§ 6. Semi-analytic sets and subanalytic sets.

Speaking in an abstract fashion, let X be a set and let Δ be a family of subsets of X. Then we often ask if Δ satisfies the following elementary set-theoretical conditions: If $A_i \in \Delta$, $i=1,2$, then

(6.1.a) $A_1 \cup A_2 \in \Delta$

(6.1.b) $A_1 \cap A_2 \in \Delta$

(6.1.c) $A_1 - A_2 \in \Delta$

Remark (6.2) (6.1.c) implies that $\emptyset \in \Delta$. If $\emptyset \in \Delta$, then (6.1.a) is equivalent to

(6.1.a') (finite union property) If $A_i \in \Delta$, $i \in I$, for a finite index set I, then

$$\bigcup_{i \in I} A_i \in \Delta$$

If $X \in \Delta$, then (6.1.b) is equivalent to

(6.1.b') (finite intersection property). If $A_i \in \Delta, i \in I$, for a finite index set I, then

$$\bigcap_{i \in I} A_i \in \Delta$$

If Δ satisfies (6.1.a') and (6.1.b'), then (6.1.c) is equivalent to

(6.1.c') (complementary property). If $A \in \Delta$, then $X - A \in \Delta$. Thus, if $X \in \Delta$, then (6.1.a) + (6.1.b) + (6.1.c) is equivalent to (6.1.a') + (6.1.b') + (6.1.c').

Remark (6.3) Let us recall that we have the following set-theoretical equalities in general.

a) $X - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X - A_i)$

b) $\bigcap_{i \in I} \bigcup_{j \in J} A_{ij}$

$= \bigcup_{k \in \mathrm{Hom}(I,J)} \bigcap_{i \in I} A_{ik(i)}$

c) $\bigcup_{i \in I} \bigcap_{j \in J} A_{ij}$

$= \bigcap_{k \in \mathrm{Hom}(I,J)} \bigcup_{i \in I} A_{ik(i)}$

Proof. a) is obvious. As for b),

$$x \in \bigcap_{i \in I} \bigcup_{j \in J} A_{ij}$$

$$\Longleftrightarrow \forall i, \exists j \text{ such that } x \in A_{ij}$$

$$\Longleftrightarrow \exists\, k \in \mathrm{Hom}(I,J) \text{ such that } \forall_i, \; x \in A_{ij}$$

$$\Longleftrightarrow x \in \bigcup_{k \in \mathrm{Hom}(I,J)} \bigcap_{i \in I} A_{ik(i)}$$

Finally, c) follows from a) + b).

Remark (6.4) Let Δ be a family of subsets of X. Assume that $\emptyset \in \Delta$. Let $\tilde{\Delta}$ be the set of those subsets of X which can be expressed in the form:

$$\bigcup_{i \in I} \bigcap_{j \in J} (A_{ij} - B_{ij})$$

where I and J are non-empty finite index set, and A_{ij} and B_{ij} are all members of Δ. Then

1) $\tilde{\Delta} \supset \Delta$

2) $\tilde{\Delta}$ satisfies the conditions (6.1.a) + (6.1.b) + (6.1.c),

3) $\tilde{\Delta}$ is the smallest among those families of subsets of X which have the properties 1) and 2).

Proof. 1) is clear as $\emptyset \in \Delta$. 3) follows 1) + 2) as is easily seen. To prove 2), we first make a general remark. Take any pair of subsets $E_\alpha, \alpha = 1,2$, belonging to $\tilde{\Delta}$, and express them as above. Say we have

$$E_\alpha = \bigcup_{i \in I} \bigcap_{j \in J_\alpha} (A_{\alpha ij} - B_{\alpha ij})$$

We then remark that we can always change their expressions so as to have $I_1 = I_2$ and $J_1 = J_2$.
In fact, let I (resp. J) be the disjoint union of I_1 and I_2 (resp. J_1 and J_2). We then extend the notation $A_{\alpha ij}$ and $B_{\alpha ij}$ for all $(i,j) \in I \times J$ as follows. For each $i \in I_\alpha$, pick one index $j_o \in J_\alpha$ and define

$$A_{\alpha ij} = A_{\alpha ij_o} \text{ and } B_{\alpha ij} = B_{\alpha ij_o}$$

for all $j \in J - J_\alpha$. After this is done, pick one index $i_o \in J_\alpha$ and define

$$A_{\alpha ij} = A_{\alpha i_o j} \quad \text{and} \quad B_{\alpha ij} = B_{\alpha i_o j}$$

for all $i \in I - I_\alpha$ and $j \in J$. Note that we then have

$$E_\alpha = \bigcup_{i\in I} \bigcap_{j \in J} (A_{\alpha ij} - B_{\alpha ij})$$

for each $\alpha = 1,2$. We now proceed to prove 2).
Let $K = \{1,2\}$. Then

$$E_1 \cup E_2 = \bigcup_{(\alpha,i)\in K\times I} \bigcap_{j\in J} (A_{\alpha ij} - B_{\alpha ij})$$

This proves (6.1.a) for $\tilde{\Delta}$. By b) of (6.3), we have

$$E_1 \cap E_2 = \bigcup_{p\in \mathrm{Hom}(K,I)} \bigcap_{(\alpha,j)\in K\times J} (A_{\alpha p(\alpha) j} - B_{\alpha p(\alpha) j})$$

This proves (6.1.b) for $\tilde{\Delta}$. Next, using a) of (6.3), we obtain

$$E_1 - E_2 = \bigcap_{i\in I} \bigcup_{j\in J} \{E_1 - (A_{2ij} - B_{2ij})\}$$

But $E_1 - (A_{2ij} - B_{2ij})$

$$= (E_1 - A_{2ij}) \cup (E_1 \cap A_{2ij} \cap B_{2ij}) .$$

Also $E_1 - A_{2ij}$

$$= \bigcup_{i'\in I} \bigcap_{j'\in J} \{(A_{1i'j'} - B_{1i'j'} - A_{2ij})\}$$

$$= \bigcup_{i'\in I} \bigcap_{j'\in J} \{(A_{1i'j'} - B_{1i'j'}) \cup C_{i'j'ij}\}$$

where $C_{i'j'ij} = A_{1i'j'} \cap B_{1i'j'} \cap A_{2ij}$. Again, by (6.1.a) and (6.1.b), we get

$$E_1 - A_{2ij} \in \tilde{\Delta} \text{ for all } (i,j) ,$$

and then

$$E_1 - E_2 \in \tilde{\Delta}, \text{ which proves (6.1.c)} .$$

<u>Definition</u> (6.5) We shall call $\tilde{\Delta}$ the elementary (set-theoretical) closure of Δ, where the notation is the same as in (6.4).

<u>Remark</u> (6.6). If Δ is a finite set, then the elementary closure $\tilde{\Delta}$ is also finite.

<u>Proof</u>. If Δ is finite, then there are only finitely many distinct subsets of the form $A_{ij} - B_{ij}$ in the expression of (6.4). So there are only finitely many distinct sub-sets of the form

$$\bigcap_{i \in J} (A_{ij} - B_{ij})$$

and hence of the form

$$\bigcup_{i \in I} \bigcap_{i \in J} (A_{ij} - B_{ij})$$

From now on, X will denote a real-analytic space. For simplicity, it will also denote its underying topological space as well as its point-set. Let U be an open subset of X.
We shall denote by $\Delta_+(U)$ (resp. $\Delta_o(U)$) the set of those subsets A of U which can be expressed as

$$A = \{x \in U \mid f(x) > 0\}$$

$$(\text{resp. } A = \{x \in U \mid f(x) \geqslant 0\})$$

with a real-analytic function f on $X|U$, i.e., $f \in \mathcal{O}_X(U)$. As $\pm 1 \in \mathbb{R} \subset \mathcal{O}_X(U)$, both $\Delta_+(U)$ and $\Delta_o(U)$ contain $\emptyset$ as well as U.
The elementary closure of $\Delta_+(U)$ coincides with the same of $\Delta_o(U)$, because if $A \in \Delta_+(U)$ is as above then

$$U - A = \{x \in U \mid g(x) \geqslant 0\} \in \Delta_o(U)$$

where $g = -f$. Moreover, if $A \in \Delta_o(U)$ is as above, then

$$A \cap (U-A) = \{x \in U \mid f(x) = 0\}$$

which is in the elementary closure, too.

<u>Definition</u> (6.7) A subset A of X is said to be semi-analytic at $x_o \in X$, if there exists an open neighborhood U of x_o in X such that $A \cap U$ belongs to the elementary closure of $\Delta_+(U)$. A is said to be semi-analytic in X if it is so at every point of X.

<u>Proposition</u> (6./) A is semi-analytic at x_o in X if and only if there exist an open neighborhood U of x_o in X and a finite system of $f_{ij}, g_{ij} \in \mathcal{O}_X(U)$ such that

$$A \cap U = \bigcup_i (A_i^o \cap A_i^+)$$

with $A_i^o = \bigcap_j \{x \in U \mid f_{ij}(x) = 0\}$

and $A_i^+ = \bigcap_j \{x \in U \mid g_{ij}(x) > 0\}$

<u>Proof</u>. The expression of $A \cap U$ implies that $A \cap U$ belongs to the elementary closure of $\Delta_+(U)$. So it is enough to show that every element of the elementary closure can be given an expression given as in (6.7). To prove this, take

$$E = \bigcup_i \bigcap_j (A_{ij} - B_{ij})$$

according to (6.4), where

$$A_{ij} = \{x \in U \mid f_{ij}(x) > 0\}$$

$$B_{ij} = \{x \in U \mid g_{ij}(x) > 0\}$$

with $f_{ij}, g_{ij} \in \mathcal{Q}_X(U)$. For each (i,j),

$$A_{ij} - B_{ij} = (A_{ij} \cap B_{ij}^-) \cup (A_{ij} \cap B_{ij}^o)$$

where $B_{ij}^- = \{x \in U \mid - g_{ij}(x) > 0\}$

$$B_{ij}^o = \{x \in U \mid g_{ij}(x) = 0\}$$

By b) of (6.3), therefore, for each i

$$\bigcap_j (A_{ij} - B_{ij})$$

can be expressed as a finite union of intersections of subsets like A_{ij}, B_{ij}^-, B_{ij}^o. It follows that E can be put in the form expressed in (6.7).

The following two propositions, complementary to each other, are due to Lojasiewicz and proven after a lemma which is also due to Lojasiewicz. Our proof of the lemma will be seen essentially different from Lojasiewicz' proof.

<u>Proposition</u> (6.8) If A is an open semi-analitic set in X, then for every $x_o \in X$ there exist an open neighborhood U of x_o in X and a finite system of $g_{ij} \in \mathcal{Q}_X(U)$, such that

$$A \cap U = \bigcup_i A_i$$

where $A_i = \bigcap_j \{x \in U \mid g_{ij}(x) > 0\}$.

Proposition (6.9) If A is a closed semi-analytic subset of X, then for every $x_o \in X$ there exist an open neighborhood U of x_o in X and a finite system of $h_{ij} \in \mathcal{O}_X(U)$ such that

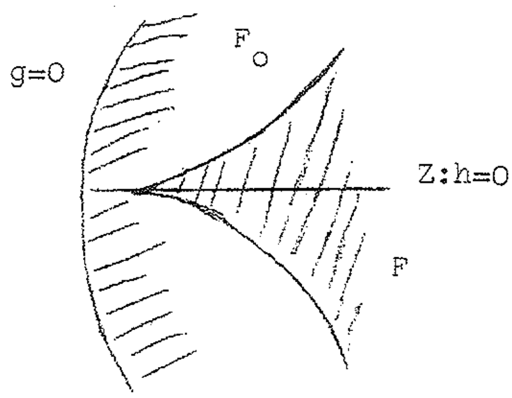
$$A \cap U = \bigcup_i \bigcap_j \{x \in U \mid h_{ij}(x) \geq 0\}$$

Note that each of these two propositions implies the other by a) + b) +c) of (6.3). We shall prove (6.8) after the following

Lemma (6.9.1) Let U be an open subset of a real analytic space X. Let g_i, $0 \leq i \leq s$, be a finite number of elements in $\mathcal{O}_X(U)$. Assume that

$$F_o = \{x \in U \mid g_i(x) > 0,\ 0 \leq i \leq s\}$$

is relatively compact in U. Let F be an open semi-analytic set in X and let



$Z = \{x \in F_o \mid h(x) = 0\}$ with $h \in \mathcal{O}_X(U)$. Assume that $Z \subset F \subset F_o$. Then there exist an integer $N > 0$ and a constant $C \in \mathbb{R}_+$ such that

$$Z \subset \{x \in F_o \mid q(x) > 0\} \subset F$$

where $q = g^{2N} - Ch^2$ with $g = \prod_{i=1}^{s} g_i$.

Proof. The question can be localized as follows

(6.9.1*) For each point $\xi \in U$, there exist an open neighborhood W_ξ of ξ in U, an integer $N_\xi > 0$ and a map $C_\xi : \mathbb{Z}_+ \to \mathbb{R}_+$ such that, if $q_{N,\xi} = g^{2N} - C_\xi(N)h^2$, then

$$Z \cap W_\xi \subset \{x \in F_o \cap W_\xi \mid q_{N,\xi}(x) > 0\}$$
$$\subset F \cap W_\xi \quad \text{for all } N \geq N_\xi .$$

In fact, assume that (6.9.1*) is verifield for each point $\xi \in U$. Then, since F_o is relatively compact in U, there exists a finite subset Δ of U such that

$$F_o \subset \bigcup_{\xi \in \Delta} W_\xi$$

Now, let N be any integer $\geq \max\{N_\xi, \xi \in \Delta\}$, and define C

to be $\max\{C_\xi(N), \xi \in \Delta\}$. As is easily seen, the claimed inclusion of (6.9.1) with these N and C follows from the corresponding inclusion relations of (6.9.1*) for all $\xi \in \Delta$.
The localized problem (6.9.1*) can be made much simpler to prove by means of a suitable modification of the ambient space X. Namely, let us apply desingularization theorems of § 5, to our situation and get a morphism $\pi : X' \to X$ having the following properties:

(a) π is proper surjective over a neighborhood of ξ in X,

(b) X' is smooth

(c) for every point ξ' of X', we can find a local coordinate system z centered at ξ', with respect to which $\pi^{-1}(F)$ is locally a union of quadrants and every one of the $g_i \circ \pi$ and $h \circ \pi$ is an invertible multiple of a monomial in z.

