§ 4. Local flatenning theorem.

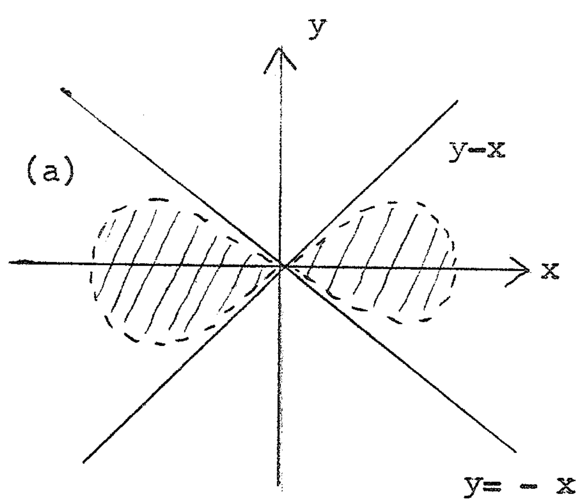
Our local flatenning theorem, the main goal of this section, was created for the purpose of studying the topological and differential nature of "subanalytic" sets. The notation of "subanalytic" will be introduced and studied in a later section. To make the main idea and the purpose of local flatenning intuitively clearer, let us bigin with some typical examples of analytic "wedges".

A subset F of a complex- (resp. real) analytic space W is often called a complex (resp. real) wedge in W if there exist a morphism of complex (resp. real) spaces $\pi : W' \to W$ and a relatively compact open subset V of W' such that $F = \pi(V)$.

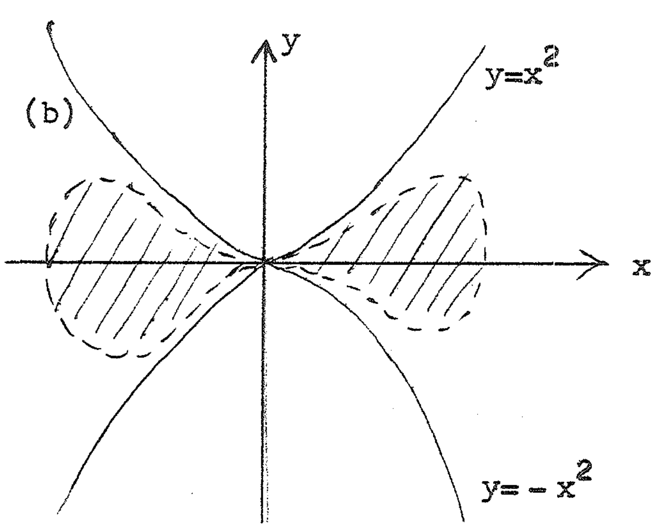
Example (4.1) Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $x = x\pi$ and $yx = y\pi$ with the standard coordinate system (x,y) in $\mathbb{R}^2$. The image F of the unit open disc in $\mathbb{R}^2$ by π is a wedge in $\mathbb{R}^2$, which looks like the shaded part of the Figure (a).

If we take $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ to be defined by $x = x\pi$ and $yx^2 = y\pi$, then the F of the same disc by this π, again a wedge in $\mathbb{R}^2$, looks like the shaded part in Figure (b). Note that the origin 0 is contained in F but not an interior point of F (while every other point of F is an interior point).

(a)



(b)



<u>Example</u> (4.2) A far more interesting (as to compare with (4.1) example of real wedge can be obtained as follows. Pick any integer $n \geq 3$ and pick a system of elements $(f_1, \dots, f_m)$, $f_i \in \mathbb{R}\{x,y\}$ (= the convergent power **series** in two variable over $\mathbb{R}$), such that $f_i(0) = 0$ for all i and the f_i, $1 \leq i \leq n$, are analytically independent in the sense that the $\mathbb{R}$-algebra homomorphism $\varphi : \mathbb{R}\{t_1, \dots, t_r\} \longrightarrow \mathbb{R}\{x,y\}$ defined by $\varphi(t_i) = f_i(x,y)$, $1 \leq i \leq n$, is injective, where $(t_1, \dots, t_n)$ is a system of n independent variables. Let $\mathbb{D}^2$ be a sufficiently small open polydisc centered at 0 in $\mathbb{R}^2$ such that all the f_i are convergent in some neighborhood of $\overline{\mathbb{D}}^2$ in $\mathbb{R}^2$. Let $f : \mathbb{D}^2 \longrightarrow \mathbb{R}^n$ be the map defined by $(f_1, \dots, f_n)$. Then the image $F = f(\mathbb{D}^2)$ is a real wedge in $\mathbb{R}^n$ having the following properties:

(1) F is a finite union of connected locally closed real-analytic submanifolds (smooth real-analytic subspaces) of dimension ≤ 2. (This fact follows from a general theorem on "stratifications of a sub-analytic set", proven in a later section).

(2) For every connected open neighborhood U of 0 in $\mathbb{R}^n$, if a real-analytic function G in U vanishes at every point of $U \cap F$, then G is identically zero in U. ((1) + (2) implies that F is not semi-analytic in the sense defined in a later section).

<u>Supplement</u> (4.2.1) A system $(f_1, \dots, f_n)$ of (4.2) can be found as follows. Take any system of algebraically indipendent elements $(g_1, \dots, g_n)$, $g_i \in \mathbb{R}\{x\}$. (This clearly exists for every $n \geq 3$). Then let $f_i(x,y) = y^{a_i} y_i(x)$, $1 \leq i \leq n$, with any system of positive integers $(a_1, \dots, a_n)$. Then for every non-zero convergent series $G \in \mathbb{R}\{t_1, \dots, t_n\}$, $G(f_1, \dots, f_n) = 0$ in $\mathbb{R}\{x,y\}$ implies a non-trivial polynomial relation among the g_j, $1 \leq j \leq n$. In fact, write $G = \sum_{k=0}^{\infty} G_k$ where G_k are weighted homogeneous polynomials of degree

k with respect to the weights $(a_1,\dots,a_n)$ for the variables $(t_1,\dots,t_n)$. Then, for at least one $k > 0$, $G_k(t) \neq 0$. For such $k > 0$, $G_k(g_1,\dots,g_n) = G_k(f_1,\dots,f_n)/y^k = 0$ would mean a non-trivial polynomial relation among the g_j. For instance, we may take $(f_1,\dots,f_n)$ to be

$$(y, ye^x, ye^{e^x}, \dots)$$

In this special case, observe that the ratio systems

$$(f_1/f_i\ ,\ f_2/f_i,\ \dots\dots\ ,\ f_n/f_i)$$

in which f_i/f_i is replaced by f_i itself, have non-trivial analytic relations respectively for every i, $1 \leq i \leq n$. Note that taking the ratio systems like these corresponds to taking the transformation of f by the blowing-up of $\mathbb{R}^n$ with center 0. What is illustrated in this simple example is the basic idea which will be developped in this section into a general tecnique of "exhibitting "hidden" analytic relation by means of blowing-ups. This technique will be formulated as "local flattering theorem".

We shall first discuss the local flattering theorem and the related in the complex-analytic case. We shall then riformulate the results in the real-analytic case by means of the method of complexification discussed in § 1.

Let $f : V \to W$ be a morphism of complex spaces. Let (U,E,π) be a local blowing-up with $\pi : W' \to W$. This induces a local blowing-up $(f^{-1}(U),\ f^{-1}(E), \varphi)$ with a morphism $\varphi: V' \to V$. We then have a unique morphism of complex spaces $f': V' \to W'$ making a commutative diagram

$$\begin{tikzcd} V' \arrow[r, "\varphi"'] \arrow[d, "f'"'] & V \arrow[d, "f"] \\ W' \arrow[r, "\pi"'] & W \end{tikzcd}$$

The unique existence of f' is shown as follows. If J denotes the ideal sheaf of E in $\mathcal{H}_W|U$, then $J\mathcal{H}_V|f^{-1}(U)$ is the ideal sheaf of $f^{-1}(E)$. So by the definition of φ, $J\mathcal{H}_{V'}$ is invertible as $\mathcal{H}_{V'}$-module. Hence, by the universal mapping property of the blowing-up π, there exists one and only one f' making the above commutative diagram.

<u>Definition</u> (4.3) The morphism $f' : V' \to W'$ obtained as above will be called the strict transform of f by local blowing-up (U,E,π). If $S = \{(U_i, E_i, \pi_i)\}_{0 \leq i < m}$ with $W_o = W$ and $\pi_i : W_{i+1} \to W_i$, is a finite sequence of local blowing-ups over W, then the strict transform $f_m : V_m \to W_m$ by S is defined by applying the above definition m times successively. (Namely, $f_o = f$ and f_{i+1} is the strict transform of f_i by (U_i, E_i, π_i), $0 \leq i < m$).

The main result of this section is the following:

<u>Theorem</u> (4.4) (<u>local flattening theorem</u>).
Let $f : V \to W$ be a morphism of complex spaces, let y be a point of W and let L be a compact subset of $f^{-1}(y)$. Assume that W is reduced. Then there exists a finite number of S_α, $1 \leq \alpha \leq a$, where each S_α is a finite sequence of local blowing-ups over W, such that

1) for every α, the centers of S_α are all nowhere dense in their respective ambient spaces (i.e. in the respective transform of W),

2) if $\pi_\alpha : W_\alpha \to W$ denote the morphism obtained by composing those in S_α, then there exists a compact subset K_α of W_α for each α, such that

$$\bigcup_{\alpha=1}^{a} \pi_\alpha(K_\alpha)$$

is a neighborhood of y in W,

3) for every α, the strict transform $f_\alpha : V_\alpha \to W_\alpha$ of f by the

sequence S_α is flat at every point of V_α corresponding to a point of $L \subset V$.

Remark (4.4.1) We have proven, in § 3, that the condition 2) is equivalent to saying that there exists an open neighborhood N of $f(z)$ in W such that

$$p_W^{-1}(N) \subset \bigcup_{\alpha=1}^{a} \mathcal{E}_{\pi_\alpha}$$

in the topological space $\mathcal{E}_W$ of étoiles over W.

Remark (4.4.2) Let us compare the diagram

$$\begin{tikzcd} V_\alpha \arrow[r, "\varphi_\alpha"'] \arrow[d, "f_\alpha"'] & V \arrow[d, "f"] \\ W_\alpha \arrow[r, "\pi_\alpha"'] & W \end{tikzcd}$$

of the theorem, with the diagram of fibre product

$$\begin{tikzcd} W_\alpha \times_W V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ W \arrow[r, "\pi_\alpha"'] & W \end{tikzcd}$$

We have a canonical closed imbedding

$$k_\alpha : V_\alpha \hookrightarrow W_\alpha \times_W V$$

compatible with $(f_\alpha, \varphi_\alpha)$. If F_α denotes the union of the inverse images in W_α of the centers in S_α, then π_α is locally isomorphic in $W_\alpha - F_\alpha$. Hence k_α is isomorphic in $V_\alpha - f_\alpha^{-1}(F_\alpha)$. So we can write

$$W_\alpha \times_W V = V_\alpha \cup V'_\alpha$$

with closed complex subspaces V_α and V'_α (as V_α is identified with its image by k_α), and f_α maps V'_α into F_α at least point-set-theoretically.

Remark (4.4.3) Some (or all) of the V_α of the theorem (4.4) can be empty. As a matter of fact, if we apply (4.4) to a complexification of the map f defined in (4.2), then all the V_α becomes empty. This would mean that the images of $W_\alpha \times_W V$ by the projection to W_α are all contained in F_α. As F_α is clearly a closed complex subspace, nowhere dense in W_α, it means that the transform of f by the base extension π_α, is given (at least locally) by a system of at most $(n-1)$ analytic, independence. (In this words, $\pi_\alpha^{-1}(F)$ with F of (4.2) has no longer the property (2) of (4.2)).

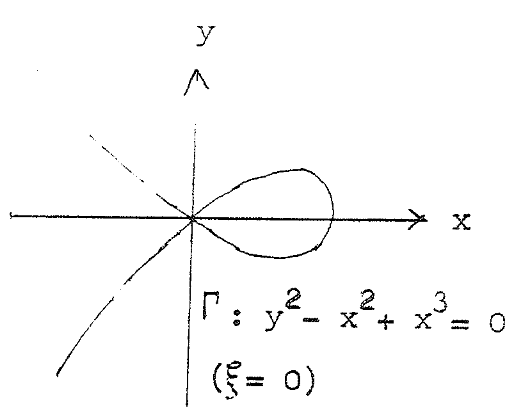
In the proof of (4.4), the key role will be played by a pair of propositions, the first asserting the existence of "local flatificator" and the second concerning the effect of blowing-up having a local flatificator as it center.

For the large part of our discussion in the sequel, we are interested in germs of complex spaces and subspaces at chosen points. With a complex-space X (or a complex subspace X in W) and a point $x \in X$, we should write $((X,x))$ for the germ represented by the pair (X,x), or the germ of X at x. A morphism $f : ((X,x)) \rightarrow ((Y,y))$ will mean the class of morphisms of complex spaces of the form $X|U \rightarrow Y|V$ such that $f(x) = y$, where U (resp. V) is some open neighborhood of x (resp. y) in X (resp. Y). Here two morphisms $f_\alpha : X|U_\alpha \rightarrow Y|V_\alpha$, $\alpha = 1,2$, belong to the same class if and only if the f_α, $\alpha = 1,2$, induce the same morphism $X|U \longrightarrow Y|V$ for some open neighborhood U (resp. V) of x(resp. y) in $U_1 \cap U_2$ (resp. in $V_1 \cap V_2$).

<u>Definition</u> (4.5) Let $f : V \rightarrow W$ be a morphism of complex spaces, let y be a point of W and let L be a non-empty compact subset of $f^{-1}(y)$. Then a germ of complex subspace (locally closed), $((P,y))$ in W is called the flatificator for (f,L) if the following condition is satisfied:

(4.5.1) For every morphism of complex spaces $h : T \rightarrow W$ and a point $t \in T$ such that $y = h(t)$, the projection $T \times_W V \rightarrow T$ is flat at every point of $t \times_W L$ if and only if h induces a morphism of germs $((T,t)) \rightarrow ((P,y))$.

<u>Example</u> (4.6) Let us take an irriducible closed complex curve Γ in $\mathbb{C}^2$ which has one and only one singular point $\xi \in \Gamma$. Assume that ξ is a node in Γ.



