The zeta function

Bill Casselman
University of British Columbia
cass@math.ubc.ca

This essay is a brief introduction to the elementary analytic properties of Riemann’s zeta function and some related $L$-functions. Its primary purpose will be to motivate a treatment of Tate’s thesis.

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1. Riemann’s zeta function

Riemann’s zeta function is defined to be

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}.$$ 

The series converges for $\Re(s) > 1$, while for $s = 1$ we have the series

$$\sum_{n>0} \frac{1}{n}$$

which is well known not to converge. For reasons that will become clear after a while, the more convenient function is

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

The importance of these functions is that unique factorization of the positive integers allows us to deduce that $\zeta(s)$ may be expanded in the Euler product

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$ 

This suggests that analytic properties of $\zeta(s)$ might imply properties of the prime numbers. As a simple example, that $\zeta(1)$ sums to $\infty$ implies that there are an infinite number of primes, and even says something, albeit rather weak, about their distribution.

Theorem 1.1. The function $\xi(s)$ extends meromorphically to all of $\mathbb{C}$, is holomorphic everywhere except for two simple poles at $s = 0$ and $s = 1$, and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$
Proof. It all comes down to the consequences of Poisson summation for the function

$$\vartheta(t) = \sum_{n>0} e^{-\pi n^2 t}.$$ 

Poisson summation tells us that

$$1 + 2\vartheta(t) = \frac{1}{\sqrt{t}}(1 + 2\vartheta(1/t)).$$

The function $\vartheta(t)$ decreases exponentially at $\infty$. As $t \to 0$ it goes off to $\infty$, and this equation tells us exactly how:

$$\vartheta(t) = \frac{1}{\sqrt{t}} \frac{1 + 2\vartheta(1/t) - \sqrt{t}}{2} \sim \frac{1}{\sqrt{t}}.$$

I recall that the Gamma function is defined to be

$$\Gamma(s) = \int_0^\infty e^{-t^s} \frac{dt}{t}.$$ 

for $\Re(s) > 0$. By a change of variable $t = \pi n^2 x$ this becomes

$$\pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \int_0^\infty e^{-\pi n^2 x^s} \frac{dx}{x}.$$

This and the asymptotic behaviour of $\vartheta$ at 0 and $\infty$ tell us that for $\Re(s) > 1$

$$\xi(s) = \int_0^\infty \left( \sum_{n>0} e^{-\pi n^2 x} \right) x^{s/2} \frac{dx}{x} = \int_0^\infty \vartheta(x) x^{s/2} \frac{dx}{x}.$$

We can now write (still for $\Re(s) > 1$)

$$\xi(s) = \int_0^\infty \vartheta(x) x^{s/2} \frac{dx}{x}$$

$$= \int_1^\infty \vartheta(x) x^{s/2} \frac{dx}{x} + \int_1^\infty \vartheta(x) x^{s/2} \frac{dx}{x}$$

$$= \int_1^\infty \vartheta(1/x) x^{-s/2} \frac{dx}{x} + \int_1^\infty \vartheta(x) x^{s/2} \frac{dx}{x}$$

$$= \int_0^\infty \left( \sqrt{x} \vartheta(x) + \frac{1}{2} x^{-s/2} \right) \frac{dx}{x} + \int_1^\infty \vartheta(x) x^{s/2} \frac{dx}{x}$$

$$= \int_0^\infty \vartheta(x) x^{(1-s)/2} \frac{dx}{x} + \int_1^\infty \vartheta(x) x^{s/2} \frac{dx}{x} + \frac{1}{2} \int_1^\infty \left( x^{-(s-1)/2} - x^{-s/2} \right) \frac{dx}{x}$$

Since $\vartheta(x)$ decreases exponentially fast, the integrals converge for all $s$. Since the expression is invariant if $s$ and $1-s$ are swapped, we are through.

Furthermore, the pole of $\xi(s)$ at $s = 1$ is simple, with residue 1.
2. Gauss sums

The next major topic will be Dirichlet $L$-functions, but in the course of development Fourier transforms on the finite ring $\mathbb{Z}/N$ will appear. I could examine them when they appear, but I prefer to deal with them here.

