The group $GL_2(\mathbb{R})$ acts on the space of all symmetric $2 \times 2$ real matrices:

$$X: S \mapsto XS'X,$$

and preserves the open cone $C$ of positive definite matrices. The quotient $PGL_2(\mathbb{R}) = GL_2(\mathbb{R})/\{\text{scalars}\}$ therefore acts on the space $\mathbb{P}(C)$, the space of such matrices modulo positive scalars. In effect, $\mathbb{P}(C)$ parametrizes the shapes of ellipses in the plane. The isotropy subgroup of $I$ is the image $O(2) = O(2)/\{\pm I\}$ in $PGL_2(\mathbb{R})$, so that $\mathbb{P}(C)$ may be identified with $PGL_2(\mathbb{R})/O(2)$. The embedding of $SL_2$ into $GL_2$ identifies this with $SL_2(\mathbb{R})/SO(2)$.

The hollow cone $C_0$ of non-negative symmetric matrices of rank one that borders $C$ is also stable under $GL_2(\mathbb{R})$. To each point of $C_0$ corresponds the null line of the corresponding quadratic form, and $\mathbb{P}(C_0)$ may be identified with $\mathbb{P}^1(\mathbb{R})$, the space of lines in $\mathbb{R}^2$. This space compactifies $\mathbb{P}(C)$.

If we choose coordinates

$$\begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix}$$

for symmetric matrices, the space $C$ is where $z, z^2 - y^2 - x^2 > 0$. The intersection of this and the plane $z = 1$ is the open disc $x^2 + y^2 < 1$, which may be identified with $\mathbb{P}(C)$. There exists a Riemannian metric on $\mathbb{P}(C)$, invariant with respect to $PGL_2(\mathbb{R})$ and unique with this property, up to a positive scalar multiple. Interesting representations of $SL_2(\mathbb{R})$ are obtained on eigenspaces of the non-Euclidean Laplacian.

The space $\mathbb{P}(C)$ is the simplest of the non-compact symmetric spaces. In general, there corresponds one of these to every semi-simple real Lie group $G$. It is isomorphic to $G/K$, where $K$ is a maximal compact subgroup of $G$, and parametrizes certain involutions of the Lie algebra of $G$.

Among many remarkable parallels between the structures of real and $p$-adic groups, one of the most remarkable is that there exists an analogue of real symmetric spaces for semi-simple $p$-adic groups. These are the buildings constructed by Bruhat and Tits. Among them is the tree on which $PGL_2(\mathbb{A})$ acts, and that is what this essay is all about. I shall define it, prove some elementary properties, and show how it can be used in harmonic analysis on $SL_2(\mathbb{Q})$ and $PGL_2(\mathbb{Q})$. Very little of what I’ll say is original, but the material is widely scattered in the literature, and sometimes available in only a sketchy manner.

For groups of higher rank, buildings generalize the trees constructed here. They are important in understanding the structure of such groups, but play a very small role in analysis. Doing analysis on the tree of $SL_2(\mathbb{A})$ offers a unique opportunity to understand things intuitively.

NOTATION. Throughout this essay, let

- $\mathbb{Q} = \text{a field with a discrete valuation, assumed to be complete}$
- $\mathfrak{o} = \text{the associated ring of integers}$
- $\mathfrak{p} = \text{the maximal ideal of } \mathfrak{o}$
- $\mathfrak{m} = \text{a generator of } \mathfrak{p}$
- $q = \text{the cardinality of } \mathfrak{o}/\mathfrak{p}, \text{assumed to be finite}$. 
Thus $p = (\pi)$ is the unique prime ideal of $\mathcal{O}$. The quotient $\mathcal{O}/p$ is isomorphic to the Galois field $\mathbb{F}_q$. Every $x \neq 0$ in $\mathfrak{t}$ can be factored as $u\pi^k$ with $u$ a unit in $\mathcal{O}$. One can define a norm on $\mathfrak{t}$:

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ q^{-k} & x = u\pi^k. \end{cases}$$

This norm satisfies the non-Archimedean conditions

$$|xy| = |x||y|$$
$$|x + y| \leq \min |x|, |y|.$$

There exists on $\mathfrak{t}$ a unique Haar measure assigning $\mathcal{O}$ measure 1. For any $x$ in $\mathfrak{t}$ we have $|x|$ is the measure of $(x)$.