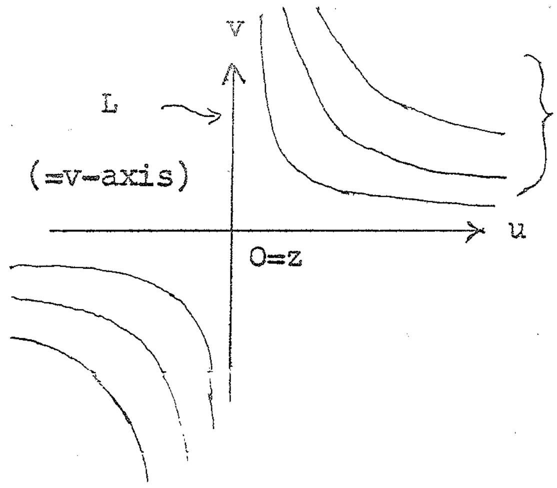
View $\mathbb{C}^2$ as a linear subspace of $\mathbb{C}^3$. We shall define a morphism of complex space $f : V \rightarrow \mathbb{C}^3$ using such a curve Γ in $\mathbb{C}^3$. Namely, take an open neighborhood N of ξ in $\mathbb{C}^3$ such that $\Gamma|N$ is a union of two irreducible smooth curves Γ_1 and Γ_2. (For instance, let N be a sufficiently small polydisc centered at ξ). Now f is defined by the following conditions.

1) Let $f_1 : V_1 \rightarrow \mathbb{C}^3|N$ be the blowing-up with center Γ_1 and let Γ_2' be the strict transform of Γ_2 by f_1, i.e., the smallest closed

curve in V_1 such that $\Gamma'_2 - f_1^{-1}(\Gamma_1) = f_1^{-1}(\Gamma_2) - f_1^{-1}(\Gamma_1)$. ($f_1$ then induces an isomorphism $\Gamma'_2 \xrightarrow{\sim} \Gamma_2$). Let $f_2 : V_2 \to V_1$ be the blowing-up with center Γ'_2. Then f induces $V|f^{-1}(N) \to \mathbb{C}^3|N$ which is isomorphic to $f_1 f_2$ (over the identity of $\mathbb{C}^3$).

2) f induces $V - f^{-1}(\xi) \to \mathbb{C}^3 - \xi$ which is the blowing-up with center $\Gamma - \xi$.
Note that the two blowing-ups, 1) over $\mathbb{C}^3|N$ and 2) over $\mathbb{C}^3 - \xi$, coincide over the common domain $N - \xi$ and can be glued together to form $f : V \to \mathbb{C}^3$.

(4.6.1) $f^{-1}(\Gamma_1) \to \Gamma_1$, induced by f, can be realized as a family of plane comes parametrized by points of Γ_1, the conics being non-singular for the points of $\Gamma_1 - \xi$ and degenerating into a union of two distinct lines meeting at a point z over the point $\xi \in \Gamma_1$. This $f^{-1}(\Gamma_1) \longrightarrow \Gamma_1$ is flat.



$uv = \eta$, a family of hyperbolas (in this real picture) parametrized by various values of η.

$: y^2 - x^2 + x^3 = \varphi_1(x,y)\varphi_2(x,y) = 0 = z$

$\Gamma_i : \varphi_i(x,y) = 0 = z$ (in $\mathbb{C}^3$), $i = 1,2$, and

$uv = \varphi_2(x,y)\circ f$ ($=\eta$), $v(\varphi_1(x,y)\circ f) = z\circ f$.

(4.6.2) Let L be the strict transform by f_2 of the fibre $f_1^{-1}(\xi)$. ($L \cong \mathbb{P}^1_{\mathbb{C}}$).

Then $f^{-1}(\Gamma_2) = f_2^{-1}(\Gamma'_2) \cup L$, f induces a $\mathbb{P}^1_{\mathbb{C}}$-bundle $f_2^{-1}(\Gamma'_2) \to \Gamma_2$, $f(L) = \xi$, and $f_2^{-1}(\Gamma'_2) \cap L = z$. Note that $f^{-1}(\Gamma_2) \to \Gamma_2$, induced by f, is not flat exactly at the point z.

(4.6.3) $((\Gamma_1, \xi))$ is the flatificator for the pair (f, z) of this example (4.6).

It should be observed, by this example, that the flatificator is not in general represented by a global complex subspace even if f is proper and surjective.

The two propositions, the key to the proof of the theorem (4.4), can be now stated as follows.

<u>Proposition</u> (4.7) Let $f : V \to W$ be any morphism of complex spaces. Let y be a point of W and L a non-empty compact subset of $f^{-1}(y)$. Then there exists a flatificator $((P,y))$ for (f,L).

<u>Proposition</u> (4.8) Let f, L and $((P,y))$ be the same as in (4.7). Pick a representative (P,y) of the germ and an open neighborhood U of y in W so that P is a closed complex subspace of $W \mid U$. Take a local blowing-up (U,P,π) with $\pi : W' \to W$ and let

$$\begin{array}{ccc} V' & \xrightarrow{\varphi} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow{\pi} & W \end{array}$$

be the diagram of the strict transform of f by (U,P,π). Then for every $y' \in \pi^{-1}(y)$, there exists at least one $z \in L$ such that the morphism of germs

$$((f'^{-1}(f'(z')), z')) \longrightarrow ((f^{-1}(f(z)), z))$$

induced by φ is not isomorphic, where $z' = y' \times z$.

<u>Remark</u> (4.8.1) Let us take y, y' and z, z' as in the proposition (4.8). ($y = f(z)$ and $y' = f'(z')$). Let $f'' : V' \times_W V \to W'$ be the projection. Then by the canonical closed imbedding

$$V' \hookrightarrow W' \times_W V$$

we obtain

$$f'^{-1}(y') \hookrightarrow f''^{-1}(y') \xrightarrow{\sim} f^{-1}(y)$$

in which the second morphism is an isomorphism as the fibre product does not change the fibres. So (4.8) asserts that the fibre $f'^{-1}(y')$ is strictly smaller than $f^{-1}(y)$ (which is identified with $f''^{-1}(y')$) at the point z'.

We shall next prove the two propositions (4.7) and (4.8) and then, using these two, we shall prove the theorem of local flattening (4.4). The original proofs of the propositions in the form stated above, given by Lejeune, Tessier and myself jointly, were based on the theory of "cônes tangents anisotropes et tropismes critiques" which had been developed by Lejeune and Teissier jointly. Here we give somewhat different proofs which are directly based on the usual Weierstrass preparation theorem in the manner that this was applied in proving the privileged neighborhood theorem of H. Cartan.

By a complex-analytic local ring, we mean a local ring $\mathcal{H}_{W,y}$ of some complex space W at some point Y. If A is such a local ring, and if $t = (t_1, \ldots, t_n)$ is a system of indeterminates, then we denote by $A\{t_1, \ldots, t_n\}$ the local ring obtained as $\mathcal{H}_{W\times\mathbb{C}^n, y\times 0}$ where $(t_1, \ldots, t_n)$ is identified with a coordinate system of $\mathbb{C}^n$ at 0.

We shall consider a direct sum of the form

$$B = \bigoplus_{i=0}^{n} A\{t_1, \ldots, t_i\}^{\nu_i}, \quad \nu_i \geqslant 0 ,$$

where $A\{t_1, \ldots, t_i\}^{\nu_i}$ means the direct sum of ν_i copies of $A\{t_1, \ldots, t_i\}^{\nu_i}$ (or a free $A\{t_1, \ldots, t_i\}$-module of rank ν_i). For another

$$B' = \bigoplus_{i=0}^{n} A\{t_1, \ldots, t_i\}^{\mu_i}$$

A homomorphism of A-modules $\Phi : B \to B'$ is said to be natural if the A-homomorphisms, induced by Φ,

$$A\{t_1, \ldots, t_i\} \longrightarrow A\{t_1, \ldots, t_j\}$$

are homomorphisms of $A\{t_1, \ldots, t_{\min(i,j)}\}$-modules for the pairs of direct summands of B and B'. (A direct sum has canonical injections from, as well as canonical projections to, various direct summands).

<u>Lemma</u> (4.9) Let $S = A\{t_1,\dots,t_n\}^m$ and let J be an $A\{t_1,\dots,t_n\}$-submodule of S. Then after a suitable non-singular $\mathbb{C}$-linear transformation among the t_i, $1 \leq i \leq n$, we can find a system of non-negative integers ν_i, $0 \leq i \leq n$, and a <u>natural</u> A-homomorphism

$$\Phi : B = \bigoplus_{i=0}^{n} A\{t_1,\dots,t_i\}^{\nu_i} \longrightarrow S$$

such that

1) Φ induces a surjective A-homorphism $\varphi : B \to S/J$ and
2) $\operatorname{Ker}(\varphi) \subset MB$ where M denotes the maximal ideal of A.

<u>Proof</u>. If $J \subset MS$, then $\Phi = \mathrm{id}$ will do (with $\nu_i = 0$ for all $i < n$ and $\nu_n = m$). Assume that $J \not\subset MS$. Given a free base $(e_1,\dots,e_m)$ of S as $A\{t_1,\dots,t_n\}$-module, after a suitable permutation among the e_j, $1 \leq j \leq m$, we can find an integer d, $1 \leq d \leq m$, such that

a) there exists at least one element

$$h_i = h_{ii}e_i + \sum_{j=i+1}^{m} h_{ij}\, e_j \in J$$

where $h_{ij} \in A\{t_1,\dots,t_n\}$ for all $j \geq i$ and $h_{ii} \notin MA\{t_1,\dots,t_n\}$, provided $1 \leq i \leq d$,

b) $$J \cap \sum_{j=d+1}^{m} A\{t_1,\dots,t_n\}\, e_j \subset M \sum_{j=d+1}^{m} A\{t_1,\dots,t_n\}\, e_j .$$

As $A\{t_1,\dots,t_n\}/MA\{t_1,\dots t_n\} \cong \mathbb{C}\{t_1,\dots t_n\}$, we can find a non-singular $\mathbb{C}$-linear transformation among the t_i, $1 \leq i \leq n$, after which

$$h_{ii} \notin (t_1,\dots,t_{n-1})\, A\{t_1,\dots t_n\}$$
$$+ MA\{t_1,\dots,t_n\}$$

for every i, $1 \leqslant i \leqslant d$. Let us simply write $A(k)$ for $A\{t_1,\dots,t_k\}$ for every k, $0 \leqslant k \leqslant n$. By Weierstrass preparation theorem, we can write

$$h_{ii} = u_i(t_n^{e_i} + g_{i1}t_n^{e_i-1} + \dots + g_{ie_i})$$

where e_i is an integer $\geqslant 0$, u_i is a unit in $A(n)$, and

$g_{ik} \in (t_1,\dots,t_{n-1})\, A(n-1) + MA(n-1)$ for all k, $1 \leqslant k \leqslant e_i$. (To make a reference to the usual Weierstrass theorem, write $A = \mathbb{C}\{u_1,\dots,u_r\}/I$ with independent variables $u_1,\dots,u_r$ and an ideal I. Then apply the theorem to a representative of h_{ii} in $\mathbb{C}\{u_1,\dots,u_r,t_1,\dots,t_n\}$). Then, by Weierstrass Division theorem (again applied to a fixed representative of h_{ii} in $\mathbb{C}\{u_1,\dots,u_r,t_1,\dots,t_n\}$ and then followed by taking module $I\,\mathbb{C}\{u,t\}$), we get an isomorphism of $A(n-1)$-modules:

$$\psi_i : A(n-1)^{e_i} \xrightarrow{\sim} A(n)/h_{ii}A(n)$$

which maps $(g_1,\dots,g_{e_i})$ to the class of

$$\sum_{j=1}^{e_i} g_j\, t_n^{e_i-j} \in A(n) \quad .$$

Let $m' = e_1 + \dots + e_d$ and $\nu_n = m-d$. Let $S_1 = \sum_{j=d+1}^{m} A(n)e_j$. Then, by means of the ψ_i for $1 \leqslant i \leqslant d$ and the inclusion $S_1 \subset S$, we get a natural homomorphism of A-modules (which is in fact, an $A(n-1)$-homomorphism)

$$\Psi \cdot A(n-1)^{m'} \oplus S_1 \longrightarrow S$$

which induces an isomorphism (of $A(n-1)$-modules)

$$A(n-1)^{m'} \oplus S_1 \xrightarrow{\sim} S / \sum_{i=1}^{d} h_i A(n)$$

Namely, as we write $A(n-1)^{m'} = \bigoplus_{i=1}^{d} A(n-1)^{e_i}$, Ψ induces the inclusion in S_1 and, in each direct summand $A(n-1)^{e_i}$, the map by

$$(g_1, \dots, g_{e_i}) \longmapsto \sum_{j=1}^{e_i} g_j t_n^{e_i - j}$$

By b), we have

c) $\Psi^{-1}(J) \cap S_1 \subset M S_1$

Let $S' = A(n-1)^{m'}$ and J' = the projection (i.e., modulo S_1) of $\Psi^{-1}(J)$ to the factor S'. If $n=0$, then all the h_{ii} are units and $S'=(0)$. Hence, in this case, the above chosen $\nu_o = \nu_n$ and $\Phi = \Psi$ have the properties required in (4.9). Now, for $n > 0$, we proceed by induction on n. Applying the induction assumption to S' and J', defined above, we obtain a natural homomorphism of A-modules

$$\Phi' : B' = \bigoplus_{i=0}^{n-1} A(i)^{\nu_i} \longrightarrow S' = A(n-1)^{m'}$$

inducing a surjective map $B' \longrightarrow S'/J'$ whose kernel is contained in MB'. Letting id denote the identity of S_1, we obtain a map

$$\Phi = \Psi \circ (\Phi' \oplus \mathrm{id}) : B = \bigoplus_{i=0}^{n} A(i)^{\nu_i} \longrightarrow S .$$

As is easily seen, this is a natural homomorphism of A-modules. It is also easy to check that Φ induces a surjective map $B \longrightarrow S/J$. Take any element $f \in \Phi^{-1}(J)$. Let f' be its projection to B'. Then $f' \in (\Phi')^{-1}(J')$ and hence $f' \in MB'$.
Hence $\Phi(f) \in J \cap (MS_2 + S_1)$ where $S_2 = \sum_{i=1}^{d} A(n)e_i$. Write

$f = \sum_{j=1}^{m} f_j e_j$, where $f_j \in MA(n)$ for all $j \leq d$. Then

$$\left(\prod_{i=1}^{d} h_{ii} \right) f - \sum_{j=1}^{d} f_j \left(\prod_{i \neq j} h_{ii} \right) h_j$$

$\in J \cap S_1$. This implies that $(\prod_{i=1}^{d} h_{ii}) f_j \in MA(n)$ for all j, because of the assumption b). Since $h_{ii} \notin MA(n)$ and $MA(n)$ is a prime ideal, so $f_j \in MA(n)$ for all j.