Now, in view of (a), the proof of (6.9.1*) at ξ can be easily reduced to that of the same local assertion at every point of $\pi^{-1}(\xi)$ for the pull-backs and inverse images in X': $g_i \circ \pi, h \circ \pi$, $\pi^{-1}(Z)$, $\pi^{-1}(F)$ and $\pi^{-1}(F_o)$. Hence it is enough to consider the case in which $\pi = id_X$ has the properties (b) and (c). Let $z = (z_1, \dots, z_r)$ be a local coordinate system having the property c). Let us reorder it and divide it into $(t_1, \dots, t_a, w_1, \dots, w_b)$, $a + b = r$, in such a way that

(d) $g_i = t^{A_i} u_i, A_i \in \mathbb{Z}_o^a$, where $A = A_1 + \dots + A_s \in \mathbb{Z}_+^a$ and u_i is an invertible real-analytic function in a neighborhood of ξ in X, and

(e) $h = t^B w^{B'} v$, $B \in \mathbb{Z}_o^a$, $B' \in \mathbb{Z}_o^b$,

where v is invertible in a neighborhood of ξ in X. (Here we are assuming that none of the g_i and h is identically zero in any neighborhood of ξ in X, for if otherwise the assertion is trivially true for any choice of N_ξ and C_ξ, provided W_ξ is chosen sufficiently small).
Furthermore, by a positive scalar multiplication on (t,w), if necessary, we may assume that

(f) the cube $U_\xi = \{ z \in X \,|\, |z_i| < 1,\ 1 \leq i \leq r \}$ is contained in and relatively compact in a coordinate neighborhood of ξ in which c), d) and e) are all verified. Let $g = \prod_i g_i$ and $u = \prod_i u_i$. We shall now prove the existence of N_ξ and D_ξ.

Case 1. Assume $B' \neq 0$. Then $F \cap U_\xi = F_o \cap U_\xi$.
In fact, pick any $(\tau,\omega) \in F_o \cap U_\xi$. Then by (d), $(\tau,o) \in F_o \cap U_\xi$. Hence by (e), $h(\tau,o) = 0$ and $(\tau,0) \in Z \subset F$. Since F is open and is a union of quadrants, F contains (τ,ω). Now, having $F \cap U_\xi = F_o \cap U_\xi$, we can prove (6.9.1*) for $W_\xi = U_\xi$ and for every $N > 0$ and every $C = C_\xi(N) \in \mathbb{R}_+$.

Case 2. Assume $B' = 0$. In this case, we claim $Z \cap U_\xi = \emptyset$. In fact, if $x \in F_o \cap U_\xi$ then $g_i(x) > 0$ for all i. Hence, by (d), none of the $t_i(x)$ is zero. So, by (e) and $B'=0$, $h(x) \neq 0$, i.e., $x \notin Z$. We shall find N_ξ and C_ξ such that $\{x \in F_o \mid q_{N,\xi}(x) > 0\} \cap U_\xi = \emptyset$ for all $N \geq N_\xi$. This clearly suffices for (6.9.1*). By (d), there exists $N_\xi \in \mathbb{Z}_+$ such that $N_\xi A - B \in \mathbb{Z}_o^a$ (i.e., all its components are non-negative). Pick such N_ξ and define

$$C_\xi(N) = \max_{x \in \bar{U}_\xi} \left\{ |u(x)^N \quad v(x)| \right\}$$

Then, for every $N \geq N_\xi$ and $C = C_\xi(N)$, $C^2 h(\tau,\omega)^2 \geq |\tau^B|^2 u(\tau,\omega)^2$

$$\geq |\tau^A|^{2N} u(\tau,\omega)^{2N} = g(\tau,\omega)^{2N}$$

for every $(\tau,\omega) \in U_\xi$. This means

$$\left\{ x \in U_\xi \mid q_{N,\xi}(x) > 0 \right\} = \emptyset$$

Proof of (6.8). The question being local, we may assume that there exists a finite system of real-analytic functions $\{g_{ij}, h_{ij}\}$ on X such that $A = \bigcup_i A_i$ where

$$A_i = \left\{ x \in X \mid h_{ij}(x) = 0,\ 1 \leq j \leq r_i,\ g_{ij}(x) > 0,\ 1 \leq j \leq s_i \right\}.$$

If all the $r_i = 0$, then this is the assertion of (6.8). If not, pick one d with $r_d > 0$. We shall then prove that there exists a real-analytic function q on X such that, within a sufficiently small neighborhood of x_o,

$$\bigcup_i A_i = \bigcup_{i \neq d} A_i \cup A'_d$$

with $A'_d = \left\{ x \in X \mid q(x) > 0,\ g_{dj}(x) > 0,\ 1 \leq j \leq s_d \right\}$.

The question being local, we may assume that X is imbedded in some $\mathbb{R}^n$. Let $(z_1, \ldots, z_n)$ be a coordinate system for $\mathbb{R}^n$, centered at x_o, and pick $\varepsilon > 0$ so small that $X \cap \left\{ \sum_{j=1}^n z_i^2 \leq \varepsilon \right\}$ is compact.
Let $g_o = \varepsilon - \sum_{j=1}^n z_j^2, g_j = g_{dj}$, $s = s_d$, $h = \sum_j h_{dj}^2$ and

$F = \{x \in A \mid g_o(x) > 0\}$. Applying (6.9.1) to these, we obtain a real-analytic function q on X such that

$$Z \subset \{x \in F_o \mid q(x) > 0\} \subset F .$$

where if $W = \{z \in \mathbb{R}^n \mid g_o(z) > 0\}$ then $Z = A_d \cap W$, $F = A \cap W$ and

$$\{z \in F_o \mid q(x) > 0\} = A'_d \cap W$$

This proves the above assertion on q and A'_d .
Now, replacing A_d by A'_d for each d with $r_d > 0$, we obtain a local expression of A as is asserted in (6.8).

Let X be a real-analytic space as before. For an open subset U of X, we denote by $\Gamma(U)$ the set of those closed subsets of U which are images of proper real-analytic maps $g : Y \to X|U$.

<u>Definition</u> (6.10) A subset A of X is said to be subanalytic at $\xi \in X$ in X if there exists an open neighborhood U of ξ in X such that $A \cap U$ belongs to the elementary closure of $\Gamma(U)$ (in the sense of (6.5)). We say that A is subanalytic in X if it is so at every point of X .

<u>Proposition</u> (6.11) A subset A of X is subanalytic at a point $\xi \in X$ if and only if there exist an open neighborhood U of ξ in X and a finite system of proper real-analytic maps $g_{ij} : Y_{ij} \to X|U$, $1 \le i \le p$ and $j = 1,2,$ such that

$$A \cap U = \bigcup_{i=1}^{p} (\mathrm{Im}(g_{i1}) - \mathrm{Im}(g_{i2})) .$$

<u>Proof</u>. It is clear that a subset expressed in this form belongs to the elementary closure of $\Gamma(U)$. Let us next prove the only-if part. By (6.4), we have an open neighborhood U of ξ in X and a finite system of proper real-analytic maps $e_{ij} : E_{ij} \to X|U$ and $f_{ij} : F_{ij} \to X|U$ such that

$$A \cap U = \bigcup_i \bigcap_j (\mathrm{Im}(e_{ij}) - \mathrm{Im}(f_{ij}))$$

For each i, let Y_{i1} be the fibre product of E_{ij} over $X|U$ for all j and let $g_{i1} = \prod_j e_{ij}$, the canonical map $Y_{i1} \to X|U$. Let Y_{i2} be the disjoint union of the fibre products $E_{ij} \times_X F_{ij}$ for all j and let $g_{i2} : Y_{i2} \to X|U$ be the canonical map. Then we have

$$A \cap U = \bigcup_i (\mathrm{Im}(g_{i1}) - \mathrm{Im}(g_{i2}))$$

Which proves (6.11).

<u>Proposition</u> 12. If A is semi-analytic at $\xi \in X$, then it is subanalytic at the same $\xi \in X$.

<u>Proof.</u> It is enough to prove that for every open subset U of X, $\Delta_o(U)$ is contained in $\Gamma(U)$. So, let f be a real-analytic function on $X|U$ and let $A = \{x \in U \mid f(x) \geq 0\}$. Let Y be the closed real-analytic subspace of $X \times \mathbb{R}$ defined by the ideaal $(t^2 - f(x))$ where t is the parameter for $\mathbb{R}$. Let $g : Y \to X|U$ be the morphism induced by the projection $X \times \mathbb{R} \to X$. It is then easy to see that g is proper and $\mathrm{Im}(g) = A$. Hence $A \in \Gamma(U)$.

§ 7. Rectilinearization of subanalytic sets.

The main purpose of this section is the rectilinearization theorem, stated below, which asserts that every subanalytic subset can be transformed "locally" into unions of quadrants in real number spaces, by means of a locally finite family of finite sequences of local blowing-ups applied to the ambient space of the given subanalytic subset. From this theorem, as we shall find in later sections, we can deduce rather easily and systematically most of the theorems of Lojasiewicz on semi-analytic sets, here generalized to the case of subanalytic sets.

Our proof of the rectilinearization theorem will be carried out in two steps. The first step is the application of local flattening theorem, proven in § 4, to proper real-analytic maps which define the given subanalytic subset. Local flattening theorem is effective in eliminating the kind of unpleasant phenomena described in Example (4.2). The result of this first step is that every subanalytic subset can be transformed into semi-analytic sets by means of a complete family of finite sequences local blowing-ups applied to the ambient space. The second step is that, after we get to the situation of semi-analytic subsets, we can apply directly the desingularization theorem of type II, (5.11) of § 5, to the real-analytic functions that appear in the definition of semi-analytic subsets. The result of this second part is that every semi-analytic set can be transformed into unions of quadrants in real number spaces, by means of finite sequences of blowing-ups applied to the ambient space.

The rectilinearization theorem can be stated as follows

Theorem (7.1) Let X be a real-analytic space and let A be a subanalytic space and let A be a subanalytic subset of X (subanalytic at every point of X). Let K be any compact subset of X. Then there exists a finite number of real-analytic maps $\{\pi_\alpha : V_\alpha \longrightarrow X\}$ such that

1) each V_α is isomorphic to $\mathbb{R}^{n_\alpha}$ (a real number space),

2) there exists a compact subset K_α of V_α, one for each α, such that

$$\bigcup_\alpha \pi_\alpha(K_\alpha)$$

is a neighborhood of K in X.

3) for each α, $\pi_\alpha^{-1}(A)$ is a union of quadrants in $\mathbb{R}^{n_\alpha}$.

<u>Remark</u> (7.1.1.) We say that a subset B of a real number space $\mathbb{R}^n$ is a <u>quadrant</u> if there exists a disjoint partition $\{1,\dots,n\} = I_\circ \cup I_+ \cup I_-$ such that B is the set of $x \in \mathbb{R}^n$ defined by $x_i = 0$ for all $i \in I_\circ$, $x_j > 0$ for all $j \in I_+$ and $x_k < 0$ for all $k \in I_-$, where $x=(x_1,\dots,x_n)$ is the usual coordinate system in $\mathbb{R}^n$.

<u>Remark</u> (7.1.2) If the ambient space X in (7.1) is smooth, then each π_α is actually obtained by a finite sequence of local blowing-ups over X whose centers are all nowhere dense in their respective ambient spaces. In particular, we may require that

4) π_α induces an open imbedding of an open dense subset of V_α into X, provided X is smooth.

<u>Remark</u> (7.1.3) In the general case in which X is singular, we apply Desingularization I, (5.10), to X and replace A by its inverse image in the derived smooth ambient space. Here we may always replace X by its restriction to an open neighborhood of the given compact subset K, so that we have a smooth real-analytic filtration $\{X^{(i)}\}_{i\geq 0}$. According to the nature of our desingularization ((5.10) with conditions a) and b)) we can find π_α of (7.1), always under the following additional condition:

5) Given $\{X^{(i)}\}$ as above, for each α there exists an index $i \geq 0$ such that π_α induces a morphism $V_\alpha \longrightarrow X^{(i)}$, that $\pi_\alpha^{-1}(X^{(i)} - X^{(i+1)}$ is

dense in V and that π_α induces an open imbedding of an open dense subset of V_α into $X^{(i)} - X^{(i+1)}$.

All we need to do this, is to apply (7.1.2) to the situation after the desingularization of X.

For the proof of (7.1), we first give the second step, as was described in the beginning of this section, and then give the proof for the first step. The first step asserts the following

Proposition (7.2) Let A be a semi-analytic subset of a real-analytic space X. Then for every point x of X, there exist an open neighborhood U of x in X and a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, hook, "j"'] \arrow[dr, "\pi"'] & & (X|U) \times \mathbb{P}^N_{\mathbb{R}} \arrow[dl, "\text{projection}"] \\ & X|U & \end{tikzcd}$$

such that

1) j is a closed imbedding and Z is smooth everywhere

2) π is surjective

3) $\pi^{-1}(A)$ is everywhere locally a union of quadrants, i.e., for every $z \in Z$, there exists an open neighborhood V of z in Z with an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ (for some $n \geq 0$) which maps $\pi^{-1}(A)|V$ to a union of quadrants in $\mathbb{R}^n$.

The proof of (7.2) will be completed as a few lemmas.

Lemma (7.2.1) In addition to the assumption of (7.2), let us further assume that X is smooth. Then there exists a diagram of (7.2) having the following properties in addition to 1), 2) and 3):

4) π is relatively algebraic; in fact, the isomorphic image of Z by j is defined by homogeneous polynomial equations (in terms of the standard homogeneous coordinate system in $\mathbb{P}^N_{\mathbb{R}}$) whose coefficients are all in $\mathcal{Q}_X(U)$.

5) π induces an isomorphism over an open dense subset of U.

Proof. By definition of semi-analyticity, we can find an open neighborhood U of x in X and a finite system of real-analytic functions $f_{\alpha\beta}$, $g_{\alpha\beta}$ on U such that

$$A \cap U = \bigcup_{\alpha} \{\xi \in U \mid f_{\alpha\beta}(\xi) = 0,\ g_{\alpha\beta}(\xi) > 0,\ \forall \beta\}.$$

Here we may assume that U is connected so that each of the $f_{\alpha\beta}, g_{\alpha\beta}$, is either identically zero on U or nowhere locally identically zero in U. Clearly we may omit those $f_{\alpha\beta}$ which are identically zero and those for which there exists at least one identically zero $g_{\alpha\beta}$. So assume that none of the $f_{\alpha\beta}$ and $g_{\alpha\beta}$ is anywhere locally identically zero in U. We then apply Desingularization II, (5.11), to the ideal $(\prod_{\alpha\beta} f_{\alpha\beta}\, g_{\alpha\beta}\, \mathcal{O}_X)|U$, with reference to the point x. As a result, if we replace U by a suitably smaller open neighborhood of x in it if necessary, then there exists a diagram of (7.2) having the properties 1),2),4),5) and the following

3') $(\prod_{\alpha\beta} f_{\alpha\beta}\, g_{\alpha\beta})\mathcal{O}_Z$ is simple everywhere.

