_i, E_i, π_i), $i=1,2$, be two local blowing-ups over a complex space W. Let (U,E,π) be the local blowing-up over W in which $U=U_1 \cap U_2$ and $E = E_1 \cap E_2$. Let $q_i \in \mathrm{Hom}\,(\pi \wedge \pi_i, \pi)$ and let $\pi: W' \to W$. Then there exist closed complex subspaces E_i', $i=1,2$, of W' such that $E_1' \cap E_2' = \emptyset$ and q_i is the blowing-up with center E_i' for $i=1,2$.

<u>Proof</u>.

$$\begin{tikzcd}
\bullet \arrow[dd] \arrow[dr, "q_1"] & & \bullet \arrow[dd] \arrow[dl, "q_2"'] \\
& {} \arrow[dd, "\pi"] & \\
\bullet \arrow[dr, "\pi_1"'] & & \bullet \arrow[dl, "\pi_2"] \\
& {} &
\end{tikzcd}$$

Let J_i be the ideal sheaf of E_i in $\mathcal{H}_W / U_i$. Then $(J_1+J_2)|U$ is the ideal sheaf of E in $\mathcal{H}_W / U$. Hence $(J_1+J_2)\mathcal{H}_{W'}$ is invertible as $\mathcal{H}_{W'}$-module.

So, for each $i=1,2$, there exists a unique coherent ideal sheaf Q_i in $\mathcal{H}_{W'}$ such that $J_i\mathcal{H}_{W'} = Q_i(J_1+J_2)\mathcal{H}_{W'}$. It is then clear that $Q_1+Q_2=\mathcal{H}_{W'}$. This means that if E_i' is the closed complex subspace of W' defined by Q_i, then $E_1' \cap E_2' = \emptyset$. By (2.16.1)-(2.16.2), we know that q_i is the blowing-up with center $\pi^{-1}(E_i)$. But the ideal sheaf of $\pi^{-1}(E_i)$ is the product of the ideal sheaf of E_i' and $(J_1+J_2)\mathcal{H}_{W'}$. This last is invertible as $\mathcal{H}_{W'}$-module. Therefore q_i is also the blowing-up with center E_i'.

<u>Proposition</u> (3.15) Let $e_i \in \mathcal{E}_W$, $i=1,2$. If $e_1 \neq e_2$ then there exists $\pi \in e_1 \wedge e_2$ such that $p_\pi(e_1) \neq p_\pi(e_2)$.

<u>Proof</u> Since $e_1 \neq e_2$, there exist $\pi_i \in e_i$, $i=1,2$, such that $\pi_i \notin e_j$ for (i,j) with $i \neq j$. Pick a finite sequence of local blowing-ups which yields π_i, for each $i=1,2$. We can then construct a net of local blowing-ups by successively adding the two projections from the join of local blowing-ups with a common target space. Each little square of the net is denoted like this. ($W_{oo} = W$, $0 \leq i < n$, $0 \leq j < m$, $\pi_1 = \pi_{oo}\pi_{1o}\cdots\pi_{n-1o}$ and $\pi_2 = \pi'_{oo}\pi'_{o1}\cdots\pi_{om-1}$).

$$\begin{tikzcd}
& W_{i+1\,j+1} \arrow[dl, "\pi'_{i+1\,j}"'] \arrow[dd, "\pi_{ij}\wedge\pi'_{ij}"] \arrow[dr, "\pi_{i\,j+1}"] & \\
W_{i+1\,j} \arrow[dr, "\pi_{ij}"'] & & W_{i\,j+1} \arrow[dl, "\pi'_{ij}"] \\
& W_{ij} &
\end{tikzcd}$$

For each (i,j), let $\pi(i,j): W_{ij} \to W$ be the composition of morphisms along the net. Then there exists at least one (i,j) such that $\pi(i,j) \in e_1 \wedge e_2$ but $\pi(i+1,j) \notin e_2$ and $\pi(i,j+1) \notin e_1$, where necessarily $0 \leq i < n$ and $0 \leq j < m$. By means of $j_{\pi(i,j)}$ of (3.7), the problem is now reduced to the following

(3.15.1) Suppose there exists a pair of local blowing-ups (U_i, E_i, π_i), $i=1,2$, such that $\pi_i \in e_i$ for $i=1,2$ and $\pi_i \notin e_j$ for any (i,j) with $i \neq j$. Then there exists π having the property of (3.15).