We are interested in the following geometric situation:

(4.10)
$$\begin{tikzcd} \underset{\underset{y\times 0}{\parallel}}{z} \in V \arrow[r, hook] \arrow[dr, "f"'] & W \times \mathbb{C}^n \arrow[d, "\text{projection}"] \\ & W \quad y=f(z) \end{tikzcd}$$

where f is induced by the projection via a locally closed imbedding $V \longrightarrow W \times \mathbb{C}^n$. Note that given any morphism of complex spaces $f: V \to W$ and a point $z \in V$, we can always find an open neighborhood of z in V, to which the restriction of V admits a locally closed imbedding into $W \times \mathbb{C}^n$ so as to make a commutative diagram (4.10) with the restricted f. In other words, so far as we are interested in the local nature of f around the given point z, we can assume (4.10) without loss of generality.

We apply (4.9) to the situation of (4.10). Let $A = \mathcal{H}_{W,y}$ and let J be the ideal of V in the local ring S of $W \times \mathbb{C}^n$ at z. By (4.9) we can choose a coordinate system $t=(t_1, \ldots, t_n)$ of $\mathbb{C}^n$ at 0 so that there exists

$$\Phi : B = \bigoplus_{i=0}^{n} A\{t_1, \ldots t_i\}^{\overset{\vee}{i}} \longrightarrow S = A\{t_1, \ldots, t_n\}$$

having the properties 1) and 2) of (4.9). Let $K = \Phi^{-1}(J)$ $(=\mathrm{Ker}(\varphi)$ with φ of (4.9)). If $g \in B$, then we have a unique expression.

$$(4.11.1) \qquad g = \sum_{i=0}^{n} \sum_{j=1}^{\nu_i} \sum_{\alpha \in \mathbb{Z}_0^i} g_{ij} \; t(i)$$

where $t(i)=(t_1,\ldots,t_i)$, $t(i)^{\alpha} = t_1^{\alpha_1} \ldots t_i^{\alpha_i}$ for $\alpha = (\alpha_1,\ldots,\alpha_i)$ $\mathbb{Z}_0^i$ and $g_{ij\alpha} \in A$ for all (i,j,α).
We then define

(4.11.2) $I(f,z)$ = the ideal in A generated by the $g_{ij\alpha}$ for all $g \in K$ and for all (i,j,α), $0 \leq i \leq n$, $1 \leq j \leq \nu_i$ and $\alpha \in \mathbb{Z}_0^i$.

Note that by 2) of (4.4), $I(f,z) \subset M$.

Lemma (4.12) Let P be a locally closed complex subspace of W such that $y \in P$ and $I(f,z)$ is the ideal of P in $\mathcal{H}_{W,y}$. Then we have

1) $f^{-1}(P) \longrightarrow P$, induced by f, is flat at z.

2) If Q is any locally closed complex subspace of W containing y such that $f^{-1}(Q) \rightarrow Q$ induced by f is flat at z, then $Q \subset P$ within a sufficiently small neighborhood of y in W.

Proof. By (4.11.2), $K = \Phi^{-1}(J) \subset I(f,z)B$. Since Φ induces a surjective map $B \rightarrow S/J$ by 1) of (4.9), it induces an isomorphism of A-modules

$$B/I(f,z)B \xrightarrow{\sim} S/(J + I(f,z)S)$$

This last local ring is equal to $\mathcal{H}_{f^{-1}(P),z}$, and

$$B/I(f,z)B \cong \bigoplus_{i=0}^{n} \bar{A}\{t_1,\ldots,t_i\}^{\nu_i}$$

with $\bar{A} = A/I(f,z)$ which is $\mathcal{H}_{P,y}$. This direct sum is clearly $\bar{A}$-flat, and hence $\mathcal{H}_{f^{-1}(P),z}$ is $\mathcal{H}_{P,y}$ - flat. Pick Q as in 2) of (4.12). Let I_1 be the ideal of Q in $\mathcal{H}_{W,y}$. Then $\mathcal{H}_{Q,y} = A/I_1$. Call this A_1.

Then Φ induces a surjective map of A_1-modules:

$$\begin{array}{ccc} \psi : \bigoplus_{i=0}^{n} A_1\{t_1,\ldots t_2\}^{\nu_i} & \longrightarrow & S/(J + J_1 S) \\ \| & & \| \\ B/I_1 B & & \mathcal{H}_{f^{-1}(Q),z} \end{array}$$

Let $B_1 = B/I_1 B$. Let $K_1 = \operatorname{Ker}(\psi)$ which is contained in $M B_1$. Since B_1 is A_1-flat, the A_1-flatness of $\mathcal{H}_{f^{-1}(Q),z}$ implies $K_1 = (0)$. In fact, if $\overline{B_1} = B_1/K_1$ is A_1-flat, we have a canonical isomorphism of A_1/M_1-modules.

$$M_1^k \overline{B_1} \,/\, M_1^{k+1} \overline{B_1} \xleftarrow{\sim} M_1^k M_1^{k+1} \otimes_{A_1} \overline{B_1}/M_1 \overline{B_1}$$

for all $k \geqslant 0$, where M_1 is the maximal ideal of A_1. Compare these with the corresponding isomorphisms for B_1 instead of $\overline{B_1}$. Since $B_1/M B_1 \cong \overline{B_1}/M \overline{B_1}$, the canonical homomorphisms

$$M_1^k B_1/M_1^{k+1} B_1 \longrightarrow M_1^k \overline{B_1} \,/\, M_1^{k+1} \overline{B_1}$$

must be all bijective. This implies $K_1 \subset M_1^k B_1$ for all integers $k > 0$. Since each $A_1\{t_1,\ldots,t_i\}$ is noetherian local, it follows that $K_1 = (0)$. This means that $K \subset I_1 B$. Hence $I(f,z) \subset I_1$, i.e., the conclusion of (2).

<u>Lemma</u> (4.13) The germ $((P,y))$ with P of (4.12) is the flatificator for (f,z) of (4.10). (cf. Def. (4.5)).

<u>Proof</u>. Since the question is local, every extension of the base W may be factored into first a base extension by a projection of the form $W \times \mathbb{C}^r \longrightarrow W$ and then an extension by a locally closed imbedding. For the case of a projection $W \times \mathbb{C}^r \rightarrow W$, we shall prove that

$((P \times \mathbb{C}^r, y \times 0))$ has the properties of (4.12) for $(f \times id_{\mathbb{C}^r}, z \times 0)$. If this is proven, then (4.12) implies that for every locally closed imbedding $y \times 0 \in W' \subset W \times \mathbb{C}^r$, $(f \times id_{\mathbb{C}^r})^{-1}(W') \longrightarrow W'$ is flat at $z \times 0$ if and only if the germ $((W', y \times 0))$ is contained in $((P \times \mathbb{C}^r, z \times 0))$, i.e., the morphism $((W', y \times 0)) \longrightarrow ((W,y))$ obtained by the projection induces $((W', y \times 0)) \rightarrow ((P,y))$. Namely that $((P,y))$ is the flatificator for (f,z) as is asserted in (4.13). To prove the above assertion for the projection $W \times \mathbb{C}^r \longrightarrow W$, we apply (4.12) to $(f \times id_{\mathbb{C}^r}, z \times 0)$. So we have a locally closed complex subspace $\tilde{P}$ of $W \times \mathbb{C}^r$ such that $y \times 0 \in \tilde{P}$ and $((\tilde{P}, y \times 0))$ has the properties 1) and 2) for $(f \times id_{\mathbb{C}^r}, z \times 0)$. These properties imply $((P \times \mathbb{C}^r, y \times 0)) \subset ((\tilde{P}, y \times 0))$. They also imply $((\tau(\tilde{P}), y \times 0)) \subset ((\tilde{P}, y \times 0))$ for every small translation τ in $\mathbb{C}^r$ (which naturally extends to a translation of the whole diagram (4.10)) because the flatness is an open condition in the source of map and the situation stays to be locally isomorphic around the given point $z \times 0$ after such a translation. On the other hand, the properties of (4.12) of $((P,y))$ imply that, via identification of $W \times 0$ with W, $((P \times 0, y \times 0)) \supset ((\tilde{P} \cap W \times 0, y \times 0))$ as the flatness is preserved by any base change. These inclusions among the two germs, imply $((P \times \mathbb{C}^r, y \times 0)) = ((\tilde{P}, y \times 0))$. This completes the proof of (4.13).

<u>Corollary</u> (4.13.1) Let L be any non-empty compact subset of $f^{-1}(y)$. Let P be the locally closed complex subspace of W such that $y \in P$ and the ideal of P in $\mathcal{H}_{W,y}$ is generated by all the $I(f,z)$ with $z \in L$. Then the germ $((P,y))$ is the flatificator for (f,L), (cf. Def. (4.5).)

<u>Proof</u>. For each $z \in L$, write P_z for P of (4.13). Then the universal mapping property of $((P_z,y))$ for each $z \in L$, all combined, implies the same for L.

Remark (4.13.2) Note that (4.13.1) proves the first proposition (4.7).

We next propose to prove the second proposition (4.8). By the universal mapping property of a flatificator, it is easy to verify that $((\pi^{-1}(P), y'))$ is the flatificator for (f'', L'') where $f'': V \times_W W' \to W'$ is the projection and $L'' = L \times_W y'$. So we can obtain $((\pi^{-1}(P), y))$ also by the procedure of (4.9), (4.10), (4.11), (4.12) and (4.13.1) for the morphism f'' (instead of f of (4.10)) and various points $z' \in L''$ (instead of z L of (4.13.1)). The ideal sheaf I' of $\pi^{-1}(P)$ in $\mathcal{O}_{W'}$ is invertible as $\mathcal{O}_{W'}$-module.
Hence $I'_{y'}$ is principal and generated by a non-zero-divisor. Since $I'_{y'}$ is generated by the ideals $I(f'', z')$ for $z' \in L''$, there exists at least one $z' \in L''$ such that $I'_{y'} = I(f'', z')\mathcal{O}_{W', y'}$. Pick and fix one such z', say $z' = z \times y'$. We shall prove that the assertion of (4.8) is true for this z. Since I' generates an invertible sheaf on V' by the definition of the strict transform f'. So the annihilators of $I'_{y'}$, i.e., that of $I(f'', z')$ in the local ring of $V \times_W W'$ at z', are contained in the ideal of V'. So all that remains to be proven is the following lemma, as we go back to the general situation of (4.10) (replacing (f'', z') by (f, z)):

Lemma (4.14) The assumptions and the notation being the same as in (4.10), (4.11) and (4.12), if $I(f, z)$ is invertible as $\mathcal{O}_{W,y}$-module then there exists at least one $h \in \mathcal{O}_{V,z}$ such that

1) h is not contained in $M\mathcal{O}_{V,z}$ (=the ideal of the fibre $f^{-1}(y)$)

2) $I(f, z)h = (0)$ in the local ring $\mathcal{O}_{V,z}$.

In particular, if V' is any locally closed complex subspace of V in a neighborhood of z such that $I(f, z)\mathcal{O}_{V', z}$ is invertible as $\mathcal{O}_{V', z}$-

module, then the inclusion

$$V' \cap f^{-1}(y) \subset f^{-1}(y)$$

is not locally isomorphic at the point z.
(For the proof of (4.8), we apply this lemma to (f'', z') instead of (f,z))

<u>Proof</u>. We follow the notation of (4.11.1)-(4.11.2). The assumption of (4.14) implies that there exist an element $g \in K = \Phi^{-1}(J)$ and a triple (i,j,α) such that $g_{ij\alpha}$ is not a zero-divisor in A and generates the ideal $I(f,z)$. Hence we can write $g = g_{ij\alpha} g'$ with $g' \in B$. Clearly $g' \notin MB = K+MB$. So if h is the class of $\Phi(g')$ modulo J, then h is an element of $\mathcal{H}_{V,z} = S/J$ but is not in $M\mathcal{H}_{V,z}$. But $\Phi(g) \in J$, i.e., $g_{ij}\ h = 0$ in $\mathcal{H}_{V,z}$. This implies $I(f,z)h = (0)$ in $\mathcal{H}_{V,z}$ because g_{ij} is a generator of $I(f,z)$. The last part of (4.14) is immediate from the proceeding, because h is not in the ideal of $f^{-1}(y)$ but is in that of V' in the local ring $\mathcal{H}_{V,z}$.

<u>Remark</u> (4.14.1) By this, we have proven the second proposition (4.8).

Having proven (4.7)-(4.8), we now proceed to prove the local flattening theorem (4.4). We want to point out that the process of proving (4.4) is as important as the result stated in (4.4), especially for its applications in later discussions. Namely the process has a certain canonicality and this important point will be made precise below and formulated into a proposition (4.16).

We begin with $f : V \to W$ with a reduced W, a point $y \in W$ and a non-empty compact subset L of $f^{-1}(y)$ as in (4.4). Then pick any étoile $e \in \mathcal{E}_W$ with $p_W(e) = y$. We shall then select an infinite sequence of local blowing-ups over W, denoted by $S(e) = \{(U_i, E_i, \pi_i)\}_{0 \leq i < \infty}$ with the diagrams of successive strict transforms f_i of f along the sequence:

$$\begin{tikzcd} V_{i+1} \arrow[r, "\varphi_i"'] \arrow[d, "f_{i+1}"'] & V_i \subset V \times_W W_i \arrow[d, "f_i"] \\ W_{i+1} \arrow[r, "\pi_i"] & W_i \end{tikzcd}$$

where $f_o=f$, $V_o=V$ and $W_o=W$. We choose S inductively under the following conditions:

(4.15.1) Let $\pi^m = \pi_o \pi_1 \cdots \pi_{m-1} : W_m \to W$. Then $\pi^m \in e$ for all $m \geq 0$. Let us write $y_o=y$ and $y_m = P_{\pi^m}(e) \in W_m$. Also $L_o=L$ and $L_m = L_o \times_W y_m$ (a non-empty subset of $V \times_W W_m$).