Namely, for every point $z \in Z$, we can find an open neighborhood V of z in Z and an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ such that

$$(\prod_{\alpha\beta} f_{\alpha\beta}\, g_{\alpha\beta})\mathcal{O}_{\mathbb{R}^n} = (t^A)\mathcal{O}_{\mathbb{R}^n}$$

where $t = (t_1, \dots, t_n)$ is the usual coordinate system in $\mathbb{R}^n$ and $A \in \mathbb{Z}_{\circ}^n$ (i.e., t^A is a monomial in t). It follows that, viewing $f_{\alpha\beta}$ and $g_{\alpha\beta}$ as real-analytic functions on $\mathbb{R}^n$ by π, we can write

$$f_{\alpha\beta} = u_{\alpha\beta}\, t^{A_{\alpha\beta}}$$

and

$$g_{\alpha\beta} = v_{\alpha\beta}\, t^{B_{\alpha\beta}}$$

where $u_{\alpha\beta}$ and $v_{\alpha\beta}$ are all non-vanishing real-analytic functions on $\mathbb{R}^n$, and $A_{\alpha,\beta}$ and $B_{\alpha\beta}$ are elements of $\mathbb{Z}_{\circ}^n$. Note that $u_{\alpha\beta}$, $v_{\alpha\beta}$ are either

positive everywhere or negative everywhere.
Hence $\pi^{-1}(A)|V$ (as a subset in $\mathbb{R}^n$) is defined by means of equalities (=0) and inequalities (>0 or < 0) on monomials in t. It follows easily from this that $\pi^{-1}(A)|V$ is a union of quadrants in $\mathbb{R}^n$.

<u>Lemma</u> (7.2.2) Let the assumptions be the same as in (7.2.1). Let K be any compact subset of X (instead of a single point $x \in X$). Then there exist an open neighborhood U of K in X and a commutative diagram as in (7.2) having the properties 1), 2) and 3).

<u>Proof</u>. We apply (7.2.1) to each point $x \in K$. Since K is compact, we can find a finite number of open subsets U_α of X and diagrams

$$\begin{tikzcd} Z_\alpha \arrow[r, hook, "j_\alpha"'] \arrow[dr, "\pi_\alpha"'] & (X|U_\alpha) \times \mathbb{P}_{\mathbb{R}}^{N_\alpha} \arrow[d, "\text{projection}"] \\ & X|U_\alpha \end{tikzcd}$$

each of which satisfies the conditions 1)-4). We have a standard affine covering $\mathbb{P}_{\mathbb{R}}^{N_\alpha} = \bigcup_{p=0}^{N_\alpha} W_{\alpha p}$ with $W_{\alpha p} \cong \mathbb{R}^{N_\alpha}$. Now pick any point $z \in Z_\alpha$.
Let $y = \pi_\alpha(z)$ and $j_\alpha(z) = y \times w$. Say $w \in W_{\alpha p}$. Let $t = (t_1, \dots, t_{N_\alpha})$ be the coordinate system for $\mathbb{R}^{N_\alpha}$ having w as its origin by translation. Let us pick any imbedding, closed, $X|U'_\alpha \hookrightarrow \mathbb{R}^m | U''_\alpha$ where U'_α (resp. U''_α) is an open neighborhood of y (resp. 0) in U_α (resp. in $\mathbb{R}^m$) and y corresponds to 0. Let $u = (u_1, \dots u_m)$ be the coordinate system of $\mathbb{R}^m$. Let $\rho(u,t) = \sum_i u_i^2 + \sum_j t_j^2$, the square of the distance from the origin in $\mathbb{R}^m \times \mathbb{R}^{N_\alpha}$. We can choose $\varepsilon > 0$ so small that

a) $\{(u,t) | \rho(u,t) \leq \varepsilon\} \subset U''_\alpha \times W_{\alpha p}$

b) if $T = \{(u,t) | \rho(u,t) = \varepsilon\}$, then T is transversal to Z_α.

c) for a neighborhood V of z in Z_α

having the property 3) for $\pi_\alpha^{-1}(A)$, V contains

$$Z_\alpha \cap \{(u,t) \mid \rho(u,t) \leq \varepsilon\}$$

and T is transversal to all the coordinate linear subspaces of V with respect to the chosen $V \xrightarrow{\sim} \mathbb{R}^n$ of 3).

Now, with such $\varepsilon > 0$, let $Z'_{\alpha,z}$ be the closed real-analytic subspace of $Z_\alpha \times \mathbb{R}$ defined by $\rho(u,t) = \varepsilon$. Let $\lambda_{\alpha,z} : Z'_{\alpha,z} \longrightarrow Z_\alpha$ be the projection. Clearly $Z'_{\alpha,z}$ is smooth and compact.

Moreover, by a)–c), $\lambda_{\alpha z}^{-1}(\pi_\alpha^{-1}(A))$ is again locally a union of quadrants in $Z'_{\alpha z}$. It is easy to see that there exists a finite set Λ of pairs (α,z) as above, such that the images of $\pi_\alpha \lambda_{\alpha z}$ with $(\alpha,z) \in \Lambda$ cover some neighborhood of K. Let Z' be the disjoint union of those $Z'_{\alpha z}$ with $(\alpha,z) \in \Lambda$. Let $\pi' : Z' \to X$ be the map defined by $\pi_\alpha \lambda_{\alpha z}$ with $(\alpha,z) \in \Lambda$.

It is easy to find a commutative diagram

$$\begin{tikzcd} Z' \arrow[rr, hook, "j'"'] \arrow[dr, "\pi'"'] & & X \times \mathbb{P}^N_{\mathbb{R}} \arrow[dl, "\text{projection}"] \\ & X & \end{tikzcd}$$

with an imbedding j'. (For instance, firstly take the natural imbedding of Z' into the product of X and all the $\mathbb{P}^{N_\alpha}_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, one for each (α,z). Then imbedd the product of these $\mathbb{P}^{N_\alpha}_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ into some $\mathbb{P}^N_{\mathbb{R}}$). Now, by restricting the above diagram over a sufficiently small open neighborhood U of K in X, we get a diagram as in (7.3) having the properties 1)–3).

<u>Proof of</u> (7.2) We first apply Desingularization I, (5.10), to X with respect to the point x. So we get a commutative diagram

$$\begin{tikzcd} X_1 \arrow[rr, hook, "j_1"] \arrow[dr, "\pi_1"'] & & X|U \times \mathbb{P}_{\mathbb{R}}^{N_1} \arrow[dl, "\text{projection}"] \\ & X|U & \end{tikzcd}$$

where U is an open neighborhood of x in X, j_1 is a closed imbedding, X_1 is smooth and π_1 is surjective. Let $A_1 = \pi_1^{-1}(A)$, semi-analytic in X_1, and let $K = \pi_1^{-1}(x)$, which is compact. We then apply (7.2.2) to $A_1 \subset X_1$ and K. By composing the result of this application with the above diagram with π_1 and j_1, and then restricting it over a sufficiently small neighborhood of x in X, we get a diagram of (7.2). (Note that π_1 being clearly proper, every neighborhood of K contains the inverse image of some neighborhood of x, and that a product of two projective spaces has a natural imbedding into some projective space).

Remark (7.2.3) The above proof (with some additional care in making choices in each step especially the choice of $X'|U'_\alpha \hookrightarrow \mathbb{R}^{m}|U''_\alpha$ in the proof of (7.2.2) in reference to its application in the proof of (7.2)) actually proves that the diagram of (7.2) so obtained has the property (4) of (7.2.1) and also the following

6) There exists a closed nowhere dense real-analytic subspace D of Z such that every point of $Z-D$ admits an open neighborhood N in Z for which π induces a locally closed imbedding of $Z|N$ into X. (In other words, π induces an immersion of $Z-D$ into X).

We will not use this stronger version of (7.2) in what follows in this section and in the later, and the details of such modification in the proof of (7.2) are left to the readers.

The remainning part of the proof for (7.1), which was the first step earlier in this section, is the following

Proposition (7.3) Let A be a subanalytic subset of a real-analytic space X and let K be a compact subset of X. Then there exist a finite number of morphisms of real-analytic spaces

$$\{ \pi_\alpha : V_\alpha \longrightarrow X \}$$

such that

1) V_α is a smooth real-analytic space,

2) there exist compact subset K_α of V_α, one for each α, such that

$$\bigcup_\alpha \pi_\alpha(K_\alpha)$$

is a neighborhood of K in X,

3) π_α induces a locally closed imbedding of an open dense subset of V_α into X, and

4) $\pi_\alpha^{-1}(A)$ is semi-analytic in V_α for every α.

<u>Remark</u> (7.3.1) The question is easily localized because of the compactness of K. Therefore, to prove (7.3), we may assume that K is a single point x X.

<u>Remark</u> (7.3.2) When X is smooth, we shall prove (7.3) in such a way that 3) is replaced by the following stronger

3^*) π_α induces an open imbedding of an open dense subset of V_α into X for every α.

<u>Remark</u> (7.3.3) Granting (7.3.2), we can easily reduce the proof of (7.3) to the case of smooth X. In fact, we apply Desingularization I,(5.10), to X with reference to the point $x \in X$. (cf. (7.3.1)). Hence there exist an open neighborhood U of x in X and a proper morphism of real-analytic spaces $g : X' \longrightarrow X|U$, such that

a) g is surjective, and X' is smooth,

b) for every connected component X'_β of X', g induces a locally closed imbedding of an open dense subset of X'_β into X.

Then $g^{-1}(A)$ is subanalytic in X'. So, by (7.3.2), we get a finite number of

$$\{h_\alpha : V_\alpha \longrightarrow X'\}$$

having the properties 1), 2) for $K = g^{-1}(x)$, 3*) of (7.3.2), and 4). Then $\{\pi_\alpha = g h_\alpha\}$ have the properties 1)-4) of (7.3).

<u>Remark</u> (7.3.4) By (7.3.1)-(7.3.3), to prove (7.3), it is enough to prove (7.3) with 3*) under the additional assumption that X is smooth and K is a single point $x \in X$.

The proof of (7.3) will be completed after a few lemmas as follows.

<u>Lemma</u> (7.3.5) (fibre-cutting lemma).
Let Y be a compact real-analytic space and let $f : Y \longrightarrow X$ be a morphism of real-analytic spaces. Then we can find a compact real-analytic space Y' and a morphism of real-analytic spaces $f' : Y' \longrightarrow X$ such that

1) $\mathrm{Im}(f') = \mathrm{Im}(f)$ (point-set-theoretically)

2) for every point η of $\mathrm{Im}(f')$, there exists at least one connected component Y'_β of Y' such that f' induces a finite real-analytic morphism $Y'_\beta \longrightarrow X$ over some open neighborhood of η in X and that $\eta \in f(Y'_\beta)$.

<u>Proof</u>. The proof will be done by induction on the T-dimension of $f(Y)$. If $\text{T-dim}\, f(Y) = 0$, then $f(Y)$ is a finite number of points and then Y can be replaced by a finite number of points. Say $\text{T-dim}\, f(Y) = r$. For each point $y \in Y$, we can apply Desingularization I,(5.10), to Y with respect to y. Hence there exist an open neighborhood U_y of y in Y and a proper surjective real-analytic map $f_y : E_y \longrightarrow Y|U_y$ such that E_y is smooth everywhere and, in each connected component of E_y, say F, f_y induces a locally closed imbedding of an open dense subset of F. Now

if K_y is any compact subset of U_y, then we can find a finite number of closed balls $B_{y\alpha}$ in E_y (each ball is defined in terms of a local coordinate system in the smooth E_y) such that the union of the interiors of $B_{y\alpha}$ for all α contain $f_y^{-1}(K_y)$. For each $B_{y\alpha}$, we can find a real-analytic map $h_{y\alpha}$ from a sphere $S_{y\alpha}$ of the same dimension as $B_{y\alpha}$, onto $B_{y\alpha}$. (If B is the ball in $\mathbb{R}^n$ defined by $\sum_{i=1}^{n} t_i^2 \leq \mathcal{E}$ with the coordinate system $(t_1, \dots t_n)$ of $\mathbb{R}^n$, then we can choose a sphere S to be defined by $\sum_{i=1}^{n} t_i^2 + u^2 = \mathcal{E}$ in $\mathbb{R}^n \times \mathbb{R}$ and h to be the map induced by the projection $\mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$). Since Y is compact, we can easily find a finite number of pairs $(y\alpha)$ for which the union of the images of $h_{y\alpha}$ contains all the points of Y.
In other words, we have shown that

(*) Given a compact real-analytic space Y, there exist a finite number of real-analytic maps

$$\{ h_\alpha : S_\alpha \longrightarrow Y \}$$

such that, for each α,

a) S_α is a sphere of some dimension

b) there exists an open dense subset of S_α in which locally around every point h_α induces a locally closed imbedding, and

c) $\bigcup_\alpha \operatorname{Im}(h_\alpha) = |Y|$

Let $g_\alpha = fh_\alpha$. We know that T-dim $g_\alpha(S_\alpha)$ is equal to the maximum of the rank of (dg_α) at the points of S_α. Call this e_α. Let $d_\alpha = \dim S_\alpha$. We shall next prove that

(**) if $e_\alpha < d_\alpha$, there exists a closed real-analytic subspace Y_α of S_α such that T-dim $Y_\alpha < d_\alpha$ and $g_\alpha(Y_\alpha) = g_\alpha(S_\alpha)$.

In fact, since S_α is embedded in some real number space, it is easy to find a real-analytic function u on S_α such that if $G_\alpha : Y \times \mathbb{R}$ is the map defined by g_α and u, then rank (dG_α) attains its maximal value $e_\alpha + 1$ at some (at least one) point of S_α. Then we know that there exists a closed real-analytic subspace Y_α of S_α such that, for $s \in S_\alpha$, we have $s \in Y_\alpha$ if and only if rank $(dG_\alpha)_s \leq e_\alpha$. It is clear that T-dim $Y_\alpha < d_\alpha = \dim S_\alpha$. Moreover, for every $\xi \in g_\alpha(S_\alpha)$, $G_\alpha^{-1}(\xi \times \mathbb{R})$ is not empty and the map $G_\alpha^{-1}(\xi \times \mathbb{R}) \longrightarrow \xi \times \mathbb{R} \cong \mathbb{R}$ induced by G_α has at least one critical point because $G_\alpha^{-1}(\xi \times \mathbb{R})$ is compact and $\mathbb{R}$ is not. This implies that for every such ξ, $Y_\alpha \cap g_\alpha^{-1}(\xi) \neq \emptyset$. In other words, $g_\alpha(Y_\alpha) = g_\alpha(S_\alpha)$. Next we can apply (*) to each one of such Y_α, and then (**) to each of the resulting g_α provided $\dim S_\alpha >$ T-dim $g_\alpha(S_\alpha)$. Repeating such arguments a finite number of times, we can achieve the situation such that

(***) there exist a finite number of real-analytic maps

$$\{ g_\alpha : S_\alpha \longrightarrow Y \}$$

such that

a) S_α is a sphere and $\dim S_\alpha = \dim g_\alpha(S_\alpha)$.

b) $\bigcup g_\alpha(S_\alpha) = f(Y)$.