If $f$ is any function on $\mathbb{Z}/N$, its Fourier transform is another function on $\mathbb{Z}/N$:

$$\hat{f}(n) = \sum_k f(k)e^{-2\pi i kn}.$$

Since

$$\sum_k e^{2\pi i kn} = \begin{cases} N & \text{if } n \equiv 0 \ (N) \\ 0 & \text{otherwise,} \end{cases}$$

you can check that the inverse to the Fourier transform is

$$f(n) = \frac{1}{N} \sum_k \hat{f}(k)e^{2\pi i kn}.$$

There is a Plancherel formula here, too:

$$\sum_k |f(k)|^2 = \frac{1}{N} \sum_\ell |\hat{f}(\ell)|^2.$$

This is a similar calculation.

Now suppose $\chi$ to be a multiplicative character of the group $(\mathbb{Z}/N)^\times$ of the units in $\mathbb{Z}/N$. It is called primitive if it is not lifted from some $\mathbb{Z}/M$ with $M$ a proper divisor of $N$, through the natural projection

$$\mathbb{Z}/N \to \mathbb{Z}/M.$$

This ring homomorphism induces a homomorphism from $(\mathbb{Z}/N)^\times$ to $(\mathbb{Z}/M)^\times$ whose kernel is the congruence group

$$\{x \in \mathbb{Z}/N \mid x \equiv 1 \ (M)\}.$$

**Lemma 2.1.** Suppose $\chi$ to be a multiplicative character of $\mathbb{Z}/N$. The following are equivalent:

(a) $\chi$ is a primitive character;
(b) $\chi(1 + m) = 1$ for all $m \equiv 0 \ (M)$ implies $m \equiv 0 \ (N)$;
(c) the space of complex-valued functions $f(n)$ on $\mathbb{Z}/N$ such that

$$f(cn) = \chi(c)f(n)$$

for all $c$ in $(\mathbb{Z}/N)^\times$ has dimension one.

If $\chi$ is a primitive character of $(\mathbb{Z}/N)^\times$, extend it to be a function on all of $\mathbb{Z}/N$ by setting it to be 0 off the units. With this extension, Dirichlet's function is

$$L(\chi, s) = \sum_{n>0} \frac{\chi(n)}{n^s}.$$

A basis for the space of $f$ in (c) is this extension. The Fourier transform of the extended character is

$$\hat{\chi}(n) = \sum_k \chi(k)e^{2\pi i kn/N}.$$
For any unit $c$ we have
\[
\hat{\chi}(cn) = \sum_k \chi(k)e^{2\pi i kcn/N} = \chi(c^{-1})e^{2\pi i \ell n/N} = \chi(c)^{-1}\hat{\chi}(n)
\]
so that by the previous Proposition the support of $\hat{\chi}$ is on the units. For $n$ a unit
\[
\hat{\chi}(n) = \chi^{-1}(n)\hat{\chi}(1) = \chi^{-1}(-n)\sum_k \chi(k)e^{2\pi i k/N} = \chi^{-1}(-n)\mathcal{G}(\chi).
\]
The sum
\[
\chi^{-1}(-n)\mathcal{G}(\chi) = \sum_k \chi(k)e^{2\pi i k/N}
\]
is called a Gauss sum. It follows from the Plancherel formula for $\mathbb{Z}/N\mathbb{Z}$ that $|\mathcal{G}(\chi)| = \sqrt{N}$, and from the Fourier inversion Formula that
\[
\mathcal{G}(\chi)\mathcal{G}(\chi^{-1}) = N\chi(-1).
\]