(4.15.2) Suppose we have already chosen

$$S_m(e) = \{(U_i, E_i, \pi_i)\}_{0 \leq i < m}$$

for some integer $m \geq 0$. If $L_m \cap V_m = \emptyset$, then let U_m be any open neighborhood of y_m in W_m, let $E_m = \emptyset$ and let $\pi_m = id_{W_m}|U_m$. Assume that $L_m \cap V_m \neq \emptyset$. Then, by (4.7), there exists a flatificator $((P_m, y_m))$ for $(f_m, L_m \cap V_m)$. Then pick any open neighborhood U_m of y_m in W_m and any representative (P_m, y_m) of the flatificator, such that P_m is a closed complex subspace of $W_m|U_m$, subject to the condition: $W_m^* - P_m = W_m|U_m - P_m$. (The existence of such W_m^* is proven in the same way as for (2.7).) Then let $E_m = W_m^* \cap P_m$. This is a closed nowhere dense complex subspace of $W_m|U_m$. Then take the local blowing-up $\pi_m : W_{m+1} \to W_m$.

Remark (4.15.3) When (U_m, E_m, π_m) is chosen as in (4.15.2), it is automatic that $\pi^m = \pi^{m-1}\pi_m \in e$. (This is so by (3.12).) In other words, (4.15.1) is a consequence of (4.15.2).

Remark (4.15.4) Since W is reduced, so are all the W_m. Therefore if P_m of (4.15.2) is nowhere dense in U_m then $W_m^* = W_m$ and $E_m = P_m$. If f_m is flat at every point of $L_m \cap V_m$, then $y_m \notin E_m$. In this case , we could choose $E_m = \emptyset$ and $\pi_m = \mathrm{id}_{W_m}|U_m$.

Remark (4.15.5) The infinite sequence $S(e)$ defined by (4.15.1)-(4.15.2) is unique up to restrictions of the domains U_i of the local blowing-ups. In fact, the germ $((P_m, y_m))$ is unique and so is $((W_m^*, y))$. Hence $((E_m, y_m))$ is uniquely determined by the given data (f, L, e).

Proposition (4.16) Given $f : V \to W$, $L \subset V$, $y \in W$ and $e \in \mathcal{E}_W$ as above, a sequence $S(e)$ selected by the conditions (4.15.2) is locally trivial except for a finite number of indices. To be precise, there exists an index $m_0 \geq 0$ such that f_m is flat at every point of $L_m \cap V_m$ (in particular, $L_m \cap V_m =$) and E_m is empty in some neighborhood of y_m in W_m for all $m \geq m_0$.

Proof. If f_m is flat at every point of $L_m \cap V_m$, then the same is true for all $m' > m$. In fact, since flatness is preserved by any base extension, if f_m is so then the projection $V_m \times_{W_m} W_{m+1} \to W_{m+1}$ is flat at every point of L_{m+1}. Hence the local rings of the fibre product at any point of L_{m+1} (belonging to the fibre product) has no torsion with respect to the multiplication by any non-zero divisor in the local ring of W_{m+1}, at y_{m+1}, and hence in particular by a generator of the ideal of $\pi_m^{-1}(E_m)$. It follows that the imbedding

$$V_{m+1} \hookrightarrow V_m \times_{W_m} W_{m+1}$$

is locally isomorphic at every point of $L_{m+1} \cap V_{m+1}$. Hence f_{m+1} is flat at every point of $L_{m+1} \cap V_{m+1}$. To prove (4.16), we shall first prove that if f_{m+1} is not flat at some point of $L_{m+1} \cap V_{m+1}$, then there exists at least one $z_{m+1} \in L_{m+1}$ such that the imbedding

$$f_{m+1}^{-1}(y_{m+1}) \hookrightarrow f_m^{-1}(y_m)$$

(via the isomorphism

$$(f_m \times_{W_m} W_{m+1})^{-1}(y_{m+1}) \xrightarrow{\sim} f_m^{-1}(y_m)$$

of the fibres by base extension) is not an isomorphism locally at z_{m+1}. Let P'_m(resp. $W^{*\prime}_m$) be the strict transform of P_m(resp. W^*_m) be the local blowing-up (U_m, E_m, π_m). Since $W_m | U_m = P_m \cup W^*_m$, W_{m+1} is a disjoint union of P'_m and $W^{*\prime}_m$. In fact, W_{m+1} is the strict transform of $P_m \cup W^*_m$ by (2.7)-(2.8) and hence equal to the union of P'_m and $W^{*\prime}_m$. The disjointness is due to (3.16.1). (It should be noted here that P'_m could be empty, as is the case if P_m is nowhere dense in $W_m | U_m$ and hence $W^*_m = W_m$ and $E_m = P_m$). We must have $y_{m+1} \in W^{*\prime}_m$. In fact, if $y_{m+1} \in P'$ then a suitable restriction of W_{m+1} to an open neighborhood of y_{m+1} is contained in P'_m and hence mapped into P_m by π_m. Since $((P_m, y_m))$ is the flatificator for $(f_m, L_m \cap V_m)$, it would imply that the base extension by fibre product of f_m by π_m is flat at every point of L_{m+1} (within the fibre product) and hence, by the same reason as above, V_{m+1} locally coincides with the fibre product and f_{m+1} is flat at every point of $L_{m+1} \cap V_{m+1}$. Thus we should have $y_{m+1} \in W^{*\prime}_m$. By (2.7), π_m induces the blowing-up $W^{*\prime}_m \to W^*_m$ with center $E_m = W^*_m \cap P_m$. By the universal mapping property of a flatificator, $((E_m, y_m))$ is the flatificator for $f_m^{-1}(W^*_m) \longrightarrow W^*_m$, induced by f_m, and for the compact set $L_m \cap f_m^{-1}(W^*_m)$. Clearly V_{m+1} is a disjoint union of $f_{m+1}^{-1}(W^{*\prime}_m)$ and $f_{m+1}^{-1}(P'_m)$ and $f_{m+1}^{-1}(W^{*\prime}_m) \to W^{*\prime}_m$, induced by f_{m+1}, is the strict transform of $f_m^{-1}(W^*_m) \to W^*_m$ by the blowing-up $W^{*\prime}_m \to W^*_m$ with center E_m. Therefore, by (2.8), $f_{m+1}^{-1}(y_{m+1}) \to f_m^{-1}(y_m)$ is not locally isomorphic at-least one point $z_{m+1} \in L_{m+1}$. (The image z_m of z_{m+1} into L_m must be in $L_m \cap V_m$). Now, if m_o of (4.16) did not exists, then we should have an infinite sequence of closed imbeddings

$$f_{m+1}^{-1}(y_{m+1}) \hookrightarrow f_m^{-1}(y_m)$$

which are strictly decreasing at some point (depending on m) corresponding to a point of compact set $L \subset V$. This is impossible by a theorem of H. Cartan (2.7.3).

<u>Proof of</u> (4.4) (local flattening theorem).
For each étoile $e \in \mathcal{E}_W$ such that $p_W(e) = y$, we choose an <u>infinite</u> sequence $S(e)$ by (4.15.1)-(4.15.2). Then, by (4.16), there exists an integer $m \geq 0$ such that f_m is flat at every point of $L_m \cap V_m$. Since the flatness is an open condition in the source of a morphism, f_m is flat in some neigh<u>h</u>borhood N_m of $L_m \cap V_m$ in V_m. Let $\varphi^m = \varphi_o \varphi_1 \cdots \varphi_{m-1} : V_m \to V$. Then f_m induces a proper map $(\varphi^m)^{-1}(L) \longrightarrow W_m$, as we have a natural closed imbedding $V_m \hookrightarrow V \times_W W_m$. Therefore we have an open neighborhood M_m of y_m in W_m such that $f_m^{-1}(M_m) \cap (\varphi^m)^{-1}(L) \subset N_m$. This means that if we denote by $f_\alpha : V_\alpha \longrightarrow W_\alpha$ the morphism $f_m^{-1}(W_m|M_m) \longrightarrow W_m|M_m$ which f_m induces, then f_α is flat at every point corrisponding to a point of $L \subset V$. We could then choose (U_m, E_m, π_m) to be $(M_m, \phi, id_{W_m}|M_m)$ so that we could say f_α is the strict transform of f by the finite sequence $S_{m+1}(e)$. We denote by π_α the composition $\pi_o \pi_1 \cdots \pi_m : W_\alpha \longrightarrow W$. Let $S_\alpha = S_{m+1}(e)$ and we know that S_α satisfies the required conditions 1) and 3) of (4.4). By packing one π_α as above for each $e \in \mathcal{E}_W$ with $p_W(e) = y$, we get

$$p_W^{-1}(y) \subset \bigcup_e \mathcal{E}_{\pi_\alpha}$$

Since p_W is proper by (3.16), we can extract a finite number of such π_α for which the union of $\mathcal{E}_{\pi_\alpha}$ contains $p_W^{-1}(y)$. Take such a finite system of π_α . Then this system has all the required properties of (4.4). Here for the property 2) of (4.4), we have only to quote (3.17). We have thus proven the local flattening theorem (4.4).

Let us next consider the case in which $f : V \longrightarrow W$ of (4.4) is given as a complexification of a morphism of real-analytic spaces. In other words, we assume that:

(4.17.1) There are given an auto-conjugation σ_W of W and σ_V of V in such a way that $f\sigma_V = \sigma_W f$ (Hence f induces a morphism of real part $f^{\mathbb{R}} : V^{\mathbb{R}} \longrightarrow W^{\mathbb{R}}$, a morphism of real-analytic spaces). Moreover, $\sigma_V(L)=L$.

The sequences $S(e)$ of (4.15.1)-(4.15.2) being essentially canonical, the morphisms $f_\alpha : V_\alpha \longrightarrow W_\alpha$ of Theorem (4.4) can be so chosen that the given pair of auto-conjugations (σ_V, σ_W) (or, for that matter, quite generally any pair of automorphisms of V and W, compatible with f, in the category of abstract ringed spaces) can be naturally extended to the derived V_α and W_α so as to be compatible with the morphisms π_α, φ_α and f_α . We shall make this fact more precise in a theorem below.

Given an auto-conjugation σ_W of W , we shall simply say that σ_W extends to a sequence of local blowing-ups over W :

$$S = \{(U_i, E_i, \pi_i)\}_{0 \leq i < m}$$

if there exists an auto-conjugation σ_i of W_i, where $\pi_i : W_{i+1} \longrightarrow W_i$ and $W_0 = W$, in such a way that $\sigma_i(U_i) = U_i$, $\sigma_i(E_i) = E_i$ and $\pi_i\sigma_{i+1} = \sigma_i\pi_i$ for all i, where $\sigma_0 = \sigma_W$. With help of the conjugation morphisms of (1.10), (2.14.1) implies that, given σ_W and S, the sequence of auto-conjugations $\{\sigma_i\}_{0 \leq i \leq m}$ is unique. Moreover, under the assumption of (4.17.1), σ_V also extends to the pull-back of the sequence S by f. Namely, if $f_i : V_i \rightarrow W_i$, $0 \leq i \leq m$, are the strict transform of f by the subsequences of S and if $\varphi_i : V_{i+1} \rightarrow V_i$ are the canonical morphisms,

$0 \leq i < m$, there exist auto-conjugations τ_i of V_i, where $\tau_o = \sigma_V$, such that $f_i \tau_i = \sigma_i f_i$, $\tau_i \varphi_i = \varphi_i \tau_{i+1}$ for all i.

Theorem (4.17) Under the additional assumption (4.17.1) to the assumption of (4.4), we can choose $\{S_\alpha\}$ having the following property in addition to 1)-3) of (4.4):

4) σ_W extends to an auto-conjugation of S_α for every α.

Consequently we obtain auto-conjugations σ_α of W_α and τ_α of V_α for each α, such that $f_\alpha \tau_\alpha = \sigma_\alpha f_\alpha$, $\pi_\alpha \tau_\alpha = \sigma_W \pi_\alpha$ and $\varphi_\alpha \tau_\alpha = \sigma_V \varphi_\alpha$ for all α.

To prove this theorem, we first make a few remarks about complex conjugation of $\mathcal{E}_W$. Quite generally, we have the complex conjugation $\rho_W : W \longrightarrow {}^*W$ of (1.10). Given a finite sequence of local blowing-ups $\{(U_i, E_i, \pi_i)\}_{0 \leq i < m}$ over W, we have its complex conjugate $\{({}^*U_i, {}^*E_i, {}^*\pi_i)\}_{0 \leq i < m}$ defined as follows. Let $\pi_i : W_{i+1} \longrightarrow W_i$ with $W_o = W$. Denote by $\rho_i : W_i \longrightarrow {}^*W_i$ the conjugation morphisms. Then ${}^*U_i = \rho_i(U_i)$ and ${}^*E_i = \rho_i(E_i)$ for all i, $0 \leq i < m$. Moreover we have a unique morphism of complex spaces ${}^*\pi_i : {}^*W_{i+1} \longrightarrow {}^*W_i$ such that ${}^*\pi_i \rho_{i+1} = \rho_i \pi_i$, $0 \leq i < m$. It is easy to see that $({}^*U_i, {}^*E_i, {}^*\pi_i)$ is actually a local blowing-up over ${}^*W_i$ for every i.

For every $\pi : W' \longrightarrow W$ belonging to $\mathcal{E}(W)$, we get a unique ${}^*\pi : {}^*W' \longrightarrow {}^*W$ belonging to $\mathcal{E}({}^*W)$ such that ${}^*\pi \rho_{W'} = \rho_W \pi$. In this way, ρ_W induces an isomorphism of categories $\mathcal{E}(W) \xrightarrow{\sim} \mathcal{E}({}^*W)$ in which $\pi \longmapsto {}^*\pi$ defined above.

If σ_W is an auto-conjugation of W, then we can identify ${}^*W$ with W by the isomorphism of complex spaces $\rho_W \sigma_W : W \xrightarrow{\sim} {}^*W$. By this identification, the conjugate $\{({}^*U_i, {}^*E_i, {}^*\pi_i)\}_{0 \leq i < m}$ can be also viewed as a sequence of local blowing-ups over W itself. In this way, σ_W induces an automorphism of the category $\mathcal{E}(W)$.