Now, to complete the proof of (7.3.5), we pick a closed real-analytic subspace T_α of S_α for each α such that for $s \in S_\alpha$, rank $(dg_\alpha)_s < \dim S_\alpha$ if and only if $s \in T_\alpha$. Then $\dim g_\alpha(T_\alpha) < \dim S_\alpha = \dim g_\alpha(S_\alpha)$. For the proof of (7.3.5), all that is left to do is to apply the induction assumption to the map from the disjoint union of all those T_α into X, induced by those g_α.

Next we make use of a theorem of Tarski-Seidenberg-Lojasiewicz, which asserts that if $f : Y \longrightarrow X$ is a relatively algebraic morphism of real-analytic spaces, i.e., there exists a diagram, commutative,

$$\begin{tikzcd} Y \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & X \times \mathbb{R}^N \arrow[dl, "\text{projection}"] \\ & X & \end{tikzcd}$$

such that the image of j is defined by polynomial equations ($=0$ or $\neq 0$) in terms of the coordinate system of $\mathbb{R}^N$, whose coefficients are elements of $\mathcal{O}_X(X)$, then $f(Y)$ is semi-analytic in X.

<u>Lemma</u> (7.3.6) If $f : Y \to X$ is a finite morphism of real-analytic spaces, then the image $f(Y)$ is semi-analytic.

<u>Proof</u>. A finite morphism is proper by definition, and hence $f(Y)$ is closed. It is therefore enough to prove the semi-analyticity of $f(Y)$ at each point of itself. If $x \in f(Y)$, then $f^{-1}(x)$ consists of a finite number of points, say $\eta_1, \ldots, \eta_s$. Let $B_i = \mathcal{O}_{Y,\eta_i}$ and $A = \mathcal{O}_{X,x}$. Then, by the finiteness of f, B_i is a finite A-module. Hence, by Weierstrass theorem, $B_i \cong A[t_1, \ldots, t_N]/I_i$ with an ideal I_i in the polynomial ring $A[t]$. This implies that there exist an open neighborhood U of x in X and an open neighborhood V_i of η_i in Y for which we have a commutative diagram:

$$\begin{tikzcd} Y|V_i \arrow[r, hook, "j"'] \arrow[dr, "f"'] & (X|U) \times \mathbb{R}^N \arrow[d, "\text{projection}"] \\ & X|U \end{tikzcd}$$

such that j is a closed imbedding and its image is defined by polynomial equations (corresponding to a finite system of generators of the ideal I_i). By the above cited theorem of T-S-L, $f(Y|V_i)$ is semi-analytic in X at x. Hence we have an open neighborhood $V = \bigcup_i V_i$ of $f^{-1}(x)$ in Y such that $f(Y|V)$ is semi-analytic in X at x. But $f(Y)$ and $f(Y|V)$ coincide in a sufficiently small neighborhood of x in X. This proves (7.3.6).

<u>Lemma</u> (7.3.7) Let $f : Y \longrightarrow X$ be a morphism of real-analytic spaces. Assume that

1) f is proper,

2) for every point $x \in f(Y)$, there exists at least one connected component Y_β of Y such that $x \in f(Y_\beta)$ and f induces a finite morphism

$$Y_\beta|f^{-1}(U) \longrightarrow X|U$$

for a sufficiently small open neighborhood U of x in X.
Under there assumptions, if E is any locally closed real-analytic subspace of X such that f induces a flat morphism

$$f^{-1}(E) \longrightarrow E$$

then there exists an open and closed real-analytic subspace Y_E of $f^{-1}(E)$ such that $f(Y_E) = f(Y) \cap E$ and f induces a finite and proper morphism:

$$Y_E \longrightarrow E$$

<u>Proof</u>. Pick any point $x \in f(Y) \cap E$.
Then we have Y_β having the property 2) for this x. Let $F_\beta = Y_\beta \cap f^{-1}(E)$. Then f induces a flat and proper morphism

$$g_\beta : F_\beta \longrightarrow E$$

such that $\mathrm{Im}(g_\beta) \ni x$ and g_β is finite over some neighborhood of x. Let V be the set of those points $y' \in F_\beta$ such that $\mathcal{O}_{F_\beta, y'}$ is finite as $\mathcal{O}_{E,x'}$ module where $x' = g_\beta(y')$. We claim that V is open and closed in F_β. In fact, the flatness of g_β implies that if the above finiteness is true at y' then it is so at every point within a sufficiently small neighborhood of y' in F_β. It also, implies that if the finiteness is not true at y' then it is not so at every point within a sufficiently small neighborhood of y' in F_β. The claim on V is now verified.
To prove (7.3.7), we simply let Y_E to be the union of $F_\beta|V$ obtained as above for all $x \in f(Y) \cap E$.

<u>Proof of</u> (7.3) Following (7.3.4), we shall assume that X is smooth and K is a single point $x \in X$. We shall prove (7.3) with 3^*) of (7.3.2). The question being local around the given point x in X, we may assume that there exist a finite number of proper real-analytic maps

$$f_{ij} : Y_{ij} \longrightarrow X, \ j=1,2 \ , \ 1 \leq i \leq m$$

such that

$$A = \bigcup_i \left(\operatorname{Im}(f_{i1}) - \operatorname{Im}(f_{i2})\right) .$$

We may further assume that each Y_{ij} is compact. In fact, if B is a closed ball centered at x in X (in terms of a local coordinate system of the smooth X at x) then there is a natural surjective real-analytic map $h:S \to B$, where S is a sphere of the same dimension as B. Then we replace Y_{ij} by the fibre product of Y_{ij} and S over X, and f_{ij} by the natural projection from the fibre product to X. By doing this, A does not change within a small neighborhood of x in X. Next we apply the fibre-cutting lemma (7.3.5) to each of the f_{ij}. By doing this, we may assume that every one of the f_{ij} satisfies the condition 1) and 2) of (7.3.7). Now let Y be the disjoint union of all the Y_{ij} and let $f : Y \to X$ be the morphism defined by the f_{ij}. Let us pick a complexification $\tilde{f} : \tilde{Y} \to \tilde{X}$ of this map f. Let $L = f^{-1}(x)$ which is a compact subset of $\tilde{f}^{-1}(x)$. We then apply the local flattening theorem to f and L in the form stated in (4.17) and (4.18).
Let $\tilde{S}_\alpha$ and $\tilde{\pi}_\alpha : \tilde{V}_\alpha \longrightarrow \tilde{X}$ be those sequences and morphisms obtained by the local flattening. By (4.17), we have auto conjugations of $\tilde{V}_\alpha$ and those complex spaces in $\tilde{S}_\alpha$ which are compatible with all the morphisms involved there. So $\tilde{\pi}_\alpha$ induces a morphism of the real parts $\pi_\alpha : V_\alpha \to X$. Since X is smooth and the centers of blowing-ups in $\tilde{S}_\alpha$ are all nowhere dense, if F_α is the union of the inverse images in V_α of the centers in

the sequence $\tilde{S}_\alpha$, then π_α induces an open imbedding of $V_\alpha - F_\alpha$ into X and $\pi_\alpha(F_\alpha)$ is nowhere dense in X. From this, we can easily deduced

[1] there exists a compact subset K_α of V_α such that the union of the closures of $\pi_\alpha(K_\alpha \cap V_\alpha - F_\alpha)$ for all α is a neighborhood of the given point x in X.

We shall next prove that

[2] $\pi_\alpha^{-1}(A)$ is semi-analytic in V_α.

In fact, we shall prove that every one of the $\pi_\alpha^{-1}(\mathrm{Im}(f_{ij}))$ is semi-analytic in V . First we look at

$$\pi_\alpha^{-1}(\mathrm{Im}(f_{ij})) \cap F_\alpha$$

To prove that this is semi-analytic, we go back to examine the way in which π_α is obtained. Let

$$\tilde{S}_\alpha = \{(\tilde{U}_{\alpha\lambda}, \tilde{E}_{\alpha\lambda}, \tilde{\pi}_{\alpha\lambda})\}_{0\le\lambda\le m}$$

with $\tilde{\pi}_{\alpha\lambda} : \tilde{V}_{\alpha\lambda+1} \longrightarrow \tilde{V}_{\alpha\lambda}$. Let $\tilde{f}_{ij\alpha\lambda} : \tilde{Y}_{ij\alpha\lambda} \to \tilde{V}_{\alpha\lambda}$ be the successive strict transforms of a complexification $\tilde{f}_{ij} : \tilde{Y}_{ij} \longrightarrow \tilde{X}$ of $f_{ij} : Y_{ij} \to X$, along the sequence $\tilde{S}_\alpha$. By (4.17), the auto-conjugation of $\tilde{X}$ extends to the autoconjugations of $\tilde{S}_\alpha$ and $\tilde{f}_{ij\alpha\lambda}$ for all i, j, α, λ. So we have their real parts, $E_{\alpha\lambda}$, $\pi_{\alpha\lambda} : V_{\alpha\lambda+1} \to V_{\alpha\lambda}$ and $f_{ij\alpha\lambda} : Y_{ij\alpha\lambda} \to V_{\alpha\lambda}$ of $\tilde{E}_{\alpha\lambda}$, $\tilde{\pi}_{\alpha\lambda}$ and $\tilde{f}_{ij\alpha\lambda}$ respectively. Let $F_\alpha^{(\lambda)}$ be the union of the inverse images of $E_\alpha^{(\mu)}$ in $V_{\alpha\lambda}$ for all $\mu < \lambda$.

Let $\pi_\alpha^{(\lambda)} = \pi_{\alpha 0} \cdots \pi_{\alpha\lambda-1}$. We shall prove, by induction on λ , that

$$(\pi_\alpha^{(\lambda)})^{-1}(\mathrm{Im}(f_{ij})) \cap F_\alpha^{(\lambda)}$$

is semi-analytic in $F_\alpha^{(\lambda)}$ and hence so in $V_{\alpha\lambda}$. (Note that $F_\alpha^{(\lambda)}$ is a closed real-analytic subset of $V_{\alpha\lambda}$). For $\lambda = 0$, $\pi_\alpha^{(\lambda)} = \mathrm{id}$ and $F_\alpha^{(\lambda)} =$ $= E_{\alpha 0}$. By (4.18), $\tilde{f}$ induces a flat morphism when induced over the center

$\tilde{E}_{\alpha 0}$. Hence f does the same when induced over $E_{\alpha 0}$. This implies that, for every (i,j), f_{ij} induces a flat morphism over $E_{\alpha 0}$. Hence, by (7.3.7),

$$\mathrm{Im}(f_{ij}) \cap E_{\alpha 0}$$

is the image of some finite morphism. Hence by (7.3.6), it is semi-analytic in $E_{\alpha 0}$ and hence so in $V_{\alpha 0}(=X)$. This proves the claim for $\lambda = 0$. Now assume $\lambda > 0$ and that the claim is verified for all smaller λ. Since the inverse image of semi-analytic subset by a real-analytic map is again semi-analytic, we have that

$$\left(\pi_{\alpha}^{(\lambda)}\right)^{-1} \mathrm{Im}(f_{ij}) \cap \left(\pi_{\lambda-1}\right)^{-1}\left(F^{(\lambda-1)}\right)$$

is semi-analytic in $V_{\alpha\lambda}$. But, outside $(\pi_{\alpha\lambda-1})^{-1}(F_{\alpha}^{(\lambda-1)})$, $\pi_{\alpha}^{(\lambda)}$ is an open imbedding and the $f_{ij\alpha\lambda}$ (obtained by the strict transform) coincides with the fibre product base extension.
So $\mathrm{Im}(f_{ij\alpha\lambda})$ coincides with $(\pi_{\alpha}^{(\lambda)})^{-1}(\mathrm{Im}(f_{ij}))$ outside $(\pi_{\alpha\lambda-1})^{-1}(F_{\alpha}^{(\lambda-1)})$.
It is easy to see that $f_{ij\alpha\lambda}$ has also the properties 1) and 2) of (7.3.7). Therefore, again by (7.3.7) and (7.3.6), $\mathrm{Im}(f_{ij\alpha\lambda}) \cap E_{\alpha\lambda}$ is semi-analytic. Thus

$$\begin{aligned}
&(\pi_{\alpha}^{(\lambda)})^{-1}(\mathrm{Im}(f_{ij})) \cap F_{\alpha}^{(\lambda)} \\
&= \left[(\pi_{\alpha}^{(\lambda)})^{-1}(\mathrm{Im}(f_{ij})) \cap (\pi_{\alpha\lambda-1})^{-1}(F_{\alpha}^{(\lambda-1)})\right] \\
&\cup \left[\mathrm{Im}(f_{ij\alpha\lambda}) \cap E_{\alpha\lambda}\right]
\end{aligned}$$

is semi-analytic. By taking $\lambda = m$, we conclude that

$$\pi_{\alpha}^{-1}(\mathrm{Im}(f_{ij})) \cap F_{\alpha}$$

is semi-analytic. According to the flattening theorem (4.4), the final strict transform of $\tilde{f}$ along $\tilde{S}_{\alpha}$ is flat. This implies that $f_{ij\alpha m}$ is flat for every (i,j). As is seen in the same way as above,

$$\pi_\alpha^{-1}(\mathrm{Im}(f_{ij}))$$
$$= \left[\pi_\alpha^{-1}(\mathrm{Im}(f_{ij}))\cap F_\alpha\right] \cup \mathrm{Im}(f_{ij\alpha m})$$

and $\mathrm{Im}(f_{ij\alpha m})$ is semi-analytic by (7.3.7) and (7.3.6). This shows that $\pi_\alpha^{-1}(\mathrm{Im}(f_{ij}))$ is semi-analytic for all i,j. This proves [2]. Namely the family $\{\pi_\alpha : V_\alpha \longrightarrow X\}$, deduced from the local flattening theorem as above, has the property 4) of (7.3). Note that this property stays to hold if we replace each π_α by $\pi_\alpha q_{\alpha\mu}$ with any real-analytic maps $q_{\alpha\mu} : V_{\alpha\mu} \rightarrow V_\alpha$. As a matter of fact, we apply Desingularization I,(5.10), to each V_α with respect to points of the compact subset K_α chosen in [1]. Namely, we have a finite system of real-analytic maps

$$\{ q_{\alpha\mu} : V_{\alpha\eta} \longrightarrow V_\alpha \}$$

for each α, such that $V_{\alpha\mu}$ are all smooth and connected, $q_{\alpha\mu}$ induces a locally closed imbedding in an open dense subset of $V_{\alpha\mu}$ and there exists a compact subset $K_{\alpha\mu}$, one for each μ, for which

$$\bigcup_\mu q_{\alpha\mu}(K_{\alpha\mu})$$

is a neighborhood of K_α in V_α. Then

$$\bigcup_{\alpha\mu} \pi_\alpha q_{\alpha\mu}(K_{\alpha\mu})$$

is a neighborhood of x in X, by [1]. Since X is smooth, the same stays to be true if we omit those $\pi_\alpha q_{\alpha\mu}$: which do not induce open imbedding of an open dense subsets of $V_{\alpha\mu}$. If we replace $\{\pi_\alpha\}$ by those $\{\pi_{\alpha\mu}\}$ that remain after the omitting, then 1),2),4) of (7.3) and 3*) of (7.3.2) are all true. This completes the proof of (7.3).