If $p_W(e_1) \neq p_W(e_2)$, then $\pi = id_W$ will do. So assume that $p_W(e_1) = p_W(e_2) = y$. Let $U = U_1 \cap U_2$, $E = E_1 \cap E_2$ and let (U,E,π) be the local blowing-up over W. Say $\pi: W' \to W$. Since $y \in U$, by (3.13) we have

$\pi \in e_1 \wedge e_2$. By (3.14), if $q_i \in \mathrm{Hom}(\pi_i \wedge \pi, \pi)$ then there exists a closed complex subspace E_i' of W', $i=1,2$, such that $E_1' \cap E_2' = \emptyset$ and q_i is the blowing-up with center E_i'. Let $y_i = p_\pi(e_i)$, $i=1,2$. We claim that $y_i \in E_j'$, $j \neq i$. In fact, if $y_1 \notin E_2'$, then q_2 is isomorphic over some open neighborhood of y_1. So $\pi q_2 \in e_1$, i.e $\pi_2 \wedge \pi \in e_1$. Hence $\pi_2 \in e_1$, contradicting the assumption. Thus $y_1 \in E_2'$. Similarly $y_2 \in E_1'$. As $E_1' \wedge E_2' = \emptyset$, $y_1 \neq y_2$.

<u>Theorem</u> (3.16) For a Hausdorff complex space W, $\mathcal{E}_W$ is also Hausdorff and the canonical map $p_W : \mathcal{E}_W \to W$ is proper.

<u>Proof</u> Hausdorff property of $\mathcal{E}_W$ is immediate from (3.15). In fact, take any $e_1 \neq e_2$ in $\mathcal{E}_W$. Then there exists $\pi \in e_1 \wedge e_2$ such that $p_\pi(e_1) \neq p_\pi(e_2)$. Say $\pi : W' \to W$. Then W' is Hausdorff. Hence there exists an open neighborhood U_i of $p_\pi(e_i)$ in W' such that $U_1 \cap U_2 = \emptyset$. So if $\pi(i) = \pi \cdot \mathrm{id}_{W'} | U_i (= \pi/U_i)$, then $\mathcal{E}_{\pi(i)} = p_\pi^{-1}(U_i)$ is an open neighborhood of e_i, and $\mathcal{E}_{\pi(1)} \cap \mathcal{E}_{\pi(2)} = p_\pi^{-1}(U_1 \wedge V_2) = \emptyset$.

We now propose to prove that p_W is proper. Take any compact subset K of W. Let $\tilde{K} = p_W^{-1}(K)$. To prove the compactness of $\tilde{K}$, let us take any family of closed subsets of $\tilde{K}$, say $\{C_\lambda\}_{\lambda \in \Delta}$ which has the finite intersection property. We then want to prove that $\bigcap_{\lambda \in \Delta} C_\lambda \neq \emptyset$.

For this purpose, we may assume that $\{C_\lambda\}_{\lambda \in \Delta}$ is maximal in the sense that if F is any closed subset of $\tilde{K}$ such that $\{F, C_\lambda\}_{\lambda \in \Delta}$ who has the finite intersection property then $F = C_\lambda$ for some $\lambda \in \Delta$. In particular, the intersection of any finite number of the numbers is also a member of the family. Let us then define a subcategory e_0 of $\mathcal{E}(W)$ which consists of exactly those $\pi : W' \to W$ such that for some relatively compact open subset V of W' the intersection

$p_\pi^{-1}(V) \cap C_\lambda \neq \emptyset$ for all $\lambda \in \Delta$. First we claim that for every $\pi \in e_o$,

(3.16.1) $$\bigcap_\lambda \overline{p_\pi(C_\lambda)} = \{y_\pi\}$$

with a single point $y_\pi \in W'$. Pick $V \subset W'$ as above. $\overline{V} \cap \overline{p_\pi(C_\lambda)}$ is compact for every $\lambda \in \Delta$. Since $\{C_\lambda\}_{\lambda \in \Delta}$ is closed under finite intersection, the family $\{\overline{V} \cap \overline{p_\pi(C_\lambda)}\}_{\lambda \in \Delta}$ has also finite intersection property. Therefore

$$\bigcap_{\lambda \in \Delta} (\overline{V} \cap \overline{p_\pi(C_\lambda)}) \neq \emptyset$$

which proves that the intersection of (3.16.1) is not empty. Let us pick any point y_π in the intersection of (3.16.1). Take any relatively compact open neighborhood U of y_π in W'. Then $U \cap p_\pi(C_\lambda) \neq \emptyset$ for every $\lambda \in \Delta$. So, if D denotes the closure of $p_\pi^{-1}(U)$ in $\mathcal{E}_W$, then $D \cap C_\lambda \neq \emptyset$ for every $\lambda \in \Delta$. By the maximality, D is also a member of the family $\{C_\lambda\}_{\lambda \in \Delta}$. This proves (3.16.1). It is also clear that:

(3.16.2) For every $\pi : W' \to W, \in e_o$, and for every open neighborhood V of y_π in W', $\pi/V \in e_o$. If $\pi' \in e$ and $q \in \mathrm{Hom}(\pi', \pi) \neq \emptyset$, then $q(y_{\pi'}) = y_\pi$. Moreover

$$\bigcap_{\substack{q \in \mathcal{E}(W') \\ \pi q \in e_o}} \mathrm{Im}(q) = \{y_\pi\}$$

Next we want to prove that e_o is in fact an étoile over W. First of all, if $\pi_\alpha \in e_o, \alpha = 1,2$, then $\pi_1 \wedge \pi_2 \in e_o$. This is easily verified by the definition of e_o. In wiew of the first statement of (3.16.2), e_o satisfies (3.1.1). Hence there exists at least one étoile $e \in \mathcal{E}_W$ containing e_o. It is clear from the third statement of (3.16.2), that for every $\pi \subset e_o$ we have $p_\pi(e) = y_\pi$. If $e \neq e_o$, then we can find $\pi : W' \to W, \in e_o$, and a local blowing-up (U', E', π') over

W' such that $\pi\pi' \in e$ but $\notin e_o$. (This is so because every element of e is obtained by a finite sequence of local blowing-ups). We have $y_\pi = p_\pi(e) \in U'$. Let F_1 be the smallest closed complex subspace of W'/U' such that $F_1 - E' = W'/U' - E'$. Let $G = F_1 \cap E'$, which is nowhere dense in W'/U'. Let $f: V \to W'$ be the morphism such that (U', G, f) is a local blowing-up. Then $\pi f \in e$ by (3.12) and $\pi f \in e_o$ because, by (3.12), $p_\pi^{-1}(U'') = p_{\pi f}^{-1}(f^{-1}(U''))$ for every open subset U'' of U' and hence there exists a relatively compact open subset N in V having $p_{\pi f}^{-1}(N) \cap C_\lambda \neq \emptyset$ for every $\lambda \in \Delta$. (For instance, pick an open relatively compact neighborhood U'' of y_π in U' and let $N = f^{-1}(U'')$.) Namely $\pi f \in e_o \cap e$. Next we need the following

<u>Lemma</u> (3.16.1) Let V_o be any complex space, M an open subset of V_o and F_i, $i=1,2$, two closed complex subspaces of V_o/M such that $|F_1| \cup |F_2| = M$. Let $G = F_1 \cap F_2$ and let (M, G, f) be a local blowing-up with $f: V \to V_o$. Let F_i' be the strict transform of F_i by f (with respect to the center G), $i=1,2$. Then $|V| = |F_1'| \cup |F_2'|$ and $|F_1'| \cap |F_2'| = \emptyset$.

In fact, if V_o' is the strict transform of $F_1 \cup F_2$ by f then $V_o' = F_1' \cup F_2'$ and $|V_o'| = |V|$. Moreover, for the ideal sheaves J_i of F_i in $\mathcal{H}_{V_o}|M$, we can write uniquely $J_i \mathcal{H}_V = Q_i(J_1 + J_2)\mathcal{H}_V$ with coherent ideal sheaves Q_i in $\mathcal{H}_V$. Then clearly $Q_1 + Q_2 = \mathcal{H}_V$ which means that the closed complex subspaces F_i'' of V defined by Q_i, $i=1,2$, are disjoint, Clearly $F_i' \subset F_i''$, $i=1,2$, and hence $|F_1'| \cap |F_2'| = \emptyset$. Thus (3.16.1) is verified.

We apply (3.16.1) to our situation with $W' = V_o$, $F_2 = E'$ and $M = U'$. Then V is disjoint union of open closed complex subspaces V_i, $i=1,2$, such that $|V_1| = |F_1'|$ and $|V_2| = |F_2'|$. By (2.16.1) and (2.16.2), if $f': V' \to V$ is the blowing-up with center $f^{-1}(E')$ then

$ff' = f \wedge \pi'$. As $f \in e$ and $\pi' \in e$, we have $ff' \in e$. Since $|V_2| = |F_2'|$ and $F_2' \subset f^{-1}(E')$, every point of V' is mapped into V_1. Therefore, in particular, $p_{\pi f}(e) \in V_1$. As $\pi f \in e_o$, $y_{\pi f} = p_{\pi f}(e)$. Hence $y_{\pi f} \in V_1$. As V_1 is open in V and $f^{-1}(E') \cap V_1$ is nowhere dense in V_1, we must have $\pi f f' \in e_o$ by the same argument as above. As $\pi f f' = (\pi f) \wedge (\pi\pi')$, it follows that $\pi\pi' \in e_o$. This is contrary to the assumption on π'. We conclude that $e = e_o$, i.e., e_o is an étoile $\in \mathcal{E}_W$. Next we want to prove that $e_o \in C_\lambda$ for all $\lambda \in \Delta$. Take any open neighborhood $\tilde{N}$ of e_o in $\mathcal{E}_W$. Then there exists $\pi: W' \to W, \in e_o$, such that $\tilde{N} \supset \mathcal{E}_\pi$. By the definition of e_o, we must have $\mathcal{E}_\pi \cap C_\lambda \neq \emptyset$ for every $\lambda \in \Delta$. This shows that every open neighborhood $\tilde{N}$ of e_o in $\mathcal{E}_W$ meets every C_λ. As C_λ are closed in $\mathcal{E}_W$, we have