In general, the isomorphism $\mathcal{E}(W) \xrightarrow{\sim} \mathcal{E}({}^*W)$ defines an isomorphism $\mathcal{E}_W \xrightarrow{\sim} \mathcal{E}_{*W}$ (i.e., a bijective and bicontinuous map). In particular, if σ_W is given as above, we get an automorphism of the topological space $\mathcal{E}_W$ (or even as a $\mathbb{C}$-ringed space). This automorphism will be denoted by $\sigma_{\mathcal{E},W}$ (which is uniquely determined by σ_W but of course depends upon the choice of σ_W). For $e \in \mathcal{E}_W$, $\sigma_{\mathcal{E},W}(e) \in \mathcal{E}_W$ will be called the <u>conjugate étoile</u> (with respect to σ_W).

<u>Proof</u> of (4.17). This will be done by choosing an infinite sequence $S(e)$ of (4.15.1)-(4.15.2) for each $e \in p_W^{-1}(y)$ in such a way that σ_W extends to $S_m(e)$ for all m. Inductively assume that, for some $m \geqslant 0$, we have chosen $S_m(e)$ of (4.15.2) so as to have an extension of σ_W to $S_m(e)$. Namely we have autoconjugations σ_i of W_i, $0 \leqslant i \leqslant m$, with $\sigma_o = \sigma_W$, such that $\sigma_i\pi_i = \pi_i\sigma_{i+1}$ for all i, $0 \leqslant i < m$, and that $\sigma_i(U_i) = U_i$ and $\sigma_i(E_i) = E_i$ for all $i, 0 \leqslant i < m$. We have $\sigma_i(y_i) = p_W({}^*e)$ with ${}^*e = \sigma_{\mathcal{E},W}(e)$. Call it ${}^*y_i$. We also have auto-conjugations τ_i of V_i, $0 \leqslant i \leqslant m$, such that $f_i\tau_i = \sigma_i f_i$, $0 \leqslant i \leqslant m$, and $\tau_i\varphi_i = \varphi_i\tau_{i+1}$, $0 \leqslant i < m$. We then want to choose (E_m, U_m) of (4.15.2) in such a way that $\sigma_m(E_m) = E_m$ and $\sigma_m(U_m) = U_m$.

<u>Case 1</u>. Assume $y_m \neq {}^*y_m$. Then pick a sufficiently small open neighborhood N of y_m in W_m such that there exists a representative (P_m, y_m) of the flatificator for $(f_m, \bar{z}_m)$, where P_m is a closed complex subspace of $W_m|N$, and, such that $N \cap \sigma_m(N) = \emptyset$. By the auto-conjugations (σ_m, τ_m), it is easy to see that $((\sigma_m(P_m), {}^*y_m))$ is the flatificator for $(f_m, {}^*L_m \cap V_m)$ with ${}^*L_m = \tau_m(L_m)$ with V_m. Let W_m^* be the smallest

complex subspace, closed, in $W_m|N$ such that $W_m^* - P_m = W_m|N - P_m$. Then let $U_m = N \cup \sigma_m(N)$ (disjoint union) and let $E_m = (W_m^* \cap P_m) \cup \sigma_m(W_m^* \cap P_m)$. Let (U_m, E_m, π_m) be the local blowing-up in the sequence S that follows $S_m(e)$. (Note that the conditions (4.15.1)-(4.15.2) are also satisfied).

Case 2. Assume $y_m = {}^*y_m$. Then ${}^*L_m = L_m$ a· $\sigma_V(L) = L$. By the universal mapping property of $((P_m, y_m))$ (and by the fact that the flatness has nothing to do with particular $\mathbb{C}$-algebra structure in the rings concerned), we then have $((P_m, y_m)) = ((\sigma_m(P_m), y_m))$. Hence we can choose P_m and U_m of (4.15.2) in such a way that $\sigma_m(U_m) = U_m$ and $\sigma_m(P_m) = P_m$. Then it follows that $\sigma_m(W_m^*) = W_m^*$ and $\sigma_m(E_m) = E_m$. Thus we get a σ_m-invariant (U_m, E_m), so that σ extends to $S_{m+1}(e)$.

By adding (U_m, E_m, π_m) so chosen in each of the two cases, to $S_m(e)$ and continuing this successively, we obtain an infinite sequence $S(e)$ to which σ_W extends.

The rest of the proof of (4.17) in quite analoguous to that of (4.4). The only one new point to be remarked here, is that if f_m is flat at a point $z_m \in V_m$ (resp. at every point of $L_m \cap V_m$) then so is f_m at $\tau_m(z_m)$ (resp. at every point of ${}^*L_m \cap V_m$). This is clear as an auto-conjugation is an isomorphism of ringed spaces.

For the sake of later applications, we remark also the following.

Theorem (4.18) In the theorems (4.4) and (4.17), we may also require (in addition to the already stated conditions):

5) For every α, if $S_\alpha = \{(U_{\alpha i}, E_{\alpha i}, \pi_{\alpha i})\}_{0 \leq i < m}$ and if $f_{\alpha i} : V_{\alpha i} \to W_{\alpha i}$ denote the strict transforms of f along this sequence S_α, $0 \leq i \leq m$, then $f_{\alpha i}^{-1}(E_{\alpha i}) \to E_{\alpha i}$, induced by $f_{\alpha i}$, is flat at every point of

$f^{-1}_{\alpha i}(E_{\alpha i})$ that corresponds to a point of $L \subset V$.

<u>Proof</u>. In the case of (4.4), we can simply choose U_m of (4.15.2) so small that f_m restricted to $f_m^{-1}(P_m)$ is flat over P_m at every point corresponding to a point of L. As for the case of (4.17), there are two cases as was shown in its proof. In Case 1, choose N sufficiently small (and restrict P_m to it), while in Case 2, choose U_m sufficiently small but subject to the condition $\sigma_m(U_m) = U_m$.

§ 5. Singularities of analytic spaces.

Let X be a complex-(resp. real-) analytic space. For a point $x \in X$, the local ring $\mathcal{O}_{X,x}$ of X at x is noetherian and hence it has a finite Krull dimension, i.e., the maximal integer $n \geqslant 0$ such that there exists a chain of prime ideals:

$$(0) \subset P_0 \subsetneqq P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n \subset \max(\mathcal{O}_{X,x})$$

(where max denotes the maximal ideal). Here if n is the Krull dimension of $\mathcal{O}_{X,x}$, then it is automatic that $P_n = \max(\mathcal{O}_{X,x})$. It is known from a theory of noetherian local rings, that the Krull dimension n of $\mathcal{O}_{X,x}$ is the smallest integer ≥ 0 such that there exists a system of n parameters for $\mathcal{O}_{X,x}$. Here a system of parameters $(f_1,\dots,f_n)$ of a noetherian local ring A means that $f_i \in \max(A)$ and $(f_1,\dots f_n)A$ is primary to $\max(A)$. The Krull dimension of $\mathcal{O}_{X,x}$ will be called Krull dimension of X at x and will be denoted by $K\text{-}\dim_x(X)$.

Definition (5.1) A complex-(resp. real-) analytic space F will be said to be finite if $|F|$ contains only a finite number of points and for every point $x \in F$, the local ring $\mathcal{O}_{F,x}$ is artinear (i.e., it satisfies the descending chain on ideals, or equivalently in this case, it is of finite dimension when viewed as a vector space over the constant field $K = \mathbb{C}$ (resp. $= \mathbb{R}$)). A morphism $f: X \longrightarrow Y$ is said to be finite if it is proper and $f^{-1}(y)$ is finite for every point $y \in Y$.

Remark (5.2) Let $(f_1,\dots,f_m)$ be a system of parameters and let N_o be any open neighborhood of x in X such that f_i extends to an element of $O_X(N_o)$ which will be also denoted by f_i, $1 \leqslant i \leqslant m$. Then there exist an open neighborhood U of 0 in K^m and an open neighborhood N of x in N_o, such that $f = (f_1,\dots,f_m)$ defines a finite morphism $X|N \longrightarrow K^m|U$.

Proof. By means of complexification, it is sufficient to prove the local finiteness of f in the complex case. The question being local, we may assume that X is a local model defined by $(V\subset\mathbb{C}^n, h_1, \dots, h_p)$ as in (1.2). Let $z = (z_1, \dots z_n)$ be the coordinate system for $\mathbb{C}^n$. Assume that $x=0\in\mathbb{C}^n$. Let $(t_1, \dots, t_m)$ be the coordinate system in $\mathbb{C}^m$. Pick $f_i' \in \mathbb{C}\{z\}$ such that f_i is the class of f_i' modulo $(h)\mathbb{C}\{z\}$. The assumption on the f_i implies that if J is the ideal in $\mathbb{C}\{t,z\}$ generated by $h_1, \dots, h_p$ and $t_i - f_i'$, $1 \leq i \leq m$, then $J + (t)\mathbb{C}\{t,z\}$ is primary to the maximal ideal. So by Weierstrass preparation theorem, there exists an element $g_1 \in J$ which is a monic polynomial in z_n with coefficient in the maximal ideal of $\mathbb{C}\{t, z_1, \dots, z_{n-1}\}$. Then $\mathbb{C}\{t, z_1, \dots z_{n-1}\} \longrightarrow \mathbb{C}\{t,z\}/(g_1)\mathbb{C}\{t,z\}$ is a finite integral extension. So if $J_1 = J \cap \mathbb{C}\{t, z_1, \dots, z_{n-1}\}$, then

$$\mathbb{C}\{t, z_1, \dots z_{n-1}\}/J_1 \longrightarrow \mathbb{C}\{t,z\}/J$$

is an injective integral extension. Hence (by going-up lemma on integral extension), $J_1 + (t)\mathbb{C}\{t, z_1, \dots, z_{n-1}\}$ is again primary to the maximal ideal.

So we can find $g_2 \in J_1$ which is a monic polynomial in z_{n-1} with coefficients in the maximal ideal of $\mathbb{C}\{t, z_1, \dots, z_{n-2}\}$. By repeating this process, we can find $g_i \in J$, $1 \leq i \leq n$, which is a monic polynomial in z_{n-i+1} with coefficients in the maximal ideal of $\mathbb{C}\{t, z_1, \dots, z_{n-i}\}$ for all i. Then we choose a polydisc $V' \subset V$ at 0 in $\mathbb{C}^n$ and a polydisc U' at 0 in $\mathbb{C}^m$, such that all the coefficients of the g_i, $1 \leq i \leq n$, extend to holomorphic functions on $U' \times V'$ and such that the g_i, so extended, all belong to the ideal sheaf generated by the h_j, $1 \leq j \leq p$, and the $t_k - f_k'$, $1 \leq k \leq m$, on $U' \times V'$. Now pick any

$U \subset U'$, so small a neighborhood of 0 that for every $\xi \in U$, the roots in z of $g_1(\xi,z) = \dots = g_n(\xi,z) = 0$ are all in V'. (This equation is viewed in $z \in \mathbb{C}^n$). Let $N_U = f^{-1}(U) \cap V'$, an open subset of X. Then f induces a finite morphism $X|N_U \to \mathbb{C}^m|U$. Given any N_0, we can make $N_U \subset N_0$ by choosing U to be sufficiently small.

Remark (5.3) Let $f : X \to Y$ be a morphism of complex spaces. Assume that we have autoconjugations σ_X (resp. σ_Y) of X (resp. Y) such that $f\sigma_X = \sigma_Y f$. In other words, f is a complexification of a real-analytic map $f^{\mathbb{R}}: X^{\mathbb{R}} \to Y^{\mathbb{R}}$. It is possible that $f^{\mathbb{R}}$ is a finite morphism but for every open neighborhood V of $X^{\mathbb{R}}$ in X, the map (of topological spaces) $f^{-1}(|Y^{\mathbb{R}}|) \cap V \to |Y^{\mathbb{R}}|$ is not proper. For instance, let X be the complex curve defined by $z_1^2 + z_2^2 - \frac{1}{2} = 0$ in the unit polydisc in $\mathbb{C}^2$ and let $f : X \to \mathbb{C}$ be the morphism by $(z_1, z_2) \mapsto z_1$. With respect to the obvious auto-conjugation, $f^{\mathbb{R}}$ is the linear projection from a circle to the first coordinate line.

Remark (5.4) Let $f : V \to W$ be a morphism of real-analytic spaces. We then say that f is relatively algebraic if we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\sim", "\varphi"'] \arrow[dr, "f"'] & V' \subset W \times \mathbb{R}^N \arrow[d, "\text{projection}"] \\ & W \end{tikzcd}$$

where φ is an isomorphism and the ideal of V' in the structure sheaf of $W \times \mathbb{R}^N$ is generated by a system of polynomials in the coordinate system of $\mathbb{R}^N$ with coefficients in $\mathcal{A}_W(W)$ (=the ring of real-analytic functions on W). For every finite morphism of real-analytic spaces, $f : V \to W$, and for every point $\eta \in W$, we can find an open neighborhood N of η in W such that the morphism $V|f^{-1}(N) \to W|N$ is relatively algebraic.

<u>Proof</u>. The question being local in W, we may assume that W is a local model in some $\mathbb{R}^m$ and $\eta = 0 \in \mathbb{R}^m$. Let $(t_1,\ldots,t_n)$ be the coordinate system in $\mathbb{R}^m$. Let $\{\xi_1,\ldots,\xi_s\} = |f^{-1}(\eta)|$. Then, by assumption, this s is finite and in each of the local rings of V at the ξ_i the ideal generated by (t) is primary to the maximal ideal. Hence, by the same argument as in the proof of (5.2), the local ring A_i of V at ξ_i is a finite module over the local ring B of W at η. So we can find an isomorphism of B-algebras

$$\theta : \bigoplus_{i=1}^{s} A_i \xleftarrow{\sim} B[z]/I$$

where I is an ideal in a polynomial ring $B[z]$ with indeterminates $z = (z_1,\ldots,z_N)$. By identifying (z) as the coordinate system on $\mathbb{R}^N$ and replacing W by its restriction to a sufficiently small neighborhood of η in W, we obtain a diagram of (5.4) in which V' is the closed real-analytic subspace defined by I (or the ideal sheaf generated by I in the structure sheaf of $W \times \mathbb{R}^N$) and the morphism φ induces the above isomorphism θ.

<u>Remark</u> (5.5) Let X be a complex-(resp. real) analytic space defined by data $(U,f_1,\ldots,f_m)$ as in (1.2)-(1.3), say $U \subset K^n$ with coordinate system $(z_1,\ldots,z_n)$. Let r be the Krull dimension of X at a point $x \in X$. Then the following conditions are equivalent to one another:

(5.5.1) $p : X \longrightarrow K^r$, induced by the projection $(z_1,\ldots,z_r) \longmapsto (z_1,\ldots z_r)$, is a local isomorphism at x.