<u>Proof of rectilinearization theorem</u> (7.1).

We first apply (7.3) to the X, A and K of (7.1). Then, for each $\pi_\alpha : V_\alpha \longrightarrow X$ of (7.3), we apply (7.2.2) to V_α, $\pi_\alpha^{-1}(A)$ and K_α (K_α being chosen as in 2) of (7.3)). The composed maps of the maps in these two steps, first (7.3) and then (7.2.2), give rise to the sought-for maps π_α of (7.1), when we take a suitable family of open subsets and isomorphisms of there with real number spaces, in each of the sources of the composed maps.

Thus the rectilinearization theorem of subanalytic subsets is now established.

§ 8. Some basic properties of subanalytic sets.

We shall first deduce from the rectilinearization theorem, § 7, basic properties of subanalyticity in regards to closure, interior and connectedness.

Proposition (8.1). Let X be a real-analytic space which is Hausdorff and countable at infinity. Let A be a subanalytic set in X. Then there exists a real-analytic map $f : Y \to X$ such that

1) f is proper and $\mathrm{Im}(f)$ is the closure $\bar{A}$ of A in X,

2) Y is a disjoint union of compact real-analytic spaces,

3) for every point $\xi \in \bar{A}$, there exists at least one connected component Y_α of Y such that $\xi \in f(Y_\alpha)$ and $f|Y_\alpha$ induces a finite real-analytic morphism over some neighborhood of ξ in X.

4) Y is smooth; in fact, every connected component of Y is real-analytically isomorphic to a sphere.

Proof. Since X is Hausdorff and countable at infinity, we can find a pair of open covering $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{U}' = \{U'_i\}_{i \in I}$ such that

a) for each i, $\overline{U_i}$ is compact and contained in U'_i, and

b) $\mathcal{U}'$ (and hence $\mathcal{U}$) is locally finite in X.

Pick one index i. We shall first prove that

c) there exist a compact real-analytic space Y_i and a real-analytic map $f_i : Y_i \to X$ such that $\mathrm{Im}(f_i) \subset \bar{A} \cap U'_i$ and $\mathrm{Im}(f_i) \cap U_i = \bar{A} \cap U_i$.

In fact, we apply the rectilinearization theorem, § 7, to A at each point of $\overline{U_i}$. Since $\overline{U_i}$ is compact and contained in the open subset U'_i, we can find a finite system of real-analytic maps

$$\{\pi_{i\alpha} : V_{i\alpha} \longrightarrow X\}_\alpha$$

such that

(i) $V_{i\alpha} \cong \mathbb{R}^{n_{i\alpha}}$

(ii) there is given a compact subset $K_{i\alpha}$ in each $V_{i\alpha}$ such that

$$\overline{U_i} \subset \bigcup_\alpha \pi_{i\alpha}(K_{i\alpha}) \subset U'_i$$

(iii) $\pi_{i\alpha}^{-1}(A)$ is a finite union of quadrants in $\mathbb{R}^{n_{i\alpha}}$ for every α.

Let $U'_{i\alpha} = \pi_{i\alpha}^{-1}(U'_i)$, which contains $K_{i\alpha}$. For each point $y \in K_{i\alpha}$, we can find a closed ball B_y centered at y in $\mathbb{R}^{n_{i\alpha}}$ such that $B_y \subset U'_{i\alpha}$ and, within some neighborhood of B_y in $\mathbb{R}^{n_{i\alpha}}$, $\pi_{i\alpha}^{-1}(A)$ coincides with a union of quadrants passing through y. It is then easy to find a real-analytic map $g_y : Y_y \to V_{i\alpha}$ such that Y_y is a finite union of disjoint spheres and $\mathrm{Im}(g_y) = B_y \cap \pi_\alpha^{-1}(A)$, since $K_{i\alpha}$ is compact, we can find a finite subset $\Delta_{i\alpha}$ of $K_{i\alpha}$ such that

$$\bigcup_{y \in \Delta_{i\alpha}} B_y \supset K_{i\alpha}$$

Now let Y_i be the disjoint union of Y_y for all α and $y \in \Delta_{i\alpha}$, and let $f_i : Y_i \to X$ be the map defined by the collection of those $\pi_\alpha g_y$, $y \in \Delta_{i\alpha}$. Then clearly the condition c) is satisfied by this f_i.

Let $f' : Y' \to X$ be the map derived from the f_i for all i, by taking Y' to be the disjoint union of the Y_i. This f' has the properties 1) and 2) of (8.1). Then, by applying the fibre-cutting lemma (7.3.5) to each connected component of Y', we can derive from f' a real-analytic map $f'' : Y'' \to X$ having the properties 1),2) and 3) of (8.1).

To realize the final condition 4), we apply the desingularization theorem to Y''. We must do this, however, without losing any of the conditions 1), 2) and 3). Let us pick a smooth real-analytic filtration $\{Y^{(i)}\}_{i\geq 0}$ of Y'', where $|Y^{(0)}| = |Y''|$. This exists by (5.8). By the desingularization theorem (5.10), for each point $y \in Y''$ we can find and fix an open neighborhood U_y of y in Y'' and a proper surjective real-analytic map $p_y : Z_y \to Y''|U_y$ such that

d) Z_y is a disjoint union of smooth real-analytic spaces $Z_y^{(i)}$,

e) $Z_y^{(i)} \cap p_y^{-1}(Y^{(i+1)})$ is nowhere dense in $Z_y^{(i)}$, and

f) p_y induces an isomorphism

$$Z_y^{(i)} - p_y^{-1}(Y^{(i+1)}) \xrightarrow{\sim} Y^{(i)}|U_y - Y^{(i+1)}$$

for every i.

Since Y'' is Hausdorff and countable at infinity, we can find a pair of open coverings of Y'', say $W = \{W_\lambda\}_{\lambda\in\Lambda}$ and $W' = \{W'_\lambda\}_{\lambda\in\Lambda}$ such that

g) for each $\lambda\in\Lambda$, W_λ is relatively compact in $W'_\lambda (W_\lambda \subset W'_\lambda)$,

h) W' (and hence W) is locally finite in Y'', and

j) for every $\lambda\in\Lambda$, there exists $y\in Y''$ such that W'_λ is contained in the open subset U_y of y chosen above.

For each $\lambda\in\Lambda$, pick one y as in j) and let $p_\lambda : Z_\lambda \to Y''|W_\lambda$ be the restriction of p_y to $p_y^{-1}(W'_\lambda)$. Let $Z_\lambda^{(i)} = Z_y^{(i)}|p_y^{-1}(W'_\lambda)$. Since Z_λ is smooth and $p_\lambda^{-1}(W_\lambda)$ is relatively compact in Z_λ, there exists

k) a finite (everywhere) morphism

$$q_\lambda : T_\lambda \longrightarrow Z_\lambda$$

for each $\lambda\in\Lambda$, where $\mathrm{Im}(q_\lambda) \supset p_\lambda^{-1}(W_\lambda)$ and T_λ is compact smooth; in fact T_λ is a finite union of disjoint spheres.

Let us now define Y to be the disjoint union of the T_λ, $\lambda \in \Lambda$, and let $f : Y \longrightarrow X$ be the map defined by $\{f'' p_\lambda q_\lambda\}$ for $\lambda \in \Lambda$. We then claim that this f has all the four properties of (8.1).

First of all, the condition 1), 2) and 4) are clearly satisfied. To verify 3), take any point $\xi \in \bar{A}$. By the property 3) for f'', there exists a connected component C of Y'' such that $\xi \in f''(C)$ and f'' induces a finite morphism $C \longrightarrow X$ over some neighborhood of ξ in X. Take the index i such that $f''^{-1}(\xi) \cap C \cap Y^{(i)} \neq \emptyset$ and $f''^{-1}(\xi) \cap C \cap Y^{(i+1)} = \emptyset$. Such i exists because $f''^{-1}(\xi)$ is compact and $\{Y^{(i)}\}$ is locally finite in Y''. Pick λ such that $f''^{-1}(\xi) \cap C \cap Y^{(i)} \cap W_\lambda \neq \emptyset$. Let $y \in Y''$ be chosen as above for this index λ. Since C is open and closed in Y'', so is $C \cap Y^{(i)} \cap W'_\lambda$ in $Y^{(i)} \cap W'_\lambda$. Let $C_\lambda^{(i)} = Z_\lambda^{(i)} | p_\lambda^{-1}(C \cap Y^{(i)} \cap W'_\lambda)$, which is therefore open and closed in $Z_\lambda^{(i)}$. Moreover, by f), p_λ induces an isomorphism

$$C_\lambda^{(i)} - p_\lambda^{-1}(Y^{(i+1)}) \xrightarrow{\sim} C \cap Y^{(i)} | W'_\lambda - Y^{(i+1)}$$

By d), $Z_\lambda^{(i)}$ is closed open in Z_λ and hence so is $C_\lambda^{(i)}$ in Z_λ. Let $S = T_\lambda | q_\lambda^{-1}(C_\lambda^{(i)})$, which is therefore closed and open in T_λ and hence so in Y. By k), $p_\lambda q_\lambda$ induces a morphism

$$S - (p_\lambda q_\lambda)^{-1}(Y^{(i+1)}) \longrightarrow C | W'_\lambda - Y^{(i+1)}$$

which is finite everywhere. Clearly, $p_\lambda q_\lambda$ induces a morphism

$$r : S \longrightarrow C$$

whose image contains $W_\lambda \cap C \cap Y^{(i)}$ and is contained in $W'_\lambda \cap C$ by the property k) of q_λ and by the definition of p_λ. Thus r is finite everywhere on $C - Y^{(i+1)}$. By our selection of the index i,

$$\mathrm{Im}(r) \cap f''^{-1}(\xi) \neq \emptyset$$

and

$$\mathrm{Im}(r) \cap f''^{-1}(\xi) \cap Y^{(i+1)} = \emptyset .$$

Therefore, as $f'': C \to X$ is finite over a neighborhood of ξ in X, $f''r : S \to X$ is also finite over some neighborhood of ξ in X. As $f|S = f''r$, we can find a connected component S_o of S, which is also such of Y, such that $S_o \cap f^{-1}(\xi) \neq \emptyset$ and $S_o \to X$ by f is finite over some neighborhood of ξ in X. Namely f has the property 3). This completes the proof of (8.1).

<u>Corollary</u> (8.1.1) If A is subanalytic in a real-analytic space X, then its closure $\bar{A}$ in X is also subanalytic. Moreover, the interior of A in X is also subanalytic.

<u>Proof</u>. The question being local in X, we may assume that X is Hausdorff and countable at infinity. Then the existence of f with the property 1) of (8.1) implies that $\bar{A}$ is subanalytic in X. The interior assertion follows immediately from the closure assertion by taking the complementary set of A in X.

<u>Proposition</u> (8.2) A subanalytic set in a real-analytic space X has a locally finite connectedness, i.e., every point $x \in X$ has a neighborhood which meets only a finite number of connected components of A. Moreover, each connected component of A is also subanalytic in X.

<u>Proof</u>. Apply the rectilinearization theorem to A around x and get a finite system of real-analytic maps $\pi_\alpha : V_\alpha \to X$ such that

(i) $V_\alpha \cong \mathbb{R}^{n_\alpha}$;

(ii) there exists a compact subset K_α in each V_α such that $\bigcup_\alpha \pi_\alpha(K_\alpha)$ is a neighborhood of x in X, and

(iii) for every α, $\pi_\alpha^{-1}(A)$ is a finite union of quadrants in $\mathbb{R}^{n_\alpha}$.

Now, each $\pi_\alpha^{-1}(A)$ has only a finite number of connected components by (iii). Hence the same is true for

$$\bigcup_\alpha \pi_\alpha(\pi_\alpha^{-1}(A))$$

which is contained in and, locally around x, coincides with A by (ii). This shows the locally finite connectedness of A in X around x. Next, to prove the second assertion of (8.2). Let A_1 be any connected component of A. Then, for each α, $\pi_\alpha^{-1}(A_1)$ is open and closed in $\pi_\alpha^{-1}(A)$. Hence, by (iii), $\pi_\alpha^{-1}(A_1)$ is also a finite union of quadrants in $\mathbb{R}^{n_\alpha}$. In the condition (ii), we may replace K_α by any compact subset of V_α containing K_α ; in particular, by a closed ball in $\mathbb{R}^{n_\alpha}$ containing K_α. So we assume that K_α is a standard closed ball in $\mathbb{R}^{n_\alpha}$. It is then easy to find a compact real-analytic space Y_α and a real-analytic map $g_\alpha : Y_\alpha \longrightarrow V_\alpha$ such that

$$\mathrm{Im}(g_\alpha) = K_\alpha \cap \overline{\pi_\alpha^{-1}(A_1)}$$

(This is also a special case of (8.1)). Let

$$A_1' = \bigcup_\alpha \pi_\alpha(K_\alpha \cap \overline{\pi_\alpha^{-1}(A_1)})$$

Then A_1' is contained in $\overline{A_1}$ and coincides with $\overline{A_1}$ within some neighborhood of x in X. If Y is the disjoint union of Y_α for all α and $f: Y \longrightarrow X$ is the map defined by $\pi_\alpha g_\alpha$ for all α, then $A_1' = \mathrm{Im}(f)$. Hence A_1' is subanalytic in X. It follows, by taking various $x \in X$, that $\overline{A_1}$ is subanalytic in X. Hence $A_1 = \overline{A_1} \cap A$ is also subanalytic in X.

Remark (8.3) A new proof, in the same type of reasoning as above, can be given also to the corresponding "semi- analytic" theorems of Lojasiewicz: The semi-analyticity of the closure and each connected component of a semi-analytic subset of a real- analytic space. In this case, we must use a result of the form (7.2) instead of the rectilinearization (7.1), noting that the imbedding j of (7.2 is obtained by a system of rational functions on $\mathbb{P}_{\mathbb{R}}^N$ whose coefficients are real-analytic in $X|U$. We must also use the generalization by Lojasiewicz (whose proof is no more difficult than the original) of the theorem of Tarski-Seidenberg:

The image of semi-analytic set, relatively semi-algebraic, is semi-algebraic.

<u>Proposition</u> (8.4) Let A be a sub-analytic set in a real-analytic space X. Let x be a point of the closure $\bar{A}$ in X.
Then there exists an open neighborhood U of x in X such that

(8.4.1) for every $y \in U \cap A$, there exists a real-analytic map $\varphi : (-1,1) \to X|U$ such that $\varphi(0) = x$ and $y \in \varphi((-1,1) - (0)) \subset A \cap U$.