$$e_o \in \bigcap_{\lambda \in \Delta} C_\lambda$$

This proves that this intersection is not empty and hence that $\tilde{Y}$ is compact.

<u>Theorem</u> (3.17) Let W be a complex space and x a point of W. Let $\{\pi_\alpha : W_\alpha \to W\}_{\alpha \in \Delta}$ be a family of morphisms $\pi_\alpha \in \mathcal{Q}(W)$. Then following conditions are equivalent to one another:

(1) $$p_W^{-1}(x) \subset \bigcup_{\alpha \in \Delta} \mathcal{E}_{\pi_\alpha}$$

(2) there exist a finite subset Δ' of Δ and a relatively compact open subset V_α of W_α for each $\alpha \in \Delta'$, such that

$$p_W^{-1}(N) \subset \bigcup_{\alpha \in \Delta'} p_{\pi_\alpha}^{-1}(V_\alpha)$$

for some neighborhood N of x in W.

(3) There exist a finite subset Δ' of Δ and a relatively compact open subset V_α of W_α for each $\alpha \in \Delta'$ such that

$$\bigcup_{\alpha\in\Delta'} \pi_\alpha(V_\alpha)$$

is a neighborhood of x in W.

(4) Write $\pi_\alpha \wedge \pi_\beta : W_{\alpha\beta} \to W$. There exists a finite subset Δ' of Δ having the following properties:

(4.a) $\bigcup_{\alpha\in\Delta'} \mathrm{Im}(\pi_\alpha)$ is a neighborhood of x in W.

(4.b) For every $\alpha\in\Delta'$ and every sequence of points $\{y_{\alpha i}\}_{i=1,2,\ldots}$ in W_α such that $\lim_{i\to\infty} \pi_\alpha(y_{\alpha i}) = x$ in W, there exist $\beta\in\Delta'$ and a sequence of points $\{y_{\alpha\beta j}\}_{j=1,2,\ldots}$ in $W_{\alpha\beta}$ such that, if $q_\gamma \in \mathrm{Hom}(\pi_\alpha \wedge \pi_\beta, \pi_\gamma)$ for $\gamma = \alpha, \beta$, then $\{q_\alpha(y_{\alpha\beta j})\}_{j=1,2,\ldots}$ is a subsequence of $\{y_{\alpha i}\}_{i=1,2,\ldots}$ and $\{q_\beta(y_{\alpha\beta j})\}_{j=1,2,\ldots}$ is convergent to a point in W_β.

<u>Proof</u>. Assume (1). For each $\alpha\in\Delta$ and each point $y\in W_\alpha$, pick one relatively compact open neighborhood $V_{\alpha,y}$ of y, in W_α. Then clearly

$$\bigcup_{\alpha\in\Delta,\, y\in W_\alpha} p_{\pi_\alpha}^{-1}(V_{\alpha,y})$$

$$= \bigcup_{\alpha\in\Delta} \mathcal{E}_{\pi_\alpha}$$

which is an open neighborhood of $p_W^{-1}(x)$ in $\mathcal{E}_W$.

Fix a compact neighborhood K of x in W and let $\{K_\mu\}_{\mu\in\Gamma}$ be the set of all compact neighborhoods of x in W. (This is closed under finite intersection). Then we obtain an open covering of $p_W^{-1}(K)$ consisting of

$$\left\{ p_{\pi_\lambda}^{-1}(V_{\alpha,y}) \cap p_W^{-1}(K),\ \alpha\in\Delta \text{ and } y\in W_\alpha \right\}$$

and $\left\{ p_W^{-1}(K-K_\mu),\ \mu \in \Gamma \right\}$. By (3.16), $p_W^{-1}(K)$ is compact and hence this covering has a finite subcovering. This implies that there exist a finite set P of pairs (α, y) and an index $\mu \in \Gamma$ such that

$$p_W^{-1}(K_\mu) \subset \bigcup_{(\alpha,y) \in P} p_{\pi_\alpha}^{-1}(V_{\alpha,y})$$