(5.5.2) $$\operatorname{rank} \left. \frac{\partial(f_1,\ldots,f_m)}{\partial(z_{r+1},\ldots,z_n)} \right|_{z=x} \geq n-r$$

(5.5.3) there exists $(i_1,\dots,i_{n-r}) \subset (1,2,\dots,n)$ such that

a) $\det \left.\dfrac{\partial(f_{i_1},\dots,f_{i_{n-r}})}{\partial(z_{r+1},\dots,z_n)}\right|_{z=x} \neq 0$ and

b) $(f_{i_1},\dots,f_{i_{n-r}})$ generates the ideal of X in $\mathcal{O}_{K^n,x}$.

The equivalence of these conditions are known as the jacobian criterion for smoothness. The proof is immediate from either implicit function theorem or from Weierstrass preparation theorem.
Note that, x being taken as the origin $0 \in \mathbb{C}^n$, the condition a) of (5.5.3) implies that the linear terms of $(f_{i_1},\dots,f_{i_{n-r}})$ and $(z_1,\dots,z_r)$ generate the same $\mathbb{C}$-vector space as $(z_1,\dots,z_n)$. This implies, by Weierstrass theorem of preparation and division, $(f_{i_1},\dots,f_{i_{n-r}})$ is an invertible linear combination, with coefficients in $\mathbb{C}\{z\}$, of a system of $n-r$ elements of the form

$$z_{r+j} - g_j(z_1,\dots,z_n)\ ,\quad 1 \leq j \leq n-r\ ,$$

with $g \in \mathbb{C}\{z_1,\dots z_r\}$. Then follows b) of (5.5.3) and (5.5.1). The other implications are easier.

<u>Definition</u> (5.6). A complex-(resp. real-) analytic space X is said to be <u>smooth</u> at $x \in X$ if the conditions (5.5.1)-(5.5.3) are satisfied with respect to some local model of X at the point x.

<u>Remark</u> (5.6) Let X be a complex space and let x be a point of X. If X is reduced at x (i.e, the local ring $\mathcal{H}_{X,x}$ has no nilpotent elements), then within a sufficently small neighborhood, open, of x in X, the set of smooth points of X is dense in X. Moreover, the set of singular points (i.e., non-smooth points) of X is defined by a finite number of holomorphic equations (or, in other words, it is

complex-analytic) within a sufficiently small neighborhood of x in X. In fact, to see this, we may assume that X is a local model defined by $(U \subset \mathbb{C}^n, f_1, \ldots f_m)$ in the sense of (1.2) and $x=0 \in U$. Let $(z_1, \ldots z_n)=z$ be the coordinate system for $\mathbb{C}^n$. Let $\{P_\alpha\}$ be the set of minimal prime ideals containing $(f_1, \ldots, f_m)\mathbb{C}\{z\}$, $1 \leq \alpha \leq s$. Let r_α be the Krull dimension of $\mathbb{C}\{z\}/P_\alpha$ (which is the local ring of the local irreducible component of X at x, corresponding to P_α). Then let J_α be the ideal in $\mathbb{C}\{z\}$ generated by the $(n-r_\alpha)\times(n-r_\alpha)$-minors of the jacobian

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_n)}$$

Then the ideal in $\mathbb{C}\{z\}$, obtained as

$$\sum_{=1}^{s} \left[\bigcap_{j \neq \alpha} P_j \cap (J_\alpha + P_\alpha) \right].$$

defines the set of singular points of X within a sufficiently small neighborhood of $x=0 \in \mathbb{C}^n$.

Remark (5.1). If a complex space is reduced at a point x, then it is so at every point in some neighborhood of x in X. If X is reduced (at every point of X), then there exists a unique closed reduced complex subspace S of X such that X is smooth at $x \in X \Leftrightarrow x \in X-S$. This S will be denoted by $\mathrm{Sing}(X)$. If X has an auto-cojugation σ, then $\sigma(\mathrm{Sing}(X))=\mathrm{Sing}(X)$. This means that there exists a closed real-analytic subspace, denoted by $\mathrm{Sing}(X^{\mathbb{R}})$, of the real-analytic space $X^{\mathbb{R}}$ (the real part of X with respect to σ) such that $\mathrm{Sing}(X)$ with the induced auto-conjugation by σ is a complexification of $\mathrm{Sing}(X^{\mathbb{R}})$. For every point $x \in \mathrm{Sing}(X^{\mathbb{R}})$, we have

$$\begin{array}{ccc} K\text{-}\dim_x(X) & > & K\text{-}\dim_x(\mathrm{Sing}(X)) \\ \| & & \| \\ K\text{-}\dim_x(X^{\mathbb{R}}) & > & K\text{-}\dim_x(\mathrm{Sing}(X^{\mathbb{R}})) \end{array}$$

<u>Proposition</u> (5.8) Let X be a real-analytic space, countable at infinity. Then there exists a sequence of real-analytic subspaces, closed, of X, say $\{X^{(i)}\}_{0\leq i<\infty}$, which has the following properties:

1) $|X^{(0)}| = |X|$ and $X^{(i)} \supset X^{(i+1)}$ for all $i \geq 0$.

2) $\{X^{(i)}\}$ is locally finite at every point $x \in X$, i.e., there exists an open neighborhood N of x in X, such that $X^{(i)} \cap N = \emptyset$ except for a finite number of indices i,

3) $X^{(i)} - X^{(i+1)}$ is smooth (everywhere).

<u>Proof</u>. Let us pick and fix a complexification $\tilde{X}$ of X with auto-conjugation σ. Then, by the theorem of H. Cartan (2.7.3), there exists the smallest closed complex subspace $\tilde{X}^{(0)}$ of $\tilde{X}$ such that $|\tilde{X}^{(0)}| \supset |X|$. It is then clear that $\tilde{X}^{(0)}$ is reduced and $\sigma(\tilde{X}^{(0)}) = \tilde{X}^{(0)}$. By this, there exists a closed real-analytic subspace $X^{(0)}$ of X such that $\tilde{X}^{(0)}$ is a complexification of $X^{(0)}$ with respect to σ. Let us assume that we have found $\tilde{X}^{(i)}$, a closed σ-invariant complex subspace of $\tilde{X}$, and $X^{(i)}$ which is the real part of $\tilde{X}^{(i)}$ with respect to σ. We also assume that $\tilde{X}^{(i)}$ is reduced. Then let $\tilde{X}^{(i+1)}$ be the smallest closed complex subspace of $\tilde{X}^{(i)}$ such that $\tilde{X}^{(i+1)} \supset |\mathrm{Sing}(X^{(i)})|$. Again $\tilde{X}^{(i+1)}$ is reduced and σ-invariant. We let $X^{(i+1)}$ be the real part of $\tilde{X}^{(i+1)}$ with respect to σ. The sequence obtained this way, $\{X^{(i)}\}$, has the properties 1) and 3). Each $X^{(i)}$ is closed in X and for every $x \in X^{(i+1)}$ we have

$$\begin{aligned} \text{K-dim}_x X^{(i+1)} &\leq \text{K-dim}_x \mathrm{Sing}(X^{(i)}) \\ &< \text{K-dim}_x X^{(i)} \end{aligned}$$

From this, 2) follows immediately.

<u>Remark</u> (5.8.1) The sequence $\{X^{(i)}\}$ defined in the proof of (5.8) is unique for a given $(\tilde{X},\sigma)$. But, in general, it depends upon the choice of $(\tilde{X},\sigma)$ (not only upon X itself). It turns out, however, that the quoted theorem of Cartan is true also for real-analytic spaces. (Here the important point is the Cartan's privileged neighborhood theorem that must be, and can be, generalized to real-analytic spaces). So we could define $X^{(i+1)}$ to be the smallest closed real-analytic subspace of $X^{(i)}$ such that $|X^{(i+1)}| \supset |\mathrm{Sing}\, X^{(i)}|$, where, to begin with, $X^{(0)}$ is taken to be the smallest closed real-analytic subspace of X such that $|X^{(0)}| = |X|$.

<u>Definition</u> (5.8.2) Any sequence $\{X^{(i)}\}_{0 \leq i < \infty}$ having the properties of (5.8) will be called a smooth real-analytic filtration of X. Remark (5.8.1) asserts the existence of a canonical smooth filtration of a given X. (We will not make use of this existence in what follows).

Here we quote two theorems from the theory of desingularization. In the article published in Annals of Mathematics Vol. 79, 1964, under the title "Resolution of singularities of an algebraic variety over a field of characteristic zero, I-II", the following two facts are verified:

I°: Let A be a reduced excellent local ring containing a field a field of characteristic zero. Then there exists a desingularization of A in the following sense. There exists a closed subscheme Z of a projective space (in the sense of scheme) $\mathbb{P}^N_A$ over A such that

a) Z is regular (i.e., every local ring of Z is regular).

b) the projection $Z \to \mathrm{Spec}\,(A)$ is birational and isomorphic outside the singular locus of Spec (A). (This means: If U is the open subset of Spec (A) consisting of regular points, then its inverse image in Z is dense in Z and mapped isomorphically onto $\mathrm{Spec}\,(A)|U$).

II°: Let A be a regular excellent local ring containing a field of characteristic zero, and let J be a non-zero ideal in A. Then there exists $Z \subset \mathbb{P}^N_A$, closed and regular, such that

c) the projection $Z \to \operatorname{Spec}(A)$ is isomorphic outside the closed subscheme $\operatorname{Spec}(A/J)$ of $\operatorname{Spec}(A)$

d) $J\mathcal{O}_Z$ is simple everywhere, i.e., for every point $z \in Z$ we can find a regular system of parameters $(t_1, \ldots, t_n)$ of $\mathcal{O}_{Z,z}$ such that $J\mathcal{O}_{Z,z}$ is generated by a monomial $t_1^{a_1} \ldots t_n^{a_n}$ with non-negative integers a_i, $1 \leq i \leq n$.

(It follows that the inverse image of $\operatorname{Spec}(A/J)$ in Z is nowhere dense and $Z \to \operatorname{Spec}(A)$ is birational).

It is known that an analytic local ring (real or complex) is excellent. So these two theorems are applicable to such a local ring.
The results of this application to real-analytic local rings can be restated more in the language of real-analytic geometry.

Definition (5.9) A morphism of real-analytic spaces $\pi : X' \to X$ is said to be algebraic (and X' is said to be relatively algebraic over X with respect to π) if we can find a commutative diagram

$$\begin{tikzcd} X' \arrow[r, hook, "j"'] \arrow[dr, "\pi"'] & X \times \mathbb{R}^N \arrow[d, "\text{projection}"] \\ & X \end{tikzcd}$$

where j is a closed imbedding of real-analytic spaces and the isomorphic image of X' by j is defined by an ideal sheaf generated by a finite number of polynomials in the coordinate system of $\mathbb{R}^N$ whose coefficients are in $\mathcal{Q}_X(X)$ (=the ring of real-analytic functions on X).

Desingularization I. (5.10) Let X be a real-analytic space and let x be a point of X. Then there exist an open neighborhood U of x in X and a morphism of real-analytic spaces $\pi : X' \to X|U$, which is proper, surjective and algebraic, and in which X' is smooth everywhere.

In more details, given any smooth real-analytic filtration $\{X^{(i)}\}_{0 \leq i < \infty}$, we can choose π in such a way that X' is a disjoint union of smooth real-analytic subspaces $X'^{(i)}$, open and closed in X', and π induces $\pi^{(i)}: X'^{(i)} \longrightarrow X^{(i)}$ having the following properties: for each i, $0 \leq i < \infty$,

a) $(\pi^{(i)})^{-1}(\mathrm{Sing}\, X^{(i)})$ is nowhere dense in $X'^{(i)}$, and

b) $\pi^{(i)}$ induces an isomorphism

$$X'^{(i)} - (\pi^{(i)})^{-1}(\mathrm{Sing}\, X^{(i)}) \longrightarrow X^{(i)}\ U - \mathrm{Sing}\, X^{(i)}$$

<u>Remark</u> (5.10.1) By taking U small enough, we may assume that $X^{(i)} \cap U = \emptyset$ except for a finite number of indices i. For every i with $X^{(i)} \cap U = \emptyset$, we have $X'^{(i)} = \emptyset$ and hence X' is in fact a union of a finite number of $X'^{(i)}$.

<u>Remark</u> (5.10.2) Given $x \in X$ and $\{X^{(i)}\}_{0 \leq i < \infty}$ as above, we can deduce (5.10) from the quoted theorem I° applied to Spec (A_i) for each i separately, where A_i is the local ring of $X^{(i)}$ at x. (Here we take only those i with $X^{(i)} \ni x$). The application for each i with $x \in X^{(i)}$ gives us $\pi^{(i)}$ having the property a)+b). As we have only a finite number of such i, the open neighborhood U of x in X can be chosen to be common for all such $\pi^{(i)}$. X' being taken to be the disjoint union of those $X'^{(i)}$, π is surjective because $|X| = \bigcup_{i=0}^{\infty} (|X^{(i)}| - |X^{(i+1)}|)$.

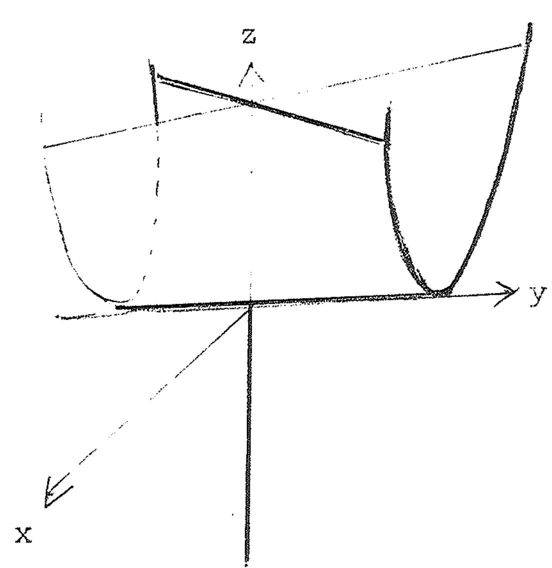
The property b) of I° implies that, when U is taken sufficiently small, $\pi^{(i)}$ is isomorphic over the open subset $|X^{(i)}| - |\mathrm{Sing}\, X^{(i)}|$, which contains $|X^{(i)}| - |X^{(i+1)}|$ for all i.