<u>Remark</u> (8.4.2) The question being local, we may assume that X is imbedded in $\mathbb{R}^n$ as a locally closed real-analytic subspace with the point x at the origin. Let $(z_1, \dots, z_n)$ be the standard coordinate system for $\mathbb{R}^n$, and let $\rho(z) = \sum_{i=1}^{n} z_i^2$. We shall then find $\varepsilon_0 > 0$ such that for every ε, $0 < \varepsilon \leq \varepsilon_0$,

$$U = U_\varepsilon = \{ z \in X \,|\, \rho(z) < \varepsilon \}$$

has the property (8.4.1).

<u>Proof</u>. We shall prove (8.4.2) which implies (8.4). Let $p : Y \to \mathbb{R}^n$ be the quadratic transformation with center at the origin. We then apply the rectilinearization theorem to $p^{-1}(A)$ (which is clearly sub-analytic in Y) at every point of $p^{-1}(x)$. Composing p with the maps resulting from this application, we obtain a finite system of real-analytic maps $\pi_\alpha : V_\alpha \to \mathbb{R}^n$ such that

1) $V_\alpha \cong \mathbb{R}^{n_\alpha}$ for every α,

2) there exists a compact subset K_α in each V_α, such that $\bigcup_\alpha \pi_\alpha(K_\alpha)$ is a neighborhood of the origin x in $\mathbb{R}^n$.

3) $\pi_\alpha^{-1}(A)$ is a finite union of quadrants in $\mathbb{R}^{n_\alpha}$, and

4) for every α,

$$(z_1, \dots, z_n)\mathcal{O}_{V_\alpha} = (t_\alpha^{E_\alpha})\mathcal{O}_{V_\alpha}.$$

where $t_\alpha = (t_{\alpha 1},\dots,t_{\alpha n_\alpha})$ is the coordinate system for $\mathbb{R}^{n_\alpha}$ and $E_\alpha \in \mathbb{Z}^{n_\alpha}$.

Let $\rho_\alpha = \rho\,\pi_\alpha$. It follows from 4) that

5) $\rho_\alpha = v_\alpha(t_\alpha) M_\alpha(t_\alpha)^2$

where v_α is an everywhere positive real-analytic function on V_α and M_α is a monomial.

Clearly we may omit those α with $\pi_\alpha^{-1}(x) = \emptyset$, so that $E_\alpha \neq (0)$, $\pi_\alpha(0) = x$ and $\deg M_\alpha > 0$ for all α. We shall then prove:

(8.4.3) For each α and for each $\tau \in \pi_\alpha^{-1}(x)$, there exist $\delta(\alpha,\tau) > 0$ and $\varepsilon(\alpha,\tau) > 0$ such that:

(*) if $0 < \delta \leq \delta(\alpha,\tau)$, if $0 < \varepsilon \leq \varepsilon(\alpha,\tau)$, and if

$$W_{\alpha,\tau}(\delta,\varepsilon) = \left\{ t \in V_\alpha \,\middle|\, \rho_\alpha(t) < \varepsilon,\ h_\alpha(t-\tau) < \delta \right\}$$

where $h_\alpha(t) = \sum_i t_i^2$, then $W_{\alpha,\tau}(\delta,\varepsilon)$ contains every linear segment in $\mathbb{R}^{n_\alpha}$ which starts at τ and ends at any given point of $W_{\alpha,\tau}(\delta,\varepsilon)$.

Assuming (8.4.3) (which will be restated and proven below as Lemma (8.4.4)), let us deduce (8.4.2). First of all, it is clear that $\delta(\alpha,\tau)$ may be replaced by any smaller positive number. So we may assume

(a) if $\tau \neq (0)$, then

$$\delta(\alpha,\tau) < \min\left\{ \tau_i^2 \,\middle|\, \tau_i \neq 0 \right\}$$

where $\tau = (\tau_1,\dots,\tau_{n_\alpha})$.

This (a) implies that, by 3), $\pi_\alpha^{-1}(A)$ is a finite union of quadrants, with respect to the coordinate system $t_\alpha - \tau$ (i.e., the origin being translate to τ) in $\mathbb{R}^{n_\alpha}$ within any one of the $W_{\alpha,\tau}(\delta,\varepsilon)$ of (*). It follows that

(b) for every $y' \in \pi_\alpha^{-1}(A) \cap W_{\alpha,\tau}(\delta,\varepsilon)$ with (δ,ε) of (*), we have a real- analytic map $\psi_{y'}:(-1,1) \to V_\alpha$ such that $\psi_{y'}(0)=\tau$ and

$$y' \in \psi_{y'}((-1,1)-(0)) \subset \pi_\alpha^{-1}(A) \cap W_{\alpha,\tau}(\delta,\varepsilon)$$

(In fact, with a sufficiently small constant $\lambda > 0$, we let $\psi_{y'}(u) = (1+\lambda)^2 u^2(y'-\tau) + \tau$).

Now, since K_α is compact, we can find a finite number of points $\{\tau(j)\}$, $\tau(j) \in \pi^{-1}(x)$, such that $K_\alpha \cap \pi_\alpha^{-1}(x)$ is contained in

$$W_\alpha = \bigcup_j W_{\alpha,\tau(j)}(\delta(\alpha,\tau(j)), \varepsilon(\alpha,\tau(j)))$$

Then choose $\varepsilon_0 > 0$ such that

$$\varepsilon_\alpha \leqslant \min_j \{\varepsilon(\alpha,\tau(j))\} \text{ and}$$

$$W_\alpha \supset K_\alpha \cap \{y \in V_\alpha \mid \rho_\alpha(y) < \varepsilon_\alpha\}$$

Let $\varepsilon_0 = \min_\alpha \{\varepsilon_\alpha\}$. We assert that this ε_0 has the required property of (8.4.2). In fact, for any ε and $U = U_\varepsilon$ as is given in (8.4.2), pick any point $y \in U \cap A$. Then we can find α, j and $y' \in W_{\alpha,\tau(j)}(\delta(\alpha,\tau(j)), \varepsilon) \cap \pi_\alpha^{-1}(A)$ such that $y=\pi_\alpha(y')$. Choose $\psi_{y'}$ as in (b) and let $\varphi = \pi_\alpha \psi_{y'}$. Then, clearly, this φ has the property (8.4.1).

Now we have only to prove (8.4.3) which we restate as follows.

<u>Lemma</u> (8.4.4) Let $(t_1,\ldots,t_m)$ be the coordinate system for $\mathbb{R}^m$. Let $M(t)$ be a non-constant monomial and let $v(t)$ be a positive real-analytic function in an open subset N of $\mathbb{R}^m$. Then for every point $\tau \in N$ such that $\rho(\tau)=0$, where $\rho = vM^2$, there exists $\delta_\tau > 0$ such that if $h(t) = \sum_{i=1}^m t_i^2$ and

$$W_\tau(\delta,\varepsilon) = \{t \in N \mid \rho(t) < \varepsilon,\ h(t-\tau) < \delta\}$$

where $0 < \delta \leq \delta_\tau$ and $0 < \varepsilon$, then $W_\tau(\delta,\varepsilon) \cap L$ is connected (an open segment) for every line L passing through τ in $\mathbb{R}^m$.

<u>Proof</u>. Let us write $M = M_1 M_2$, product of two monomials such that $M_1(\tau) \neq 0$ and M_2 is a monomial of only those t_i with $t_i(\tau)=0$. Let $v_2 = vM_1^2$ and let $t' = t-\tau$. Then v_2 is positive in an open neighborhood N' of τ in N. Therefore, by replacing t, M, v, N by t', M_2, v_2, N' respectively, we may assume, from the beginning, that τ is the origin of $\mathbb{R}^m$. Now, let $v_i = \partial v/\partial t_i$ and pick $\delta_\tau > 0$ so small that for all t with $h(t) < \delta_\tau$ we have $t \in N$ and, with $p = \deg M$,

$$h(t) \sum_{i=1}^{m} v_i(t)^2 < (2pv(t))^2$$

Pick any line through the origin in $\mathbb{R}^m$:

$$L = \{au \mid u \in \mathbb{R}\}$$

where a is a unit vector in $\mathbb{R}^m$. Pick any δ, $0 < \delta \leq \delta_\tau$, and any $\varepsilon > 0$. We have

$$L \cap \{t \in \mathbb{R}^m \mid h(t) < \delta\} = \{au \mid u \in (-\sqrt{\delta}, \sqrt{\delta})\} .$$

So, to prove (8.4.4), it is enough to prove that $\rho(au)$ is monotone increasing (resp. decreasing) for $u \in (0, \sqrt{\delta})$ (resp. for $u \in (-\sqrt{\delta}, 0)$), provided $M(a) \neq 0$. In fact, if $M(a) = 0$, then $\rho(t)$ is identically zero on $L \cap N$ and hence

$$W_\tau(\delta,\varepsilon) \cap L = \{au \mid u \in (-\sqrt{\delta}, \sqrt{\delta})\}$$

Now, assume $M(a) \neq 0$. Then

$$\frac{d\rho(au)}{du} = M(a)^2 u^{2p-1} \left\{ 2pv(au) + u \sum_{i=1}^{m} a_i v_i(au) \right\}$$

But for u with $h(au) = u^2 < \delta \leq \delta_\tau$,

$$2pv(au) + u \sum_{i=1}^{m} a_i v_i(au)$$

$$\geqslant \left|2p\, v(au)\right| - \sqrt{h(au)}\, \sqrt{\sum_{i=1}^{m} v_i(au)^2} > 0$$

where the last inequality is by the selection of δ_τ. Therefore,

$$\frac{d\rho(au)}{du} \begin{cases} > 0 & \text{if } u \in (0, \sqrt{\delta}) \\ < 0 & \text{if } u \in (-\sqrt{\delta}, 0) \end{cases}$$

This completes the proof of (8.4.4). Note that the proof of (8.4) is also completed.

§. 9. Lojasiewicz' inequalities.

In his work on the division of distributions by analytic functions, and in his subsequent works on semi-analytic sets, Lojasiewicz proved various fundamental inequalities among values of real-analytic functions and distances from semi-analytic sets related to them. These are the inequalities which are deeply characteristic of real-(or complex-) analyticity. The primary purpose of this section is somewhat more systematically to reproduce his inequalities. Our proofs, given here, are essentially different from his and are based on the techniques of blowing-ups, in contrast to his use of linear projections and discriminants of Weierstrass polynomials. Our proofs are based upon the theorems of desingularization by blowing-ups (which are not entirely elementary in themselves) but, after granting these, the inequalities become much geometric in nature and conceptually clearer. Our proofs give more general theorems, mainly thank to our theorems of local flattening and rectilinearization theorem. Namely, the inequalities will be proven for subanalytic sets to which, due to the lack of analytic equations and analytic inequalities that define such sets, the technique of linear projections and discriminants is not directly applicable as in the case of semi-analytic sets.

Inequality I. Let f be a real-analytic function on an open subset U in $\mathbb{R}^n$. Let us denote by $\mathfrak{Z}_U(f)$ the set of zeros of f in U, i.e.,

$$\mathfrak{Z}_U(f) = \{ x \in U \mid f(x) = 0 \}$$

Let g be any real-analytic (or, more generally, any C^{∞}) function on U such that $g(x) = 0$ for all $x \in \mathfrak{Z}_U(f)$. Then, for every compact subset K in U, we can find $N \in \mathbb{Z}_+$ (= the positive integers) and $C \in \mathbb{R}_+$ (= the positive real numbers) such that

$$|g|^N \leq C|f|$$

at every point of K.

<u>Proof</u>. The question of inequalities of this type (for this one and for all subsequent ones in this section) can be localized as follows.

(I.1) Given $K \subset U$, there exist $N_o \in \mathbb{Z}_+$ and a map $C : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that for every $N \geq N_o$

$$|g(x)|^N \leq C(N)|f(x)|$$

for all $x \in K$.

The point of this reformulation is that if $K = \cup_\alpha K_\alpha$, a finite union with compact subsets K_α of K, and if $N_{\alpha o}$ and C_α have the property (I,1) for K_α, then $N_o = \max_\alpha\{N_{\alpha o}\}$ and $C = \max_\alpha\{C_\alpha\}$ has (I,1) for K. This is the way in which the question is localized. By this type of localization, (I,1) can be reduced to the case in which Desingularization theorem II of § 5 is applicable to U and f. (Here those connected components of U, in which f is identically zero, should be omitted. There, in fact, the inequalities are trivially true). Thus we have a proper morphism of real-analytic spaces $\pi : V \longrightarrow U$ such that V is smooth, that π is almost everywhere isomorphic and that $(f)\mathcal{O}_V$ is locally simple everywhere. This last property means that we have an open covering $V = \bigcup_\alpha V_\alpha$ together with isomorphisms $V_\alpha \cong \mathbb{R}^n$, such that

$$(f)\,\mathcal{O}_V|V_\alpha = (z^{A_\alpha})\mathcal{O}_V|V_\alpha$$

where $z = (z_1, \dots, z_n)$ is the coordinate system for $\mathbb{R}^n = V_\alpha$ and $A_\alpha \in \mathbb{Z}_o^n$. For each α, write $A_\alpha = (a_{\alpha 1}, \dots, a_{\alpha n})$ and let Δ_α be the set of those i with $a_{\alpha i} > 0$. (Δ_α may be empty for some α). The assumption on g implies that $g \circ \pi$ vanishes identically on each of the coordinate hyperplane in $V_\alpha \cong \mathbb{R}^n$ defined by $z_i = 0$ with $i \in \Delta_\alpha$. It is then easy to see that g is divisible by the product of those z_i with $i \in \Delta_\alpha$. Hence we can write $(g \circ \pi)^N = h_{\alpha N} f$ on V_α with a real-analytic function $h_{\alpha N}$ (or, if g is C^∞-function, then so is $h_{\alpha N}$) for all $N \geq N_{\alpha o} = \max\{a_{\alpha i}, i \in \Delta_\alpha\}$. Let us then pick a compact subset K_α in V_α for each α, such that the union of $\pi(K_\alpha)$ for a finite number of α contain the given K. Say

$\bigcup_{\alpha\in\Lambda}\pi(K_\alpha)\supset K$, where Λ is a finite index set. (This is possible because $\pi^{-1}(K)$ is compact). Now let $C_\alpha(N)$ be the maximum of $|h_{\alpha N}(z)|$ for those $z\in K_\alpha$. (Here, for $N<N_{\alpha o}$, choose any positive real number for $C_\alpha(N)$.) Let $N_o=\max\{N_{\alpha o},\alpha\in\Lambda\}$ and let $C(N)=\max\{C_\alpha(N),\alpha\in\Lambda\}$. Then for every $N\geq N_o$, we have

$$|(g\circ\pi)(z)|^N\leq C(N)|(f\circ\pi)(z)|$$

for all $z\in K_\alpha$ and for all $\alpha\in\Lambda$. Immediately follows that

$$|g(x)|^N\leq C(N)|f(x)|$$

for all $x\in K$, i.e., (I,1).