3. Dirichlet’s functions: even characters

Suppose $N > 1$ and $\chi$ a primitive character of $(\mathbb{Z}/N\mathbb{Z})^\times$. Extend it to be $0$ on integers that are not relatively prime to $N$. The associated Dirichlet $L$-function is
\[
L(\chi, s) = \sum_{n > 0} \frac{\chi(n)}{n^s}.
\]
It converges absolutely for $\text{RE}(s) > 1$ and conditionally for $\text{RE}(s) > 0$. In these regions it has the Euler product
\[
L(\chi, s) = \prod_{(p,N)=1} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \cdots \right) = \prod_{(p,N)=1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.
\]
We want to associate to this a function analogous to $\xi(s)$, but we have to be a bit careful, because things go differently according to whether $\chi(-1) = \pm 1$. I’ll look at the case $\chi(-1) = 1$ first. Things develop very much like they did for $\zeta(s)$. First of all, define
\[
\xi(\chi, s) = \pi^{-s/2}\Gamma(s/2)L(\chi, s).
\]
There is a functional equation relating $\xi(\chi, s)$ to $\xi(\chi^{-1}, s)$ but there is a complication I do not want to explain until it appears in the course of its discovery.

We first define the function
\[
\vartheta(\chi, t) = \sum_{n > 0} \chi(n)e^{-\pi n^2 t}
\]
and easily get the formula
\[
\xi(\chi, s) = \int_0^\infty \vartheta(\chi, t)t^{s/2} \frac{dt}{t}.
\]
We must next find a functional equation for $\vartheta(\chi, t)$, but it’s a bit trickier than for $\xi(s)$. We write
\[
\vartheta(\chi, t) = \sum_{\mathbb{Z}/N} \chi(k)\sum_n e^{-\pi(Nn+k)^2 t}
\]
and consider
\[ \vartheta_k(x) = \sum_n e^{-\pi(Nn+k)^2t}. \]

We therefore want to apply Poisson summation to the function \( f(x) = e^{-\pi(Nx+k)^2t} \). We know we can compute its Fourier transform, in principle at least, because of Proposition 2.4. But I'll do the calculation directly.

\[
\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi(Nx+k)^2t} e^{-2\pi i xy} dx \\
= \frac{1}{N} \int_{-\infty}^{\infty} e^{-\pi(z+k)^2t} e^{-2\pi i zy/N} dz \\
= \frac{1}{N} \int_{-\infty}^{\infty} e^{-\pi(w^2t) e^{-2\pi i (w-k)y/N} dw \\
= \frac{e^{2\pi i ky/N}}{N} \int_{-\infty}^{\infty} e^{-\pi(w^2t) e^{-2\pi i w} e^{2\pi i ky/N} 1/\sqrt{t} e^{-\pi y^2/N^2t}.}
\]

Poisson summation now tells us that
\[
\vartheta(\chi, t) = \sum_{\mathbb{Z}/N} \chi(k) \sum_n e^{-\pi(Nn+k)^2t} \\
= \frac{1}{N\sqrt{t}} \sum_{\mathbb{Z}/N} \chi(k) \sum_n e^{2\pi i kn/N} e^{-\pi n^2/N^2t} \\
= \frac{1}{N\sqrt{t}} \sum_n e^{-\pi n^2/N^2t} \sum_{\mathbb{Z}/N} \chi(k) e^{2\pi i kn/N} \\
= \frac{\Theta(\chi)}{N\sqrt{t}} \sum_n \chi^{-1}(n) e^{-\pi n^2/N^2t} \\
= \frac{\Theta(\chi)}{N\sqrt{t}} \vartheta(\chi^{-1}(n), 1/N^2t). 
\]

As with \( \xi(s) \), we eventually get:

**Proposition 3.1.** If \( \chi \) is a primitive character of \( (\mathbb{Z}/N)^\times \), then \( \xi(\chi, s) \) is a holomorphic function of \( s \), and satisfies the functional equation

\[
\varepsilon(\chi) N^{-s/2} \xi(\chi, s) = N^{-(1-s)/2} \xi(\chi^{-1}, 1-s)
\]

where
\[
\varepsilon(\chi) = \frac{\Theta(\chi)}{\sqrt{N}}.
\]

4. Dirichlet’s functions: odd characters

In this case I set
\[ \xi(\chi, s) = \pi^{(s+1)/2} \Gamma((s + 12)/2) L(\chi, s). \]

Then
\[
\varepsilon(\chi) N^{-s/2} \xi(\chi, s) = N^{(1-s)/2} \xi(\chi^{-1}, 1-s)
\]

where
\[
\varepsilon(\chi) = \frac{i\Theta(\chi)}{\sqrt{N}}.
\]
5. Some arithmetic applications

**Theorem 5.1.** There are an infinite number of primes in every arithmetic progression \(an + b\) for relatively prime \(a, b\).