The ring $\mathcal{O}$ is the set of $x$ with $|x| \leq 1$. For $x \neq 0$ in $\mathcal{O}$, the index of $(x)$ in $\mathcal{O}$ is $|x|$. I shall write $\ast \pi^n$ for a general element of $\mathfrak{t}$ of norm $q^{-n}$.

The assumption that $\mathfrak{t}$ be complete means that it is complete in the topology defined by the norm. This happens if and only if $\mathcal{O}$ is the projective limit of quotients $\mathcal{O}/p^n$, and in this case $\mathfrak{t}$ is locally compact. A case in which completeness does not hold is that in which $\mathcal{O}$ is the ring $\mathbb{Z}(p)$, the ring of fractions $a/b$ with $b$ relatively prime to the prime number $p$. One point of a less restrictive assumption is that one might want to implement algorithmically some of the results presented here.

Although I shall be working with a limited range of reductive groups—the groups $\text{SL}_2$, $\text{GL}_2$, $\text{PGL}_2$ and tori contained in them—it will be convenient to use some notation from the general theory.

If $T$ is an algebraic torus, let $X_*(T)$ be the lattice of algebraic homomorphisms from the multiplicative group $\mathbb{G}_m$ to $T$, defined over an algebraic closure. If $T$ is the group of diagonal matrices in $\text{SL}_2$, for example, this is the free module of rank one spanned by the map $\alpha^\vee: x \mapsto \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}$.

If $T$ is not split over $\mathfrak{t}$ it is completely characterized by the action of the Galois group on $X_*(T)$, and the maximal split torus $A$ in $T$ is that whose group $X_*(A)$ is the sublattice whose elements are fixed by the Galois group. The group $X_*(T)$ is canonically dual to the lattice $X^*(T)$ of algebraic characters of $T$. I shall use additive notation for $X_*(T)$, so that $\lambda$ maps $x$ to $x^\lambda$.

If $G$ is one of the three groups $\text{SL}_2$, $\text{GL}_2$, or $\text{PGL}_2$, then

- $P_G$ = the subgroup of upper triangular matrices in $G$
- $A_G$ = the subgroup of diagonal matrices in $P$
- $N_G$ = the subgroup of unipotent matrices in $P$
- $\alpha_G$ = the character of the adjoint action of $A_G$ on the Lie algebra $\mathfrak{n}_G$
- $\delta_G = |\alpha_G|$
- $\overline{N} =$ the lower triangular unipotent matrices oin $G$
- $X^*_{\overline{N}}(A) = \{ \lambda \in X_*(A) \mid \langle \alpha, \lambda \rangle \leq 0 \}$
- $K_G = G(\mathcal{O})$
- $w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $H$ is any subgroup of $G$, I’ll let $H(\mathcal{O})$ be the matrices in $H$ with entries in $\mathcal{O}$, and $H(\mathcal{O}^m)$ the subgroup of $h$ in $H(\mathcal{O})$ with $h \equiv I \mod \mathfrak{p}^m$. 
I shall often leave out a subscript if the context warrants it. The valuation on $\ell$ allows one to identify $X_*(A)$ with $A/A(\sigma)$, taking $\lambda^\vee$ to $\lambda^\vee(\varpi)$. The image of $X^{-\sigma}(A)$ is contained in

$$A^{++} = \{ a \in A \mid \alpha(a) \geq 1 \}.$$ 

Thus for $SL_2$ the monoid $X_{-\sigma}(A)$ is identified with the matrices

$$\begin{bmatrix} 1/\varpi^m & 0 \\ 0 & \varpi^m \end{bmatrix} \quad (m \geq 0).$$

The notations $++$ and $--$ are adapted from Ian Macdonald, for whom $+$ and $-$ refer to obtuse rather than acute cones.

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Contents

I. Geometry
   1. Lattices
   2. The tree
   3. The action of $K$
   4. Apartments
   5. Iwahori subgroups
   6. Orbits of $N$ and the Iwasawa factorization
   7. A fixed point theorem
   8. Appendix. Using a map to navigate in the tree
   9. Appendix. Centrefold
   10. References

II. Analysis on the tree
   11. The Hecke algebra
   12. Spherical functions
   13. Deconstruction
   14. References
Part I. Geometry

The standard reference for the material in this part is Chapitre II of [Serre:1977].