Inequality II. Let f be the same as in the Inequality I. Then for each compact subset K of U, we can find $N\in\mathbb{Z}_+$ and $C\in\mathbb{R}_+$ such that

$$C|f(x)|\geq\operatorname{dist}(x,\mathcal{Z}_U(f))^N$$

for all $x\in K$, where $\operatorname{dist}(x,\mathbb{Z})$ denote the shortest distance from x to the closure of Z in $\mathbb{R}^n$ (with respect to the usual distance).

Proof. Again the question can be localized by formulating the question as

(II.1) Given $K\subset U$, there exists $N_o\in\mathbb{Z}_+$ and a map $C:\mathbb{Z}_+\to\mathbb{R}_+$ such that

$$C(N)|f(x)|\geq\operatorname{dist}(x,\mathcal{Z}_U(f))^N$$

for all $x\in K$ and for all $N\geq N_o$.

To prove this, we may again assume the existence of $\pi:V\to U$ having the same property as in the proof of Inequality I. Also pick an open covering $V=\bigcup_\alpha V_\alpha$ with $V_\alpha=\mathbb{R}^n$, such that $(f)\mathcal{C}_V|V_\alpha=(z^{A_\alpha})\mathcal{C}_V|V_\alpha$. With the same Δ_α as before, $\pi^{-1}(\mathcal{Z}_U(f))\cap V_\alpha$ (which is $\mathcal{Z}_{V_\alpha}(f\circ\pi)$) is equal to the union of those coordinate hyperplanes $H_{\alpha i}$ defined by $z_i=0$ with

$i \in \Delta_\alpha$. For each α and $i \in \Delta_\alpha$, let us define

$$\varphi_{\alpha i}(z) = \operatorname{dist}\,(\pi(z),\ \pi\psi_{\alpha i}(z))^2$$

for $z \in V_\alpha$, where $\psi_{\alpha i}$ denotes the orthogonal projection from $V_\alpha = \mathbb{R}^n$ to $H_{\alpha i}$. Since $\operatorname{dist}\,(x,x')^2$ is a real-analytic function on $\mathbb{R}^n \times \mathbb{R}^n$ and hence in $U \times U$, it is easy to deduce that $\varphi_{\alpha i}$ is a real-analytic function on V_α for every $i \in \Delta_\alpha$. Let $\varphi_\alpha(z)$ be the product of $\varphi_{\alpha i}(z)$ for all $i \in \Delta_\alpha$. Then

1) φ_α is real-analytic in V_α

2) $\varphi_\alpha(z) \geq \operatorname{dist}(\pi(z), \mathcal{Z}_U(f))^{2n(\alpha)}$ where $n(\alpha)$ denotes the number of elements in Δ_α.

For this inequality 2), it is enough to check $\pi\psi_{\alpha i}(z) \in \pi(H_{\alpha i}) \subset \mathcal{Z}_U(f)$ if $i \in \Delta_\alpha$. Now let us pick a finite index set Λ and a compact subset $K_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$, such that $\bigcup_{\alpha \in \Lambda} \pi(K_\alpha) \supset K$. It is easy to see that

$$\mathcal{Z}_{V_\alpha}(\varphi_\alpha) \supset \bigcup_{i \in \Delta_\alpha} H_{\alpha i} = \mathcal{Z}_{V_\alpha}(f \circ \pi)$$

Therefore, by Inequality I, there exist $N_{\alpha o} \in \mathbb{Z}_+$ and a map $C_\alpha : \mathbb{Z}_+ \to \mathbb{R}_+$ such that

$$C_\alpha(N)\,|f \circ \pi(z)| \geq |\varphi_\alpha(z)|^N$$

for all $N \geq N_{\alpha o}$ and all $z \in K_\alpha$. By 2), it follows that

3) $C_\alpha(N)\,|f \circ \pi(z)| \geq \operatorname{dist}\,(\pi(z), \mathcal{Z}_U(f))^{2n(\alpha)N}$ for all $N \geq N_{\alpha o}$ and all $z \in K_\alpha$. Let N_o be any positive integer such that $N_o \in 2n(\alpha)\mathbb{Z}_+$ and $N_o/2n(\alpha) \geq N_{\alpha o}$ for all $\alpha \in \Lambda$ with $n(\alpha) \neq 0$. Then 3) implies that

4)
$$\frac{\operatorname{dist}(x, \mathcal{Z}_U(f))^{N_o}}{|f(x)|} \leq \max\{C_\alpha(N_o/2n(\alpha))\} < \infty$$

for all $x \in K - \mathfrak{Z}_U(f)$, where this maximum is taken for those $\alpha \in \Lambda$ with $n(\alpha) \neq 0$. This boundedness 4) implies the same boundedness, when N_o is replaced by any integer $N \geq N_o$. Now, define $C : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ in such a way that, for each $N \geq N_o$,

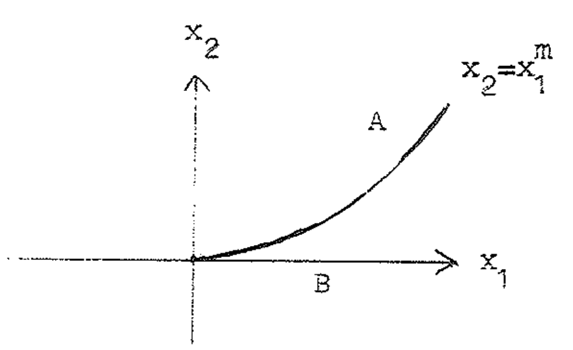
$$C(N) = \sup_{x \in K - \mathfrak{Z}_U(f)} \left\{ \frac{\operatorname{dist}(x, \mathfrak{Z}_U(f))^N}{|f(x)|} \right\}$$

(and $C(N)$ is any number $\in \mathbb{R}_+$ for those $N < N_o$). This N_o and C have the required property (II,1).

<u>Inequality</u> III. Let A and B be closed subanalytic subsets of $\mathbb{R}^n$, such that $A \cap B \neq 0$. Then, for each compact subset K of $\mathbb{R}^n$, we can find $N \in \mathbb{Z}_+$ and $C \in \mathbb{R}_+$ such that for all $x \in K$,

$$C(\operatorname{dist}(x,A) + \operatorname{dist}(x,B)) \geq \operatorname{dist}(x, A \cap B)^N$$

<u>Examples</u>. Let us take very simple examples just to show that, in general, we cannot expect N to be 1 in the inequality III. I do not know any exact condition (in terms of set-theoretical tangent cones of A and B at the points of $A \cap B$) under which the inequality III holds with $N=1$ (and with some constant $C \in \mathbb{R}_+$) for all compact $K \subset \mathbb{R}^n$, where A and B are assumed to be subanalytic and closed as above.



Let A be the part with $x_1 \geq 0$ of the curve $x_2 = x_1^m$ with an integer $m \geq 1$. Let B be the part $x_1 \geq 0$ of the x_1-axis. $A \cap B$ is the origin. Then we have Inequality III, for every compact $K \subset \mathbb{R}^2$, in which $N=m$. If we take $x_2^n = x_1^m$ with positive integers m and $n < m$, instead of $x_2 = x_1^m$, then we have the same with $N = m/n$. Though this N is not an integer in general, the inequality of type III makes sense. Of course, we can alway replace it by any integer $\geq m/n$ (and the constant $\dot{C}$ accordingly).

<u>Proof of Inequality III</u>. We apply the rectilinearization theorem for A, B and $A \cap B$, at each point of K. Since K is compact, we obtain a finite number of real-analytic maps $\{\pi_\alpha : V_\alpha \longrightarrow \mathbb{R}^n\}$ such that

1) $V_\alpha = \mathbb{R}^n$ and π_α is locally isomorphic in a dense open subset of V_α,

2) there exists a compact subset K_α in each V_α for which

$$\bigcup_\alpha \pi_\alpha(K_\alpha) \supset K$$

3) $\pi_\alpha^{-1}(A)$, $\pi_\alpha^{-1}(B)$ and hence $\pi_\alpha^{-1}(A \cap B)$ are unions of quadrants in V_α.

When the rectilinearization is stated in this form, we can replace each π_α by 2^n maps $\pi_{\alpha\varepsilon} : V_{\alpha\varepsilon} \longrightarrow \mathbb{R}^n$ which are derived from π_α in the following way. Let ε be any map from $\{1, \ldots, n\}$ to $\{-1, 1\}$ and let $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be the map defined by

$$h_\varepsilon(z_1, \ldots, z_n) = (\varepsilon(1)z_1^2, \varepsilon(2)z_2^2, \ldots, \varepsilon(n)z_n^2)$$

where $(z_1, \ldots, z_n)$ denotes the coordinate system in $\mathbb{R}^n$. Then let $\pi_{\alpha\varepsilon} = \pi_\alpha h_\varepsilon$. Now, when π_α is replaced by the $\pi_{\alpha\varepsilon}$ with all ε and K_α by the inverse images of K_α by those h_ε, the conditions 1) and 2) are not affected. Moreover, since $\pi_\alpha^{-1}(A)$, $\pi_\alpha^{-1}(B)$ and $\pi_\alpha^{-1}(A \cap B)$ are all closed in V_α, the condition 3) turns out to be strengthened by the replacement as follows.

4) $\pi_\alpha^{-1}(A)$, $\pi_\alpha^{-1}(B)$ and hence $\pi_\alpha^{-1}(A \cap B)$ are unions of coordinate linear subspaces of $V_\alpha = \mathbb{R}^n$.

For the proof of Inequality III, a finite system $\{\pi_\alpha\}$ satisfying 1), 2) and 4) will be used. Now let S (resp. T) be the disjoint union of those coordinate linear subspaces of V_α, for all α, which are irreducible components of $\pi_\alpha^{-1}(A)$ (resp. $\pi_\alpha^{-1}(B)$). The $\{\pi_\alpha\}$ defines natural morphisms $p : S \longrightarrow \mathbb{R}^n$ and $q : T \longrightarrow \mathbb{R}^n$, whose images coincide with A and B within K at least, respectively. Let us define a function D_α on $S \times T \times V_\alpha$ for each α as follows:

$$D_\alpha(s,t,z) = \mathrm{dist}(p(s), \pi_\alpha(z))^2 + \mathrm{dist}(q(t), \pi_\alpha(z))^2$$

This is clearly a real-analytic function on $S \times T \times V_\alpha$. Let $\{H_{\alpha i}\}_{i\in\Delta_\alpha}$ be the irreducible components of $\pi_\alpha^{-1}(A \cap B)$, which are coordinate linear subspaces of $V_\alpha = \mathbb{R}^n$ by 4). Let $\psi_{\alpha i}$ be the orthogonal projection from V_α to $H_{\alpha i}$ and let

$$\varphi_{\alpha i}(z) = \mathrm{dist}\,(\pi_\alpha(z), \pi_\alpha \psi_{\alpha i}(z))^2$$

and $\varphi_\alpha(z) = \prod_{i\in\Delta_\alpha} \varphi_{\alpha i}(z)$, where $z \in V_\alpha$.

Let Φ_α be the pull-back of φ_α to $S \times T \times V_\alpha$ by the projection, i.e., $\Phi_\alpha(s,t,z) = \varphi_\alpha(z)$. As before, Φ_α is a real-analytic function on $S\times T\times V_\alpha$ and we have the inequality

$$\Phi_\alpha(s,t,z) \geq \mathrm{dist}(\pi_\alpha(z), A\cap B)^{2n(\alpha)}$$

where $n(\alpha)$ is the number of elements in Δ_α.

Now $D_\alpha(s,t,z)=0$ if and only if $p(s)=\pi_\alpha(z)=q(t)$ and hence $\pi_\alpha(z)\in A\cap B$. $\Phi_\alpha(s,t,z)=0$ if $\pi_\alpha(z)\in A\cap B$. Therefore we have $\mathfrak{Z}(\Phi_\alpha) \supset \mathfrak{Z}(D_\alpha)$ where $\mathfrak{Z}$ denotes the set of zeros in $S\times T\times V_\alpha$. Let L be the union of the inverse images of K_α for all α by the natural maps from the components of S into various V_α. Let M be the analoguous union for T instead of S. Then L (resp. M) is a compact subset of S (resp. T). Let $\tilde{K}_\alpha = L\times M\times K_\alpha$, compact in $S\times T\times V_\alpha$. We now apply Inequality I to the functions Φ_α and D_α with respect to $\tilde{K}_\alpha$. So we find $N_{\alpha 0} \in \mathbb{Z}_+$ and $C_\alpha : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that

$$C_\alpha(N)\,|D_\alpha(s,t,z)| \geq |\Phi_\alpha(s,t,z)|^N$$

for all $N \geq N_{\alpha 0}$ and all $(s,t,z)\in\tilde{K}_\alpha$. This implies

$$C_\alpha(N)\left\{\operatorname{dist}(p(s),\ \pi_\alpha(z))^2 + \operatorname{dist}(q(t),\pi_\alpha(z))^2\right\}$$
$$\geq \operatorname{dist}\ (\pi_\alpha(z),\ A\cap B)^{2n(\alpha)N}$$

Since $p(L) \supset A \cap K$ and $q(M) \supset B \cap K$, it follows that

$$\frac{\operatorname{dist}(\pi_\alpha(z), A\cap B)^{2n(\alpha)N}}{\operatorname{dist}(\pi_\alpha(z), A\cap K)^2 + \operatorname{dist}(\pi_\alpha(z), B\cap K)^2}$$

is bounded for $z \in K_\alpha - \pi_\alpha^{-1}(A\cap B)$. Since A and B are closed in $\mathbb{R}^n$ and K is compact, we can deduce from this (by the usual reasoning of testing by means of sequences of points in K) that

$$\frac{\operatorname{dist}(\pi_\alpha(z),\ A\cap B)^{2n(\alpha)N}}{\operatorname{dist}(\pi_\alpha(z), A)^2 + \operatorname{dist}\ (\pi_\alpha(z),\ B)^2}$$

and hence

$$\frac{\operatorname{dist}(\pi_\alpha(z),\ A\cap B)^{n(\alpha)N}}{\operatorname{dist}(\pi_\alpha(z),\ A) + \operatorname{dist}\ (\pi_\alpha(z), B)}$$

is bounded for $z \in K_\alpha - \pi_\alpha^{-1}(A\cap B)$. Since $K \subset \bigcup_\alpha \pi_\alpha(K_\alpha)$, we can deduce from this that

$$\frac{\operatorname{dist}\ (x, A\cap B)^N}{\operatorname{dist}(x,A) + \operatorname{dist}\ (x,B)}$$

is bounded for all $x \in K - A\cap B$, provided N is an integer sufficiently large. The inequality III is now proven.

<u>Inequality IV</u>. Let M_i, $i=1,2$, be locally closed, smooth, connected real-analytic subspaces of $\mathbb{R}^n$. Assume that $\overline{M_1} \supset M_2$ and $M_1 \cap M_2 = \emptyset$. Assume further that both of the M_i are subanalytic in $\mathbb{R}^n$. In this situation, if (M_1, M_2) satisfies Whitney condition (at every point of M_2) then it satisfies strict Whitney conditions.