Remark (5.10.3) Each $\pi^{(i)}$ is obtained with a commutative diagram

$$\begin{tikzcd} X'^{(i)} \arrow[r, hook, "j^{(i)}"'] \arrow[dr, "\pi^{(i)}"'] & (X^{(i)}|U) \times \mathbb{P}^N_{\mathbb{R}} \arrow[d, "\text{projection}"] \\ & X^{(i)} \end{tikzcd}$$

where $j^{(i)}$ is a closed imbedding whose image is defined by a finite number of homogeneous polynomials (in terms of a fixed homogeneous coordinate system in $\mathbb{P}^N_{\mathbb{R}}$) whose coefficients are in $\mathcal{O}_{X^{(i)}}(U)$. $\pi^{(i)}$ is then algebraic in the sense of (5.9) because we can always imbed $\mathbb{P}^N_{\mathbb{R}}$ into a real number space $\mathbb{R}^{N'}$ by means of a finite number of polynomial functions on $\mathbb{P}^N_{\mathbb{R}}$.

Example (5.10.4) Let us take a surface X in $\mathbb{R}^3$ defined by $x^2 - zy^2 = 0$ with the coordinate system (x,y,z). Then $X^{(o)} = X$ and $X^{(1)} =$ the z-axis (defined by the ideal $(x,y)\mathcal{O}_{\mathbb{R}^3}$) make a smooth real-analytic filtration of X.



Let $X'^{(o)} \longrightarrow X^{(o)}$, called $\pi^{(o)}$, be the real part of the blowing-up over the complex surface defined by the same equation in $\mathbb{C}^3$, having the complex z-axis as its center. Let $\pi^{(1)}: X'^{(1)} \to X^{(1)}$ be the identity. Let X' be the disjoint union of $X'^{(0)}$ and $X'^{(1)}$, and let

$\pi : X' \to X$ be the morphism defined by $\pi^{(i)}$, $i=0,1$. Then π is a desingularization of X in the sense of Theorem (5.10). For comparison, if we replace X by $X-0$, then $X^{(o)} = X_{z>o} \cup (z\text{-axis})_{z<o}$ and $X^{(1)} = (z\text{-axis})_{z>0}$ make a smooth real-analytic filtration of the new X. A desingularization of this X, say $\pi : X' \to X$ again, is obtained by taking the disjoint union of $\pi^{(o)} : X'^{(o)} \to X^{(o)}$, which is the blowing-up with $(z\text{-axis})_{z>o}$ as its center, and $\pi^{(1)} : X'^{(1)} \to X^{(1)}$, which is the identity of $X^{(1)}$.

<u>Desingularization</u> II (5.11) Let X be a smooth real-analytic space, and let J be a coherent ideal sheaf in $\mathcal{O}_X$. Let $x \in X$. Then there exist an open neighborhood U of x in X and a morphism of real-analytic spaces $\pi : X' \to X|U$, which has the following properties:

1) π is proper, surjective and algebraic.

2) X' is smooth everywhere

3) if Y is the closed real-analytic subspace of X defined by J, then $X' - \pi^{-1}(Y)$ is dense in X' and π induces an isomorphism

$$X' - \pi^{-1}(Y) \xrightarrow{\sim} X|U - Y$$

4) $J\mathcal{O}_{X'}$ is locally simple, i.e. for every point x' of X' there exists a local coordinate system $(z_1, \ldots z_n)$ of X', centered at x', such that $J\mathcal{O}_{X',x'}$ is generated by a monomial $z_1^{a_1} \ldots z_n^{a_n}$ with non-negative integers a_i. Here n is the dimension of X' locally at x'.

<u>Remark</u> (5.11.1) The condition 4) is often expressed by saying that $\pi^{-1}(Y)$ has <u>normal crossings</u> (everywhere in X'). For the theorem (5.11), we are only interested in the case of $J_x \neq (0)$. (If $J_x = (0)$, we can choose U to be sufficiently small and X' to be empty).

<u>Remark</u> (5.11.2) Consider the case in which we are given a finite number of coherent ideal sheaves J_α in $\mathcal{O}_X$. Let Y_α be the closed real-analitic subspace of X defined by J_α for each α. Then we can find U and $\pi : X' \longrightarrow X|U$, which has the properties 1), 2) of (5.11) and

3') the condition 3) for the union Y of those Y_α for which $J_{\alpha,x} \neq (0)$,

4') $J_\alpha \mathcal{O}_{X'}$ are locally simultaneously simple for those α with $J_{\alpha,x} \neq (0)$, i.e., for every point x' of X' there exists a local coordinate system $(z_1, \dots, z_n)$ of X', centered at x', such that for every α as above the ideal $J_\alpha \mathcal{O}_{X',x'}$ is generated by a monomial in the z_i, $1 \leq i \leq n$.

This seemingly more general statement is actually an immediate consequence of (5.11). In fact, let J be the product of those J_α with $J_{\alpha,x} \neq (0)$, and apply (5.11) to this J, in which U should be so small that $J_y \neq (0)$ for all $y \notin U$.

Let X be a complex-(resp. real-) analytic space. Let $X = \bigcup_\alpha X_\alpha$ with local models X_α of complex-(resp. real-) analytic spaces as were defined in (1.2)-(1.3). Let N be the union of $|X_\alpha \times X_\alpha|$ for all α in $|X \times X|$. This N is an open subset of $X \times X$. The <u>diagonal</u> Δ_X of $X \times X$ is, by definition, the closed complex- (resp. real-) analytic subspace of $X \times X|N$ such that, for every α, the restriction of Δ_X to the open subset $|X_\alpha \times X_\alpha|$ is defined by the ideal sheaf on $X_\alpha \times X_\alpha$ generated by the elements of the form $(fp_{\alpha 1})-(fp_{\alpha 2})$ for all $f \in \mathcal{O}_{X_\alpha}(X_\alpha)$, where $p_{\alpha i} : X_\alpha \times X_\alpha \longrightarrow X_\alpha$ denote the first and second projections, $i=1,2$. Let us denote by $p_i : X \times X \longrightarrow X$, $i=1,2$, the first and the second projections. Then it is clear that each p_i induces an isomorphism $\Delta_X \xrightarrow{\sim} X$. Let J be the ideal sheaf of Δ_X on $X \times X|N$. Then J/J^2 can be viewed as a coherent sheaf on Δ_X and hence the first Projection defines

(5.12) $$\Omega_X = (p_1)_* (J/J^2)$$

This is a coherent $\mathcal{O}_X$-module and called the <u>sheaf of differentials</u> on X.

For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{F})$ can be naturally identified as the <u>sheaf of derivations</u> from $\mathcal{O}_X$ into $\mathcal{F}$. Namely, for $h \in \mathcal{O}_X(U)$ with an open subset U of X, we denote by δh the element of $\Omega_X(U)$ represented by $(hp_2)-(hp_1)$ modulo J^2. Then each element $e \in \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \mathcal{F})(U)$ defines a map

$$d_e : \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$$

by $d_e(h) = e(\delta h)$. This d_e has the usual properties of derivations, trivial on the constant field $K = \mathbb{C}$ (resp. $\mathbb{R}$). ($d_e(h_1 h_2) = h_1 d_e(h_2) + h_2 d_e(h_1)$ for $h_1, h_2 \in \mathcal{O}_X(U)$.) Moreover, every derivation $\mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$ can be obtained by an element e as above. In particular, if we take $\mathcal{F}$ to be the residue field of the local ring $\mathcal{O}_{X,x}$ with a point $x \in X$ ($\mathcal{F}$ is zero on $X-x$), we get the duality between the fibre $\Omega_X(x)$ $(=\Omega_{X,x}/\max(\mathcal{O}_{X,x})\Omega_{X,x})$ and the module of all derivations $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\max(\mathcal{O}_{X,x})$. Here $\Omega_X(x)$ and this module of derivations $\mathcal{D}er(\mathcal{O}_{X,x}, K)$ are viewed as vector spaces over K.

We can define a vector fibre space

(5.13) $$\tau_X : T_X \longrightarrow X$$

associated with the coherent $\mathcal{O}_X$-module Ω_X as follows. Let $G=\mathrm{Sym}(\Omega_X)$, the symmetric tensor algebra of the $\mathcal{O}_X$-module Ω_X. It is a graded $\mathcal{O}_X$-algebra of finite presentation, locally everywhere on X. In fact, if U is an open subset of X such that $\Omega_X|U$ is generated by a finite number of elements in $\Omega_X(U)$, say $\omega_1, \dots, \omega_s$, then $G|U$ is isomorphic to $(\mathcal{O}_X|U)[w_1, \dots, w_s]/H$, where the w_i are independent variables and H is the ideal sheaf generated by the kernel of the $\mathcal{O}_X$-homomorphism $\varphi : \sum_{i=1}^{s} (\mathcal{O}_X|U)w_i \longrightarrow \Omega_X|U$ defined by $\varphi(w_i) = \omega_i$, $1 \leq i \leq s$. So we have a well-defined complex space T_X with the projection $\tau_X : T_X \longrightarrow X$ such that for every U as above, $T_X \mid \tau_X^{-1}(U)$ is naturally imbedded in $(X|U) \times K^s$ in such a way that $(w_1, \ldots, w_s)$ is identified as the coordinate system of K^s and H generates the ideal sheaf of the isomorphic image of $T_X|\tau_X^{-1}(U)$ in $(X|U) \times K^s$.

$$\begin{tikzcd} T_X|\tau_X^{-1}(U) \arrow[r, hook] \arrow[dr, "\tau_X"'] & (X|U) \times K^s \arrow[d, "\text{projection}"] \\ & X|U \end{tikzcd}$$

(commutative diagram)

Thus we have

(5.13.1) If X is a smooth complex-(resp. real-) analytic space, then $\tau_X : T_X \longrightarrow X$ is the usual tangent vector bundle (locally trivial) of X. In this case, Ω_X is canonically identified as the dual of the $\mathcal{O}_X$-module of complex-(resp. real-) analytic sections of τ_X. In other words, it is the $\mathcal{O}_X$-module consisting of complex-(resp. real-) analytic functions on $\tau_X^{-1}(U)$ which are linear along the fibres (the fibres being vector spaces), where U denotes variable open subset of X .

(5.13.2) If U is any open subset of a complex-(resp. real-) analytic space X and $j : X|U \rightarrow Z$ is a closed imbedding into a smooth space Z, then we have a canonical closed imbedding

$$T_X \mid \tau_X^{-1}(U) \hookrightarrow T_Z$$

the image of which is defined by the ideal sheaf generated by the kernel of the natural epimorphism $\Omega_Z \longrightarrow \Omega_X | U$.

Remark (5.14) If $f : X \rightarrow Y$ is a morphism of complex-(resp. real-) analytic spaces. Then the morphism $f \times f : X \times X \rightarrow Y \times Y$ induces a morphism $f_\Delta : \Delta_X \rightarrow \Delta_Y$ and a f_Δ -homomorphism of coherent sheaves

$J_Y/J_Y^2 \longrightarrow J_X/J_X^2$, where J_Y (resp. J_X) denotes the ideal sheaf of Δ_Y (resp. Δ_X). Hence we get an f-homomorphism

$$\delta f : \Omega_Y \to \Omega_X$$

This then induces a morphism of complex-(resp. real-) analytic spaces df making a commutative diagram

$$\begin{array}{ccc} T_X & \xrightarrow{df} & T_Y \\ \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

For each $x \in X$, df induces a morphism of fibres:

$$\begin{array}{cccc} (df)_x : & T_{X,x} & \longrightarrow & T_{Y,f(x)} \\ & \| & & \| \\ & \tau_X^{-1}(x) & & \tau_Y^{-1}(f(x)) \end{array}$$

This $(df)_x$ is a homomorphism of K-vector spaces and is called the <u>differential</u> of f at x. The df, with respect to various f, has the usual functorial properties.

<u>Definition</u> (5.14.1) The dimension of the image vector space of $(df)_x$ will be called the <u>rank</u> of df at x and denoted by rank $(df)_x$.

Throughout the rest of this section, we are mostly interested in the structure of the images of proper real-analytic morphisms. The differentials of morphisms, defined above, will be used to study such images.

The first task is to find a good smooth filtration of such an image.

__Definition__ (5.14.1) The dimension of the image vector space of $(df)_x$ will be called the __rank__ of df at x and denoted by $\operatorname{rank} (df)_x$.

Throughout the rest of this section, we are mostly interested in the structure of the images of proper real-analytic morphisms. The differentials of morphisms, defined above, will be used to study such images.

The first task is to find a good smooth filtration of such an image.

__Definition__ (5.15) Let A be a subset of a real-analytic space X. By a __smooth sub-analytic filtration__ of A in X will mean a sequence $\{A^{(i)}\}_{0 \leq i < \infty}$ such that

1) $A^{(0)} = A$, $A^{(i+1)} \subset A^{(i)}$ for all $i \geq 0$,

2) $\{A^{(i)}\}_{0 \leq i < \infty}$ is locally finite in X,

3) for every i, $A^{(i)} - A^{(i+1)}$ can be given a structure of locally closed smooth real-analytic subspace of X. (Such a structure is unique).

4) for every i, there exists a proper morphism of real-analytic spaces $f_i : T_i \longrightarrow X$ such that $A^{(i)} = \operatorname{Im}(f_i)$.

__Remark__ (5.15.1) From the point of wiew of topological spaces, a smooth real-analytic filtration defined by (5.8.2) is a special case of a smooth subanalytic filtration. To be precise, let $A = |X_0|$ with a closed real-analytic subspace X_0 of X, countable at infinity. Then by (5.8), there exists a smooth real-analytic filtration $\{X^{(i)}\}_{0 \leq i < \infty}$ of X_0 in the sense of (5.8.2). If we let $A^{(i)} = |X^{(i)}|$ for every i, then $\{A^{(i)}\}$ is a smooth subanalytic filtration of A in the sense of (5.14.1).

__Remark__ (5.16) Let A be a subset of a real-analytic space X. Assume that

(5.16.1) A is a union of subsets A_i, where $\{A_i\}$ is locally finite

in X and, for every i, A_i can be given a structure of locally closed smooth real-analytic subspace of X.
Then we define the topological dimension of A at a point $x \in X$ to be

$$\text{T-dim}_x A = \inf_U \Big\{ \sup_{U \cap A_i \neq \emptyset} \dim (A_i \cap U) \Big\}$$

where U ranges through all open neighborhood of x in X and $\dim(A_i \cap U)$ is well-defined because $A_i \cap U$ is a disjoint union of smooth connected real-analytic subspaces of X. It is true that the definition of $\text{T-dim}_x A$ is independent of the choice of the expression $A = \bigcup_i A_i$ subject to the condition (5.16.1). Moreover $\text{T-dim}_x A$ is always finite (in fact; $\leq \text{T-dim}_x X$).