1. Lattices

A lattice in \( \mathfrak{t}^2 \) is any finitely generated \( \mathfrak{o} \)-submodule that spans \( \mathfrak{t}^2 \) as a vector space, for example \( \mathfrak{o}^2 \).

1.1. Proposition. Every lattice in \( \mathfrak{t}^2 \) is free over \( \mathfrak{o} \) of rank 2.

Proof. The proof will be constructive. The lattice is finitely generated, so we may suppose given \( m \) generators of the \( \mathfrak{o} \)-submodule \( L \), and suppose \( M = M_L \) to be the \( 2 \times m \) matrix whose columns are those generators. Multiplying by matrices in \( \text{GL}_n(\mathfrak{o}) \) on the right does not change the lattice the columns generate.

The Proposition is therefore a consequence of:

1.2. Lemma. Every \( 2 \times n \) matrix of rank 2 with entries in \( \mathfrak{t} \) may be reduced through multiplication on the right by a matrix in \( \text{GL}_n(\mathfrak{p}) \) to one whose non-zero columns are of the form

\[
\begin{bmatrix}
\varpi^m & x \\
\varpi^n & \end{bmatrix}.
\]

The integers \( m, n \) are unique, and the entry \( x \) is unique modulo \( \mathfrak{p}^m \).

Proof of the Lemma. I'll specify precisely what multiplications need to be carried out. These will be what I call integral column operations.

There are three types of integral column (or, for that matter, row) operations:

(a) permuting columns (rows);
(b) multiplying one column (row) by a unit of \( \mathfrak{o} \);
(c) adding to any column (row) an integral multiple of another.

These column operations may be effected through multiplication on the right by matrices in \( \text{GL}_m(\mathfrak{o}) \), typically

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & u
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & u
\end{bmatrix},
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}.
\]

Multiplication on the right by any one of these does not change the lattice generated by the columns.

Row operations can be carried out through multiplication on the left by matrices in \( \text{GL}_2(\mathfrak{o}) \), and amount to a change of basis in \( \mathfrak{t}^2 \).

Now to start the description of the process. Because the lattice has rank at least 2, there exists at least one non-zero entry in the second row. One among them will have maximal norm, and we may swap columns if necessary to get it into the lower left corner. By an operation of type (b), we may make it \( \varpi^n \) for some \( n \), and then we may apply operations of type (c) to reduce the rest of the second row to 0.

We now look at the first row. Again beginning with a swap if necessary, possibly followed by a unit column multiplication, we may get an entry in position \((1, 2)\) of the form \( \varpi^m \) and of maximal norm in columns \( c \geq 2 \). We may then apply operations of type (c) to make the first row in columns 3 to \( m \) vanish. The only non-zero entries are now in columns 1, 2, giving us this:

\[
\begin{bmatrix}
x & \varpi^m \\
\varpi^n & 0
\end{bmatrix}.
\]

A column swap will conclude.
1.3. **Corollary.** Every matrix in \( \text{GL}_2(k) \) can be expressed as

\[
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega^m & \circ \\
\circ & \omega^n
\end{bmatrix}
k
\]

with \( k \in K \), unique \( m, n \), and \( x \) in \( \mathfrak{t} \) unique modulo \( p^m \).

Later, we shall see how this can be interpreted in terms of the geometry of the tree.

Another consequence:

1.4. **Corollary.** The group \( K \) is generated by the matrices corresponding to integral column operations.

In fact, we can be a bit more explicit.

1.5. **Proposition.** The group \( K \) is the disjoint union \( N(\omega)wP(\omega) \sqcup N(\mathfrak{p})P(\mathfrak{p}) \).