The definition of the terms used here will be explained with an example, before we go into the proof of the inequality.

<u>Remark</u> (IV,1). Given $(x,y) \in M_1 \times M_2$, let $\overline{xy}$ denote the line connecting x and y in $\mathbb{R}^n$. Let us identify $\mathbb{R}^n$ with the tangent space of $\mathbb{R}^n$ at x so that $x \in \mathbb{R}^n$ corresponds to the zero of the tangent space. In this way, the tangent space Tx of M_1 at x will be identified with a linear subspace through x in $\mathbb{R}^n$. Now let $\theta(Tx, \overline{xy})$ denote the angle between $\overline{xy}$ and its orthogonal projection into Tx. Define

$$\underline{W}(x,y) = \sin\left[\theta(Tx, \overline{xy})\right]$$

$(0 \leq W(x,y) \leq 1)$. We shall call $W(x,y)$ the <u>Whitney function</u> on $M_1 \times M_2$. Let $\tilde{M} = M_1 \cup M_2$ with the induced topology from $\mathbb{R}^n$. Let Δ_1 denote the diagonal of $M_1 \times M_1$ which is contained in $M_1 \times \tilde{M}$. Now Whitney condition can be stated as follows.

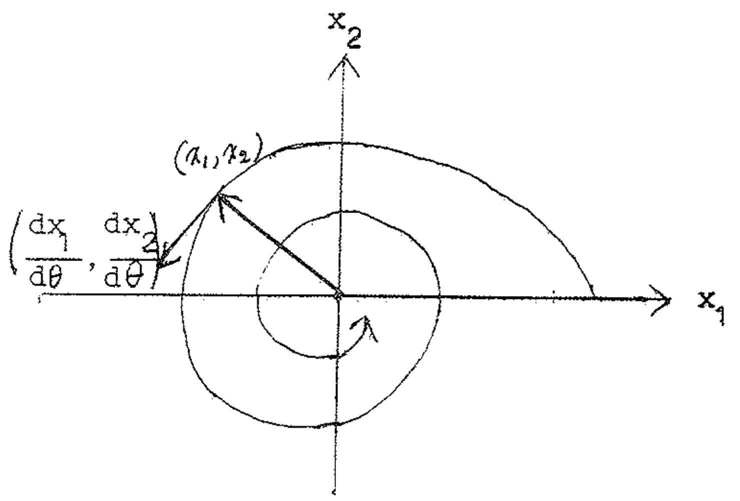
(10.1) (M_1, M_2) satisfies <u>Whitney condition</u> in $\mathbb{R}^n$ (at every point of M_2), if and only if $\underline{W}$ extends to a continuous function $\underline{\tilde{W}}$ on $M_1 \times \tilde{M}$ and $\underline{\tilde{W}}$ vanishes identically on Δ_1.

(10.2) (M_1, M_2) satisfies <u>strict Whitney condition</u> in $\mathbb{R}^n$ (at every point of M_2) if and only if for every compact subset K of M_2, there exist $N \in \mathbb{Z}_+$ and $C \in \mathbb{R}_+$ such that

$$\underline{W}(x,y)^N \leq C \operatorname{dist}(x,y)$$

for all (x,y) $M_1 \times M_2$ with $y \in K$. This is actually what we call Inequality IV.

<u>Example</u> (10.3) Let M_1 be the curve in $\mathbb{R}^2$ defined by $r = e^{-\theta^2}$ in terms of the polar coordinates (r, θ) in $\mathbb{R}^2$, where $0 < \theta < \infty$. (If (x_1, x_2) denotes the usual coordinate system in $\mathbb{R}^2$, then $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$). Let M_2 be the origin. Clearly $M_1 \cap M_2 = \emptyset$ and $\overline{M_1} \supset M_2$.



Whitney function $\underline{W}$ on $M_1 \times M_2$ is given by

$$\underline{W}(x,0) = \frac{\left|\frac{dz_1}{d\theta}x_2 - \frac{dz_2}{d\theta}x_1\right|}{\sqrt{\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dx_2}{d\theta}\right)^2}\sqrt{x_1^2+x_2^2}}$$

$$= \frac{1}{\sqrt{4\theta^2+1}},$$ which tends to zero

as θ does to ∞ (i.e., as $x \to M_2$). Namely (M_1,M_2) satisfies Whitney condition. It does not, however, satisfy the strict Whitney condition (10.2), because for every $N \in \mathbb{Z}_+$

$$\frac{\underline{W}(x,0)^N}{\operatorname{dist}(x,0)} = e^{\theta^2} \Big/ (4\theta^2+1)^{N/2}$$

is not bounded on M_1 (it tends to ∞ as $\theta \longrightarrow \infty$).

<u>Remark</u> (10.4) Let (M_1,M_2) be a pair of real-analytic (or, C^∞-) submanifolds of $\mathbb{R}^n$ such that $M_1 \cap M_2 = \emptyset$ and $\bar{M}_1 \supset M_2$. We can then find a (sufficiently small) open neighborhood U_o of M_2 in $\mathbb{R}^n$, such that for every point $x \in U_o \cap M_1$ there exists a unique point $\pi(x) \in M_2$ with $\operatorname{dist}(x,\pi(x)) = \operatorname{dist}(x,M_2)$. Pick such U_o and let $V_o = U_o \cap M_1$. We can then define a vector field on V_o,

$$v = v(M_1,M_2;\ U_o,\mathbb{R}^n)$$

as follows. For each x, take the unit vector heading toward $\pi(x)$ on the line $\overline{x\pi(x)}$, i.e.,

$$\frac{\overrightarrow{x\,\pi(x)}}{\operatorname{dist}(x,\pi(x))}$$

and let $v(x)$ be its orthogonal projection to $T_{M_1,x}$. Now, if (M_1,M_2)

satisfies the strict Whitney condition in $\mathbb{R}^n$, then we can prove that there exists an open neighborhood U_1 of M_2 in U_o such that for each point $x_o \in M_1 \cap U_1$, there exists a unique integral curve through x_o of the vector field v on $M_1 \cap U_o$, say $x(t)$, $0 \leq t < a \in \mathbb{R}_+$, having the following properties:

1) there exists $y_o = \lim_{t \to a} x(t) \in M_2$.

2) the tangent lines to the integral curve is convergent for $t \to a$, i.e., there exists

$$\lim_{t \to a} v(x(t)) \in T_{\mathbb{R}^n, y_o}$$

This fact shows that under the strict Whitney condition, the whirling phenomena as was seen in Example (10.3) cannot happen.

<u>Proof of Inequality IV</u>. In this proof, it is essential that M_1 is subanalytic in $\mathbb{R}^n$. To prove an inequality of type (10.2) for each compact subset K of M_2, we may assume that M_1 is bounded in $\mathbb{R}^n$. (Since $0 \leq W(x,y) \leq 1$, the question concerns points within a sufficiently small neighborhood of K in $\mathbb{R}^n$). Since M_1 is bounded and subanalytic in $\mathbb{R}^n$, we can find a real-analytic map $f : Y \longrightarrow \mathbb{R}^n$ such that

(a) Y is smooth and compact.

(b) $\mathrm{Im}(f) = \overline{M_1}$ and $f^{-1}(M_1)$ is dense in Y.

(c) f induces a submersion in an open dense subset of $f^{-1}(M_1)$ as a map into M_1.

(d) $\mathrm{Im}(\delta f)$ is locally free on Y, where $\delta f : f^*\Omega_{\mathbb{R}^n} \to \Omega_Y$ denotes the canonical homomorphism of the sheaves of differential forms associated with f. In fact, first of all, we know that there exists a proper morphism of real-analytic spaces $f : Y \to \mathbb{R}^n$ such that $\mathrm{Im}(f) = \overline{M_1}$.
Then Y is necessarily compact. By desingularization theorem applied to Y, we may assume that Y is smooth everywhere. We may omit those connected

components of Y in which $f^{-1}(M_1)$ is not dense. (This does not change the image of f). For each connected component Y' of Y, there exists a closed real-analytic subspace $Y'_{(d-1)}$, $d = \dim M_1$, such that for $y \in Y'$, $\operatorname{rank}(df)_y \leq d-1$ if and only if $y \in Y'_{(d-1)}$. Then either $Y'_{(d-1)} = Y'$ or $Y'_{(d-1)}$ is nowhere dense in Y'. If $Y'_{(d-1)} = Y'$, then $f(Y')$ is a union of smooth locally closed real-analytic subspaces of pure dimensions $\leq d-1$. Hence $f(Y')$ is nowhere dense in M_1. Therefore we can omit such Y' without changing $\operatorname{Im}(f)$. Namely we may assume (C). By (C), $\operatorname{Im}(\delta f)$ is locally free of rank d on $f^{-1}(M_1)$. Hence, by a suitable blowing-up $f_1: Y_1 \to Y$ whose center is nowhere dense (and in fact, is defined by a suitable Fitting ideal of the coherent sheaf $\operatorname{Im}(\delta f)$), we get a surjective homomorphism $\tau : f_1^*(\operatorname{Im}(\delta f)) \to F$ such that F is locally free of rank d everywhere. By the same reasoning as above, we can find a proper real-analytic map $f_2: Y_2 \to Y_1$ such that ff_1f_2 satisfies (a)-(c) once again. If we replace f by ff_1f_2, then (d) follows. (d) implies in its dual,

(e) $f^*T_{M_1}$ (the base extension of the tangent bundle of M_1) extends to a vector subbundle G_1 of the vector bundle $f^*T_{\mathbb{R}^n}$.

Let Γ be the fibre product of Y and M_2 over $\mathbb{R}^n$ with respect to f and the inclusion. Then Γ is a closed real-analytic subspace of $Y \times M_2$. In fact, Γ is the inverse image of the diagonal $\Delta_{\mathbb{R}^n}$ in $\mathbb{R}^n \times \mathbb{R}^n$ by the map $Y \times M_2 \to \mathbb{R}^n \times \mathbb{R}^n$. Now, by Desingularization II applied to Y in $Y \times M_2$, we can find a finite number of real-analytic maps $\pi_\alpha: W_\alpha \to Y \times M_2$ such that

(f) each W_α is smooth, and $\pi_\alpha^{-1}(f^{-1}(M_1) \times M_2)$ is dense in W_α.

(g) there exists a compact subset K_α in each W_α, such that $\bigcup_\alpha \pi_\alpha(K_\alpha) \supset Y \times K$. (Recall that Y and K are compact).

(h) $\pi_\alpha^{-1}(\Gamma)$ has normal crossings everywhere.

Here in (h), all we need is that $\pi_\alpha^{-1}(\Gamma)$ is defined by an ideal sheaf which is invertible as module. This implies that for every α, we have a commutative diagram:

$$\text{(i)}\qquad \begin{tikzcd} W_\alpha \arrow[rr, "(f\times id)\pi_\alpha"] \arrow[dr, "p_\alpha"'] & & \mathbb{R}^n \times \mathbb{R}^n \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$$

where q is the blowing-up with center $\Delta_{\mathbb{R}^n}$. Let L_o be the secant line bundle on $\mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n}$, i.e., if E denotes the pull-back of $T_{\mathbb{R}^n}$ by the first projection $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, L_o is the line subbundle of $E|\mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n}$ such that for every (x,y) the fibre $L_o(x,y)$ is the line $\overline{xy}$ in $E(x,y) = T_{\mathbb{R}^n,x}$. If q is the blowing-up as above, then $q^*(L_o)$ extends to a line subbundle of q^*E on the entire Z. Hence, by (i), the pull-back $\{(f\times id)\pi_\alpha\}^*(L_o)$ extends to a line subbundle of $\{(f\times id)\pi\}^*E$ on W_α for every α. Call this extension $G_{2\alpha}$. Let $G_{1\alpha}$ be the vector bundle on W_α obtained as the pull-back of G_1 by the canonical map $W_\alpha \longrightarrow Y$ via projection. Thanks to (b) and (f), Whitney condition implies that

$$G_{2\alpha}(\gamma) \subset G_{1\alpha}(\gamma)$$

(inclusion of fibres naturally identified as vector subspaces of $T_{\mathbb{R}^n,y}$ where $y \in M_2$ is the point corresponding to γ) for every point $\gamma \in \pi_\alpha^{-1}(\Gamma)$. $G_{1\alpha}$ and $G_{2\alpha}$ are vector subbundles of the pull-back of $T_{\mathbb{R}^n}$ by the canonical map $W_\alpha \longrightarrow \mathbb{R}^n$ via the first projection $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Call this pull-back $\tilde{E}$ and the map $g_\alpha : W_\alpha \longrightarrow \mathbb{R}^n$. Let $g'_\alpha : W_\alpha \longrightarrow \mathbb{R}^n$ be the map via the second projection $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Whitney function $\underline{W}$ on $M_1 \times M_2$ yields a function $\underline{W} \circ g_\alpha$ on $g_\alpha^{-1}(M_1 \times M_2) = \pi_\alpha^{-1}(f^{-1}(M_1) \times M_2)$. This extends to a real-analytic function $\tilde{\underline{W}}_\alpha$ on W_α. Namely, for every $z \in W_\alpha$,

$$\tilde{\underline{W}}_\alpha(z) = \sin\left[\theta\,(G_{1\alpha}(z), G_{2\alpha}(z))\right]$$

where $G_{1\alpha}(z)$, $G_{2\alpha}(z)$ are vector subspaces of $\widetilde{E}(z) = T_{\mathbb{R}^n, g_\alpha(z)}(= \mathbb{R}^n)$. Whitney condition implies that $\underline{\widetilde{W}}_\alpha$ vanishes identically on $\pi_\alpha^{-1}(\Gamma)$ for every α. Let D_α be the function on W_α defined by

$$D_\alpha(z) = \operatorname{dist}\,(g_\alpha(z), g'_\alpha(z))^2$$

This D_α is again a real-analytic function, and we have

$$\mathcal{Z}(D_\alpha) = \left|\pi_\alpha^{-1}(\Gamma)\right| \mathcal{G}(\underline{\widetilde{W}}_\alpha) .$$

Thus, with respect to K_α of (g), there exist $N_{\alpha o} \in \mathbb{Z}_+$ and a map $C_\alpha : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that

$$\left|\underline{\widetilde{W}}_\alpha(z)\right|^N \leq C_\alpha(N)\left|D_\alpha(z)\right|$$

for all $N > N_{\alpha o}$ and for all $z \in K_\alpha$. In view of (b) and $g(y)$, we can deduce from this in the same way as in the proof of earlier inequalities, that there exist $N_o \in \mathbb{Z}_+$ and $C : \mathbb{Z}_+ \longrightarrow \mathbb{R}_+$ such that

$$\left|\underline{W}\,(x,y)\right|^N \leq C(N)\ \operatorname{dist}\,(x,y)$$

for all $N \geq N_o$ and for all $(x,y) \in M_1 \times M_2$ with $y \in K$.