Let Δ' be the set of those $\alpha \in \Delta$ for which at least one $(\alpha, y) \in P$ exists. Then for each $\alpha \in \Delta'$, let V_α be the union of those $\overline{V}_{\alpha,y}$ with $(\alpha, y) \in P$. Then V_α is relatively compact in W_α for each $\alpha \in \Delta'$ and

$$\bigcup_{\alpha \in \Delta'} p_\pi^{-1}(V\ )$$

contains a neighborhood K_μ of x in W, which is (2). Assume (2). Since p_W is surjective,

$$\bigcup_{\alpha \in \Delta'} \pi_\alpha(V_\alpha) = \bigcup_{\alpha \in \Delta'} p_W(p_{\pi_\alpha}^{-1}(V_\alpha))$$

$$\supset p_W(p_W^{-1}(N)) \supset N \ .$$

This means (3). Let us next prove (3) $\Longrightarrow$ (1).
For each $\alpha \in \Delta'$, $p_{\pi_\alpha}^{-1}(\overline{V}_\alpha)$ is compact by (1.16) applied to p_{W_α}). Hence, so is

$$\tilde{K} = \bigcup_{\alpha \in \Delta'} p_{\pi_\alpha}^{-1}(\overline{V}_\alpha)$$

If $\tilde{K} \supset p_W^{-1}(x)$, then (1) is clear. So let us assume that $\tilde{K}$ does not contain $p_W^{-1}(x)$. Since $\tilde{K}$ is closed there exists $\pi \in e \in p_W^{-1}(x)$ such that $\tilde{K} \cap \mathcal{E}_\pi = \emptyset$. Say $\pi : W' \to W$. Since π is strict, there exists an open dense subset U' of W' in which π is locally isomorphic. As $y = p_\pi(e) \in W'$ and $\pi(y) = x$, if N is any open neighborhood of x in W then $\pi^{-1}(N) \cap U' \neq \emptyset$. Close N which is contained in the union of (3). Pick $z \in \pi^{-1}(N) \cap U'$ and an open neighborhood N' of z

in $\pi^{-1}(N) \cap U'$ such that $\pi|N' = id_W|\pi(N')$. By the assumption on N, there exists α such that $\pi_\alpha(V_\alpha)$ contains $\pi(z)$. So there exists $e \in p_{\pi_\alpha}^{-1}(V_\alpha)$ such that $p_W(e) = \pi(z)$. This implies $\pi|N' \in e$ and hence $\pi \in e$. In other words, $\mathcal{E}_\pi$ contains e. But $e \in p_{\pi_\alpha}^{-1}(V_\alpha) \subset \tilde{K}$, contradictory to $\tilde{K} \cap \mathcal{E}_\pi = \emptyset$. (3) $\Rightarrow$ (1) is now established. Consequently, we now know that all the first 3 conditions of (1.17) are equivalent to one another. Assume (2). Then with the same Δ', (4.a) is obvious. We shall prove (4.b). For each i, pick $e_i \in \mathcal{E}_{\pi_\alpha}$ such that $p_{\pi_\alpha}(e_i) = y_{\alpha i}$. Since $\lim_{i \to \infty} \pi_\alpha(y_{\alpha i}) = x$ and p_W is proper, there exists a subsequence $\{e_{i(j)}\}_{j=1,2,..}$ of $\{e_i\}$ such that $\lim_{j \to \infty} e_{i(j)} = \bar{e}$ exists in $\mathcal{E}_W$. It is clear that $\bar{e} \in p_W^{-1}(x)$. By the assumption (2), there exists $\beta \in \Delta'$ such that $\bar{e} \in p_{\pi_\beta}^{-1}(V_\beta) \subset \mathcal{E}_{\pi_\beta}$. Since this last set is open, we can replace the subsequence so as to have $e_{i(j)} \in \mathcal{E}_\pi$ for all j. Since $e_{i(j)} \subset \mathcal{E}_{\pi_\alpha} \cap \mathcal{E}_{\pi_\beta}$ for all j, we get a point $y_{\alpha\beta j} = p_{\pi_\alpha \wedge \pi_\beta}(e_{i(j)})$ for each j. Then $y_{\beta j} = q_\beta(y_{\alpha\beta j}) = p_{\pi_\beta}(e_{i(j)})$, $i=1,2,\ldots$, form a convergent sequence with limit $p_{\pi_\beta}(\bar{e})$ in W_β. This completes the proof of (2) $\Rightarrow$ (4). Finally we shall prove (4) $\Rightarrow$ (3). First of all, (4) implies the following condition:

(5) For every sequence y_i $i=1,2,..$, $y_i \in W$, which is convergent to x in W, there exist $\beta \in \Delta'$ and a sequence $\{y_{\beta j}\}_{j=1,2,..}$, $y_{\beta j} \in W$, which is convergent to a point in W_β and such that $\{\pi_\beta(y_{\beta j})\}_{j=1,2,\ldots}$ is a subsequence of $\{y_i\}_{i=1,2,..}$.