**Proof.** The following will be a basic equation in things to come:

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
det/c & a \\
1 & c
\end{bmatrix} \begin{bmatrix}
\circ & -1 \\
1 & \circ
\end{bmatrix} \begin{bmatrix}
1 & d/c \\
\circ & 1
\end{bmatrix} \quad (c \neq 0).
\]

If the matrix \( g \) lies in \( K \) and \( c \) is a unit, it implies that \( g \) lies in \( N(\omega)wP(\omega) \). If \( c \) is not a unit it will lie in \( \mathfrak{p} \), and \( a \) will be a unit. Then

\[
g = \begin{bmatrix} 1 & b \\
c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\
\circ & d - bc/a \end{bmatrix}.
\]

The group \( \text{GL}_2(\mathfrak{t}) \) acts transitively on bases of \( \mathfrak{t}^2 \), hence also on the set of lattices. The stabilizer of \( \omega^2 \) is \( K \), so with that choice of base lattice the set of lattices may be identified with \( \text{GL}_2(\mathfrak{t})/K \).

1.7. **Proposition.** Given an invertible \( 2 \times 2 \) matrix \( g \) with coefficients in \( \mathfrak{t} \), there exist matrices \( k_1, k_2 \) in \( K \) and a diagonal matrix

\[
d = \begin{bmatrix}
\omega^m & \circ \\
\circ & \omega^n
\end{bmatrix}
\]

with \( m \leq n \) such that

\[
g = k_1 dk_2.
\]

The diagonal matrix \( d \) is unique.

**Proof.** The proof is a variation on that of the previous Proposition. By column and row permutations, we may assume that the left corner entry is that of maximal norm in the entire matrix, and by a unit column multiplication we may assume it to be \( \omega^m \). Row and column operations of type (3), followed by a unit column multiplication, make it of the right form.

As for uniqueness, the greatest common divisor of the entries of the matrix is \( \omega^m \), and \( \omega^{m+n} \) is its determinant.

1.8. **Corollary.** (Principal divisor theorem) If \( L \) and \( M \) are two lattices, there exists a basis \((e, f)\) of \( L \) and integers \( m \leq n \) such that \((\omega^me, \omega^n f)\) is a basis of \( M \).

In these circumstances I call \([\omega^m : \omega^n]\) the matrix index of the pair \((L, M)\) and \(q^{m+n}\) the index. If \( m, n \geq 0 \) this last is indeed the index, the size of \( L/M \). If \( L = \omega^2 \) and \((e, f)\) form an \( \omega \)-basis of \( M \), this is also

\[
| \det [e \ f] |^{-1}.
\]
The Bruhat-Tits tree of \( G = \text{SL}_2(\mathbb{t}) \) is a graph \( X \) on which the group \( \text{PGL}_2(\mathbb{t}) \) acts. The geometry of this graph encodes much of the group structure.

- **The nodes of the tree** are the lattices in \( \mathbb{t}^2 \) modulo similarity.

These are the analogues of the points of the real symmetric space. One point of similarity (so to speak) is that a point of the real symmetric space corresponds to a Euclidean metric (modulo similarity) on \( \mathbb{R}^2 \), whereas the choice of a lattice \( L \) in \( \mathbb{t}^2 \) determines a norm on \( \mathbb{t}^2 \):

\[ \|v\|_L = \inf_{v \in L} |c| \, . \]

In effect, the choice of \( L \) here is roughly the same as specifying a unit (elliptical) disk in the Euclidean case.

For each lattice \( L \) let \( \langle L \rangle \) be the corresponding node of the tree, or in other words its equivalence class, the set of lattices \( \{\lambda \circ L \} \).

If \( L \) and \( M \) are lattices, the principal divisor theorem asserts that we may find a basis \( (e, f) \) of \( L \) such that \( (\lambda^m e, \lambda^n f) \) is a basis of \( M \), for some integers \( m \leq n \). The difference \( n - m \) is an invariant of the similarity class of \( M \), so that the definition \( \text{inv}(\langle L \rangle; \langle M \rangle) = n - m \) makes sense. This invariant is 1 if and only if the two nodes possess representatives \( L \) and \( M \) with \( L/M \cong \mathfrak{o}/\mathfrak{p} \), or equivalently

\[ \mathfrak{w} L \subset M \subset L \, . \]

In this case, I’ll call them **neighbours**.

- **There is an edge** of the Bruhat-Tits tree between two nodes if and only if they are neighbours.

The nodes linked by an edge to \( \langle L \rangle \) thus correspond to lines of \( L/\mathfrak{w} L \cong (\mathbb{F}_q)^2 \), and there are \( q + 1 \) of them.