Proof. It is enough to prove the following

(5.16.2) Let $A' = \bigcup_j A'_j$ be another subset of X in which the union satisfies the condition (5.16.1). If $A \subset A'$, then

$$\text{T-dim}_x A \leq \text{T-dim}_x A'$$

where T-dim's are defined by the given expression as union in each case.

Note that if (5.16.2) is proven, then by taking $A=A'$ (and by symmetry) we get the uniqueness of $\text{T-dim}_x A$. Moreover, by taking $A' = |X|$ and a smooth subanalytic filtration of X in the sense of (5.15.1), we get the finiteness of $\text{T-dim}_x A$. (Namely it is bounded by $\text{K-dim}_x X$ by (5.8).)

Let us prove (5.16.2). The question being local, we may assume that X is a local model given in $\mathbb{R}^n$ in the sense of (1.3). It is clear that every connected component of A_i and A'_j has dimension $\leq n$. Hence each of the A_i and the A'_j can be expressed as a finite union (of at most n) of locally closed smooth pure-dimensional real-analytic subspaces of X. (Group together those connected components having equal dimension). Therefore, in (5.15.1), we may assume that each A_i is pure-dimensional. By the local finiteness of (5.16.1), we can find an open neigh-borhood U of x in X

such that

$$T\text{-}\dim_x A = \max_{U \cap A_i \neq \emptyset} \{ \dim A_i \}$$

and such that $U \cap A'_j = \emptyset$ except for a finite number of indices j. Pick one i, such that $T\text{-}\dim_x A = \dim A_i$ and $U \cap A_i \neq \emptyset$. If $\dim A'_j < \dim A_i$, then $A'_j \cap U \cap A_i$ is nowhere dense in $U \cap A_i$. ($U \cap A_i$ is a smooth pure-dimensional real-analytic space. $A'_j \cap U \cap A_i$ is a locally closed real-analytic subspace of $U \cap A_i$. If it is dense somewhere in $U \cap A_i$, then it must contain a non-empty open subset of $U \cap A_i$. This is impossible by $\dim A'_j < \dim A_i$). Since there are only a finite number of j with $A'_j \cap U \neq \emptyset$, we must have at least one A'_j with $A'_j \cap U \neq \emptyset$ and $\dim A'_j \geq \dim A_i$, in order to have $A_i \cap U \subset A' = \bigcup_j A'_j$. This proves (5.16.2) and hence (5.16).

Remark (5.16.3) It is clear from (5.16) that every subset A of a real-analytic space X has a well-defined $T\text{-}\dim_x A$ for each $x \in X$, if it admits at least one smooth subanalytic filtration in the sense of (5.15).

Proposition (5.17) Let $f : V \longrightarrow X$ be a proper real-analytic morphism. Assume that X is Hausdorff and countable at infinity. Assume, moreover, that

$$d = \max_{y \in V} \{ \operatorname{rank}(df)_y \} < \infty \quad .$$

Then there exists a smooth subanalytic filtration

$$\{ A^{(i)} \}_{0 \leq i \leq d}$$

of the subset $f(V)$ in X, such that for each i, $A^{(i)} - A^{(i+1)}$ $(A^{(d+1)} = \emptyset)$ has pure dimension $d-i$ (or it is empty).

The proof of (5.17) will be given after a few lemmas.

<u>Lemma</u> (5.17.1). Let $f : V \to X$ be a morphism of real-analytic spaces. Then for each integer $i \geq 0$, there exists a closed real-analytic subspace $V^{(i)}$ of V such that, for $y \in V$, we have $y \in V^{(i)}$ if and only if $\dim T_{V,y}\text{-rank}(df)_y \geq i$.

<u>Proof</u> We have a canonical f-homomorphism $\delta f : \Omega_X \to \Omega_V$, i.e., a homomorphism of $\mathcal{O}_V$-modules $f^*\Omega_X \to \Omega_V$. Let $C = \mathrm{Coker}\,(\delta f)$ which is a coherent $\mathcal{O}_V$-module. Let $J^{(i)}$ be the ideal sheaf of the annihilators in $\mathcal{O}_V$ of the exterior power $\Lambda^i C$. Then $J^{(i)}$ is coherent. Let $V^{(i)}$ be the closed real-analytic subspace of V defined by $J^{(i)}$. We claim that this $V^{(i)}$ has the property of (5.17.1). Take any point $y \in V$.
Then we have an exact sequence of fibres

$$\begin{tikzcd} \Omega_X(fy)) \arrow[r, "(\delta f)_y"] \arrow[d, equal] & \Omega_V(y) \arrow[r] & C(y) \arrow[r] & 0 \\ (f^*\Omega_X)(y) \arrow[ur] & & & \end{tikzcd}$$

(This equality means the canonical isomorphism). This exactness comes from the fact that tensor product is a right exact functor. The $(\delta f)_y$ is the dual of the homomorphism of vector spaces $(df)_y : T_{V,y} \to T_{X,f(y)}$. So $\dim T_{V,y} - \mathrm{rank}(df)_y \geq i$ if and only if $\dim C(y) \geq i$. This is true if and only if $\Lambda^i C(y) \neq (0)$. But the exterior power and taking fibre at a point commute with each other. Hence it is so if and only if $(\Lambda^i C)(y) \neq (0)$. This is clearly so if and only if $y \in V^{(i)}$ with the $V^{(i)}$ defined above. (Use Nakayama's lemma for this).

<u>Corollary</u> (5.17.2) In (5.17.1), assume that V is smooth. Then for each $j \geq 0$, there exists a closed real-analytic subspace $V_{(j)}$ of V such that, for $y \in V$, $y \in V_{(j)}$ if and only if $\mathrm{rank}\,(df)_y \leq j$.

<u>Proof</u> Apply (5.17.1) to each connected component V_α of V for the integer $i = \dim V_\alpha - j$. Here it should be noted that $\dim V_\alpha = \dim T_{V,y}$ for all $y \in V_\alpha$.

<u>Lemma</u> (5.17.3) Let $f : V \to X$ be a morphism of real-analytic spaces. Assume that V is smooth (everywhere), that f is proper and that $\mathrm{rank}(df)_y$ is equal to a constant $d \geq 0$ for all $y \in V$. Then $f(V)$ is locally a finite union of smooth real-analytic subspaces of equal dimension d at every point of $f(V)$. In particular, when X is Hausdorff, $f(V)$ has a natural structure of closed real-analytic subspace of X.

<u>Proof</u>. The second statement is an immediate consequence of the first. For the first, which is local in X, we may assume that X is smooth by taking a local model of X locally around the given point, say $x \in X$, and then by replacing it by its ambient space. If y is any point of $f^{-1}(x)$, then the implicit function theorem is applicable by the assumption on rank $(df)_y$. The theorem then implies that there exist an open neighborhood N of y in V and an isomorphism $h : N \xrightarrow{\sim} N_1 \times N_2$, where N_i are connected smooth real-analytic spaces for $i=1,2$, such that $f|N$ is given as $q\,p_1\,h$ where $p_1 : N_1 \times N_2 \to N_1$ is the projection and $q : N_1 \to X$ is a locally closed imbedding. Since f is proper, it is easy to find a finite number of such $q(N_1)$, the union of which is equal to $f(V)$ within a sufficiently small open neighborhood of x in X. Clearly each $q(N_1)$ is locally closed, smooth and of dimension $d = \mathrm{rank}\,(df)_y$.

<u>Lemma</u> (5.17.4) Under the same assumption as in (5.17.3), $f(V)$ will be viewed as a real-analytic sub-space of X in the way described in (5.17.3). Let W be the fibre product $V \times_X V$(= the inverse image of Δ_X by $f \times f - V \times V \to X \times X$) and let $g : W \to X$ be the canonical morphism. Let $g' : W' \to W$ be any surjective morphism of real-analytic spaces with

a smooth W'. Let $W'_{(d-1)}$ be the closed real-analytic subspace of W' defined by (5.17.2) with repsect to gg'. Then $gg'(W'_{(d-1)})$ contains all the singular (i.e., non-smooth) points of $f(V)$.

<u>Proof</u> Take any singular point x of $f(V)$.
Then, in view of (5.17.3), we can find two points $y_i \in f^{-1}(x)$, $i=1,2$, and an open neighborhood N_i of y_i in V, such that $f(N_i)$ is a smooth connected locally closed real-analytic subspace of dimension d in X for each i and $f(N_1) \neq f(N_2)$ within any neighborhood of x in X. Take any point $w' \in W'$ such that $g'(w') = y_1 \times y_2$. Then there exists an open neighborhood M of w' in W' such that $gg'(M) \subset f(N_1) \cap f(N_2)$. This implies, by the implicit function theorem, rank $d(gg')_w < d$. In other words, $w' \in W'_{(d-1)}$ and $x \in gg'(W'_{(d-1)})$.

<u>Remark</u> (5.17.4*) It can be easily proven that if x is a smooth point of $f(V)$, then W is smooth at every point of $g^{-1}(x)$ and rank $(dg)_y = d$ for all $y \in g^{-1}(x)$. (This is by the fact that the fibre product of any two submersions is again a submersion). So, for instance, if g' of (5.17.4) is so chosen that it is locally isomorphic above every smooth point of W, then $gg'(W'_{(d-1)})$ is exactly the singular locus of $f(V)$. Also, without using such g', we can get the exact singular locus of $f(V)$ as follows. Let $W_{(d-1)}$ be the closed real-analytic subspace of W such that for every connected component W_k of W, $W_{(d-1)} \cap W_k = W_k^{(n_k-d+1)}$ defined by (5.17.1) with respect to $g|W_k$, where n_k = T-dim W_k. Then $g(W_{(d-1)} \cup \mathrm{Sing}(W))$ is exactly the singular locus of $f(V)$. We will not use these facts in the sequel.

We need one more technical lemma.

<u>Lemma</u> (5.17.5) Let V be a paracompact real-analytic space. Then there exists a morphism $\pi : V' \longrightarrow V$ such that π is proper and surjective and

such that V' is a disjoint union of spheres (with standard real-analytic structure). Moreover, we can choose π in such a way that for each connected component V'_α of V' (which is a sphere), π induces an immersion of a dense open subset of V'_α into V. (A morphism of real-analytic spaces $f : A \to B$ is said to be an <u>immersion</u> if every point $a \in A$ admits an open neighborhood N in A such that f induces a locally closed imbedding $A|N \to B$).

<u>Proof</u>. For each point $y \in V$, let us pick an open neighborhood U_y of y in V for which a desingularization $\pi_y : V'_y \to V|U_y$ of (5.10) including the additional conditions a) and b)) exists. We may assume that U_y is Hausdorff and there exists a relatively compact open neighborhood N_y of y in U_y. Since V is paracompact and since we may always replace U_y by a smaller open neighborhood of y, and accordingly N_y, we may assume that a subset E of $|V|$ can be found in such a way that $\{U_y\}_{y \in E}$ is locally finite in V and $|V| = \bigcup_{y \in E} N_y$. The properties of π_y, as stated in (5.10), imply that $\pi_y^{-1}(N_y)$ is relatively compact and V'_y is smooth everywhere. So for each $y \in E$, we can choose a finite number of relatively compact open balls (in reference to local coordinate systems) $B_{y,\alpha}$ in V'_y such that $\bigcup_\alpha B_{y,\alpha}$ contains $\pi_y^{-1}(N_y)$. We then have an obvious real-analytic map $\lambda_{y,\alpha} : S_{y,\alpha} \to V'_\alpha$ such that $S_{y,\alpha}$ is a sphere of the same dimension as $B_{y,\alpha}$ and $\mathrm{Im}(\lambda_{y,\alpha})$ is the closure of $B_{y,\alpha}$. Now let V' be the disjoint union of all these spheres $S_{y,\alpha}$, $y \in E$, and let $\pi : V' \to V$ be the morphism defined by $\{\pi_y \lambda_{y,\alpha}\}$. This π has all the required properties of (5.17.5). (Here, to say that π is proper, it is necessary and sufficient that the images of those $\pi_y \lambda_{y,\alpha}$ are locally finite in V).

<u>Proof of</u> (5.17) The proof will be done by induction on d. Let us note, in general, that the assumption of (5.17) implies V is paracompact.

Hence, by (5.17.5), the proof of (5.17) can be always reduced to the case in which V is smooth, or in particular a disjoint union of spheres. If $d=0$ and V is smooth, then by (5.17.3) the image $f(V)$ must be a discrete set of points in X and the assertion is obviously true. Say $d > 0$ and assume that V is smooth. Let $V_{(d-1)}$ be the closed real-analytic subspace of V defined by (5.17.2) with respect to f. Let $W = V \times_X V$, the fibre product with respect to f. Let $g : W \to X$ be the canonical morphism. By (5.17.5), we have a proper and surjective morphism $g' : W' \to W$, where W' is smooth everywhere. Let $W'_{(d-1)}$ be the closed real-analytic subspace of W' defined by (5.17.2) with respect to gg'. Let V' be the disjoint union of $V_{(d-1)}$ and $W'_{(d-1)}$. Let $f' : V' \to X$ be the morphism defined by f and gg'. Then f' is proper and rank $(df')_{y'} \leqslant d-1$ for all points $y' \in V'$. Lemma (5.17.3) is applicable to $V - f^{-1}(f(V_{(d-1)})) \to V - f(V_{(d-1)})$, induced by f, and (5.17.4) is applicable to the same together with $W' - (gg')^{-1}(f(V_{(d-1)})) \longrightarrow$

$\longrightarrow W - g^{-1}(f(V_{(d-1)}))$, induced by g'. Such applications of (5.17.3)-(5.17.4) prove that $f(V) - (f(V_{(d-1)}) \cup gg'(W'_{(d-1)})) = f(V) - f'(V')$ is a smooth locally closed real-analytic subspace of pure dimension d in X. (5.19) follows from this by applying the induction assumption to the morphism f' defined above.

<u>Remark</u> (5.18) For every proper real-analytic morphism $f : V \to X$, the image set $A = f(V)$ has the property (5.16.1) locally around every point x of X. (By (5.17)). Hence we have a well defined integer $T\text{-}\dim_x A$, the topological dimension of A at x.