The proof will reduce to this:

**Lemma 5.2.** If \(\chi\) is a primitive character of \((\mathbb{Z}/N)\times\) then \(L(\chi, 1) \neq 0\).

First I’ll show why the Lemma implies the Proposition.

In order to avoid technical difficulties, but still give some idea of how things work, I’ll just look at the cases \(a = 4\), \(b = 1\) or \(3\). where \(\chi\) is the unique character of \((\mathbb{Z}/4)\times\) of order two:

\[
\chi(n) = \begin{cases} 
1 & \text{if } n \equiv 1 \pmod{4} \\
-1 & \text{if } n \equiv -1 \pmod{4} \\
0 & \text{if } n \text{ is even.}
\end{cases}
\]

Thus

\[
L(\chi, 1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots > 0.
\]

6. The Gaussian integers

Let \(R = \mathbb{Z}[i]\). This is called the ring of **Gaussian integers** because they were first introduced by Gauss in his exploration of biquadratic reciprocity. The ring \(R\) is a principal ideal domain, and in fact a Euclidean domain, since geometry easily implies that for two integers \(a, b\), we can write

\[a = qb + r\] with \(|r| < |b|\).

The units in \(R\) are the powers of \(i\). Thus each non-trivial ideal in \(R\) is generated by a unique \(a + ib\) in the strict positive quadrant

\[Q = \{a + ib \in R | a > 0, b \geq 0\}\].

Functions analogous to \(\zeta(s)\) and \(L(\chi, s)\) may be defined by formulas very much like those for \(\mathbb{Z}:\)

\[
\zeta_R(s) = \sum_{r \in Q} \frac{1}{|r|^s}.
\]

Here \(|a + ib| = a^2 + b^2\). The analogue of Riemann’s \(\xi\) is \(\pi^{-s} \Gamma(s) \zeta_R(s)\), which satisfies a functional equation

\[X \xi_R(s) = \xi_R(1 - s)\].

A Dirichlet character of \(R\) is a multiplicative character of some finite ring \(R/N\) and then

\[L(\chi, s) = \sum_{(r,N) = 1} \frac{\chi(r)}{|r|^s}\].

Dealing with these is relatively straightforward, following the model for \(\mathbb{Z}\). But in addition there is a new type of L-function:

\[L(\mu^n, s) = \sum_{r \neq 0} \frac{\mu^n(r)}{|r|^s}\].

Here \(\mu(a + ib) = (a + ib)/\sqrt{a^2 + b^2}\) and \(n\) is in \(\mathbb{Z}\). After looking at analytic properties of \(L(\mu^n, s)\) in this section, in the next I’ll use them to prove that there are an infinite number of prime numbers of \(R\) in every angular sector.
7. Distribution of Gaussian primes

8. In terms of idèles

Let $F$ be a global field, $\mathfrak{o}$ its ring of integers. Suppose $F$ to have class number 1. In these circumstances

$$A_F^\times = F^\times \cdot \left( F^\times_{\mathbb{R}} / \mathfrak{o}^\times \right) \cdot \prod \mathfrak{o}_v^\times.$$

Thus in this case a Grössencharakter amounts to a finite collection of characters of the local units of $\mathfrak{o}_v$, together with a character of $F^\times_{\mathbb{R}}$ trivial on the image of $\mathfrak{o}^\times$. By Dirichlet’s theorem, we know exactly what this quotient is.

For any Grössencharakter $\chi$ we may define the corresponding $L$-function

$$\sum_{\alpha \neq 0 \pmod{\mathfrak{o}}^{\mathfrak{o}}} \frac{\chi(\alpha)}{|\alpha|^s}.$$

This definition has the advantage that it is very elementary.

9. References