In fact, by (4.a) and by the finiteness of Δ', we can replace $\{y_i\}$ by a subsequence so that there exists $\alpha \in \Delta'$ and $y_{\alpha i} \in W_\alpha$ for every i such that $\pi_\alpha(y_{\alpha i}) = y_i$ for all i. Then (4.b) applied to this $\{y_{\alpha i}\}$ implies (5). We shall prove that (5) implies (3) with the same Δ'. Suppose this implication were false. Let us pick a fundamental system of countably many neighborhoods $\{U_i\}_{i=1,2,\ldots}$ of the point x in W, such that $U_{i+1} \subset U_i$ and $\overline{U_i}$ is compact for all i. For each $\alpha \in \Delta'$, $\pi_\alpha^{-1}(U_1)$ is countable at infinity and hence we can find a sequence $\{V_{\alpha i}\}_{i=1,2,\ldots}$ of open subsets in W_α such that $V_{\alpha i}$ is relatively compact in $V_{\alpha i+1}$ for all i and $\bigcup_{i-1}^{\infty} V_{\alpha i} = \pi_\alpha^{-1}(U_1)$. Under our hypothesis, $\bigcup_{\alpha\in\Delta'} \pi_\alpha(V_{\alpha i})$ is not a neighborhood of x in W for any α. Therefore, for each $i=1,2,\ldots$, we can find a point $y_i \in U_i - \bigcup_{\alpha\in\Delta'} \pi_\alpha(V_{\alpha i})$. It is then clear that $\lim_{i\to\infty} y_i = x$ in W. By assumption (5), there should then exist $\beta \in \Delta'$ and a sequence $\{y_{\beta j}\}$ convergent to a point y_β in W_β, such that $\{\pi_\beta(y_{\beta j})\}$ is a subsequence of $\{y_i\}$. Then clearly $y_\beta \in \pi_\beta^{-1}(U_1)$. Then we must have an index p such that $y_{\beta j} \in V_{\beta p}$ for all but a finite number of j. But $\pi_\beta(V_{\beta p})$ cannot contain y_i with $i > p$ and hence it cannot contain $\pi_\beta(y_{\beta j})$ except for a finite number of j. This is absurd. Thus $(5) \Rightarrow (3)$.

Although our interest in $\mathcal{E}_W$ is only as a topological space so far as the applications in later sections of this work are concerned, $\mathcal{E}_W$ can be given a natural structure of $\mathbb{C}$-ringed space and we now give a few remarks about it. Their proofs will be left to the readers.

<u>Definition</u> (3.18) An étoile $e \in \mathcal{E}_W$ gives rise to an inductive partially ordered system $\{\pi_\alpha \in e\}$ in which $\pi_\alpha \geqslant \pi_\beta$ if and only if $\mathrm{Hom}(\pi_\alpha, \pi_\beta) \neq \emptyset$. Write $\pi_\alpha : W_\alpha \to W$ and $\mathcal{H}(W_\alpha) = \mathcal{H}_{W_\alpha}(W_\alpha)$. If $q_\alpha \in \mathrm{Hom}(\pi_\alpha, \pi_\beta)$, then q_α defines a homomorphism of $\mathbb{C}$-algebras $\theta_{\alpha\beta} : \mathcal{H}(W_\beta) \to \mathcal{H}(W_\alpha)$. We have $\theta_{\alpha\beta}\theta_{\beta\gamma} = \theta_{\alpha\gamma}$ whenever the first two are defined.

Thus we can define a local $\mathbb{C}$-algebra:

(3.18.1) $\mathcal{H}_e = \varinjlim_{\pi_\alpha \in e} \mathcal{H}(W_\alpha)$, $\Theta_{\alpha,e} : \mathcal{H}(W_\alpha) \longrightarrow \mathcal{H}_e$,

where the limit is defined by the $\Theta_{\alpha\beta}$ for all $\pi_\alpha, \pi_\beta \in e$ such that $\mathrm{Hom}(\pi_\beta, \pi_\alpha) \neq \emptyset$. We call $\mathcal{H}_e$ the local ring of $\mathcal{E}_W$ at e. We can put these local rings together into a sheaf of $\mathbb{C}$-algebras, denoted by $\mathcal{H}_{\mathcal{E},W}$, as follows:

(3.18.2) For every open subset U of $\mathcal{E}_W$, $\mathcal{H}_{\mathcal{E},W}(U)$ is the $\mathbb{C}$-subalgebra of $\prod_{e \in U} \mathcal{H}_e$ defined by: $f = \{f_e\}_{e \in U} \in \mathcal{H}_{\mathcal{E}_W}(U)$ if and only if for every $e_0 \in U$ there exist $\pi_\alpha \in e_0$ and $f_\alpha \in \mathcal{H}(W_\alpha)$ such that $\mathcal{E}_{\pi_\alpha} \subset U$ and $f_e = \Theta_{\alpha,e}(f_\alpha)$ for all $e \in \mathcal{E}_{\pi_\alpha}$. From now on, we shall view $\mathcal{E}_W$ as a $\mathbb{C}$-ringed space with the structure sheaf $\mathcal{H}_{\mathcal{E},W}$.

<u>Definition</u> (3.19) To compare with $\mathcal{E}_W$, we can define a limit space of global blowing-ups, Γ_W, as follows. Let us consider the subcategory $\Gamma(W)$ of $\mathcal{E}(W)$, which consists of all the morphisms $\pi : W' \longrightarrow W$ obtained as blowing-ups whose centers are closed nowhere dense complex subspaces of W. It follows from (2.16.1) that if $\pi_\alpha \in \Gamma(W)$, $\alpha = 1, 2$, then the join $\pi_1 \wedge \pi_2 \in \Gamma(W)$. Hence $\{\pi_\alpha \in \Gamma(W)\}$ form an inductive partially ordered system in the same way as in (3.18). Hence we can define a $\mathbb{C}$-ringed space

$$\Gamma_W = \varprojlim_{\pi_\alpha \in \Gamma(W)} W_\alpha$$

(which means

$$|\Gamma_W| = \varprojlim_{\pi_\alpha \in \Gamma(W)} |W_\alpha| \text{ (as topological space)}$$

and $\mathcal{H}_{\Gamma,W} = \varinjlim_{\pi_\alpha \in \Gamma(W)} \mathcal{H}_{W_\alpha}$ (as sheaf of $\mathbb{C}$-algebras)

which is the structure sheaf of Γ_W).

<u>Remarks</u> (3.20) We have a natural morphism of $\mathbb{C}$-ringed space. $\mathcal{E}_W \to W$ which induces the map of topological spaces p_W. It will be also denoted by p_W. We also have a natural morphism $q_W : \Gamma_W \to W$ (the projection from the limit). By (3.12) we get a natural map $\mathcal{E}_W \to W'$ for every $W' \to W$ belonging to $\Gamma(W)$ and hence a natural map

$$|p_W| : |\mathcal{E}_W| \longrightarrow |\Gamma_W|$$

We also easily find a natural $|p_W|$- homomorphism

$$\Theta_W : \mathcal{H}_{\Gamma, W} \to \mathcal{H}_{\mathcal{E}, W}$$

These two define a canonical morphism of $\mathbb{C}$-ringed space.

$$p_W : \mathcal{E}_W \to \Gamma_W$$

which has the commutativity $q_W\, p_W = p_W$.
Furthermore, if $\pi : W' \to W$ belongs to $\Gamma(W)$, then π induces a canonical isomorphism

$$i_\pi : \Gamma_{W'} \xrightarrow{\sim} \Gamma_W$$

The map j_π of (3.7) canonically extends to an isomorphism of $\mathbb{C}$-ringed spaces $\mathcal{E}_{W'} \xrightarrow{\sim} \mathcal{E}_W$ and

$$i_\pi p_{W'} = p_W j_\pi .$$

<u>Remark</u> (3.21) For ringed space X, we denote by $\mathrm{Pic}(X)$ the group of invertible $\mathcal{O}_X$-modules ("invertible" means "locally isomorphic to $\mathcal{O}_X$") where $\mathcal{O}_X$ denotes the structure sheaf of X. (Here the $\mathcal{O}_X$-modules are counted only up to isomorphisms, and the group law in $\mathrm{Pic}(X)$ is defined by tensor product over $\mathcal{O}_X$). $\mathrm{Pic}\,(X)$ are often thought of as the group of line boundles (locally trivial) on X. Now the morphism $p_W : \mathcal{E}_W \to \Gamma_W$ induces a natural homomorphism of groups

$$\mathrm{Pic}(P_W) : \mathrm{Pic}(\Gamma_W) \to \mathrm{Pic}(\mathcal{E}_W)$$

defined by "pull-back". We can prove that $\mathrm{Pic}(P_W)$ is never isomorphic (i.e., never surjective) unless $\dim W \leq 2$. This shows that $\mathcal{E}_W$ is substantially "bigger" than Γ_W (even when W is an algebraic variety or a Stein variety such as a local model of complex space). Let us first take a simple situation as follows.

(3.21.1) Let D_1 be an open bounded **polydisc** centered at 0 in $\mathbb{C}^n$ with $n \geq 3$. Let D_2 be another such that $\overline{D_2} \subset D_1$. Take a connected smooth closed complex curve Γ in D_2 such that there exists no closed complex subspace of D_1 which has dimension 1 and contains Γ. Let $\pi_o : D_2' \to D_2$ be the blowing-up with center Γ and pick a point $\xi' \in D_2'$ such that $\pi_o(\xi') \in \Gamma$. Let $\pi_o' : D_2'' \to D_2'$ be the blowing-up with center ξ'. Let $E'' = (\pi_o')^{-1}(\xi')$. Let J'' be the ideal sheaf of E'' in $\mathcal{H}_{D_2''}$. Now, given any W and a point $x \in W$ at which W has dimension $n \geq 3$, there exist an open neighborhood N of x in W and a surjective finite (in particular, proper) morphism $f : W|N \to D_1$. Let $(f^{-1}(D_2), f^{-1}(\Gamma), \pi)$ be the local blowing-up with $\pi : W' \to W$. Then there exists a canonical morphism $f' : W' \to D_2'$, which is again surjective and finite. (f' is the unique morphism with $\pi_o f' = f\pi$). Let $E' = (f')^{-1}(\xi')$ and let $\pi' : W'' \to W'$ be the blowing-up with center E'. (E' is not, in general, reduced). Let $f'' : W'' \to D_2''$ be the canonical morphism. Let $I'' = J''\mathcal{H}_{W''}$ with respect to f''. This is the ideal sheaf of $(\pi')^{-1}(E')$. We obtain an element of $\mathrm{Pic}(\mathcal{E}_W | \mathcal{E}_{\pi\pi'})$ which is the pull-back of I'' by the projection $P_{\pi\pi'}$. This element is trivial outside the compact subset $P_\pi^{-1}(E')$ of $\mathcal{E}_{\pi\pi'}$ and hence it extends to an element $\mathfrak{F} \in \mathrm{Pic}(\mathcal{E}_W)$. We claim that this $\mathfrak{F}$ is not in the image of $\mathrm{Pic}(P_W)$.

(3.21.2) Let $\mathcal{F}$ be any element of Pic (Γ_W) and let U be any relatively compact open subset of W. Then we can find a closed complex subspace E_α of W, nowhere dense in W, and an element $\mathcal{G} \in \mathrm{Pic}\,(W_\alpha)$ where $h_\alpha : W_\alpha \longrightarrow W$ is the blowing-up with center E_α, such that $\mathcal{F}|U$ coincides with the pull-back of $\mathcal{G}|U$ with respect to the morphism $i^{-1}_{h_\alpha} p'_{W_\alpha}$. (Here $|U$ denotes the restriction to the inverse image of U into the respective space).

(3.21.3) Let $\mathcal{F}$ be any element of $\mathrm{Pic}(\mathcal{E}_W)$ and let U be any relatively compact open subset of W. Then, thanks to the properness of p_W, we can find a finite number of morphisms $g_\alpha : V_\alpha \to W, \in \mathcal{E}(W)$ for all α, and $\mathcal{G}_\alpha \in \mathcal{E}(W)$ such that

a) $\mathcal{F}|\mathcal{E}_{g_\alpha}$ is the pull-back of $\mathcal{G}_\alpha$ by p_{g_α} for every α,

b) $p_W^{-1}(U) = \bigcup_\alpha \mathcal{E}_{g_\alpha}$

Conversely, let $\{g_\alpha : V_\alpha \longrightarrow W\}$ be any family of morphisms $\in \mathcal{E}(W)$ and $\{\mathcal{G}_\alpha\}$ be a family with $\mathcal{G}_\alpha \in \mathrm{Pic}(V_\alpha)$, one for each α, such that

c) $\mathcal{E}_W = \bigcup_\alpha \mathcal{E}_{g_\alpha}$

d) With $q_{\alpha\beta} \in \mathrm{Hom}\,(g_\alpha \wedge g_\beta, g_\beta)$, $g^*_{\alpha\beta}(\mathcal{G}_\beta) = g^*_{\beta\alpha}(\mathcal{G}_\alpha)$ (i.e., the two pull-backs are isomorphic) for all (α,β).

Then there exists $\mathcal{F} \in \mathrm{Pic}(\mathcal{E}_W)$ such that a) is true for all α.

These statements on the $\mathbb{C}$-ringed space structure on $\mathcal{E}_W$ and on Pic $(\mathcal{E}_W)$ are given only to indicate the difference between $\mathcal{E}_W$ and Γ_W (or "local blowing-ups" and "global blowing -up" in terms of their limits). They will not be used in any part of the subsequent sections.