If \( u \) and \( v \) form a basis of \( \mathfrak{t}^2 \), let \( [u, v] \) be the lattice they span and \( \langle u, v \rangle \) the corresponding node. Fix basis vectors and particular nodes

\[
\begin{align*}
u_0 &= (1, 0) \\
v_0 &= (0, 1) \\
\nu_m &= \langle \lambda^m u_0, v_0 \rangle = \langle u_0, \lambda^{-m} v_0 \rangle \quad (m \in \mathbb{Z})
\end{align*}
\]
The Bruhat–Tits tree

so that \( \nu_0 \) is the equivalence class of \( \sigma^2 \).

Conventionally, the space \( \mathbb{P}^1(\mathbb{F}_q) \) may be identified with the union of a copy of \( \mathbb{F}_q \) and a point called \( \infty \). The first are the lines in \( \mathbb{F}_q^2 \) through the points \((x, 1)\), and the second the single line through \((1, 0)\). This implies:

- The neighbours of \( \langle \langle u, v \rangle \rangle \) are \( \langle \langle \sigma u, v + xu \rangle \rangle \) and the \( \langle \langle u, \sigma v \rangle \rangle \) as \( x \) ranges over \( \mathbb{F}_q^* \).

If \( g \) is in \( \text{GL}_2(\mathfrak{o}) \), it takes a lattice \([u, v]\) to the lattice \([gu, gv]\). The group \( \text{GL}_2(\mathfrak{o}) \) preserves equivalence of lattices, and it also preserves the lattice pair invariant. Hence it transforms edges to edges, and therefore acts on the graph \( X \). By definition, this action factors through \( \text{PGL}_2(\mathfrak{o}) \). The group \( \text{PGL}_2(\mathfrak{o}) \) acts transitively on nodes of the tree. The stabilizer in \( \text{PGL}_2(\mathfrak{o}) \) of the node \( \nu_0 \) is the maximal compact subgroup \( \text{PGL}_2(\mathfrak{o}) \), which is therefore the analogue in \( \text{PGL}_2(\mathfrak{o}) \) of the image of \( \text{O}(2) \) in \( \text{PGL}_2(\mathbb{R}) \).

If \( \alpha = \left[ \begin{array}{cc} 1 & \sigma \\ \sigma & \varpi \end{array} \right] \) then \( \alpha(\nu_n) = \nu_{n+1} \) for all \( n \).

The principal divisor theorem gives us the Cartan decompositions

\[ G(\mathfrak{o}) = KA^{++}K \]

and hence a bijection of \( K \setminus G(\mathfrak{o}) / K \) with \( X^{--}(A) \).

Suppose that \( L = \sigma^2 \) and that the matrix index of \( [L; M] \) is \([\varpi^m; \varpi^n]\). I call \( [M] \) even or odd depending on the parity of \( n - m \). The action of \( \text{SL}_2(\mathfrak{o}) \) preserves this parity, and in fact there are exactly two orbits of the group \( \text{SL}_2(\mathfrak{o}) \) among the nodes of the tree, each one corresponding to lattices of a given parity.

A chain in the tree \( X \) is a finite or half-infinite sequence of nodes linked by edges. Every chain may be represented by a sequence of lattices \( L_0 \supset L_1 \supset \cdots \supset L_n \supset L_{n+1} \cdots \) with \( L_n \supset L_{n+1} \supset \varpi L_n \) for all \( n \). A standard chain is one of the form

\[ \nu_0 \rightarrow \nu_1 \rightarrow \nu_2 \rightarrow \cdots \]

whether finite or infinite. I’ll call a chain simple if, like this one, it does not back-track.

2.1. Proposition. Every simple chain in the building may be transformed to a standard one by an element of \( \text{GL}_2(\mathfrak{o}) \).

Proof. The proof is by induction on the length of the associated chain of lattices

\[ L_0 \supset L_1 \supset \cdots \supset L_n \]

in which we may assume \( L_k \supset L_{k+1} \supset \varpi L_k \) for all \( k \). It will be constructive.

Since \( \text{GL}_2(\mathfrak{o}) \) acts transitively on nodes, we may assume that \( L_0 = \sigma^2 \).

If \( n = 1 \), the image of \( L_1 \) in \( L_0 / \varpi L_0 \) is a line. We can find a matrix \( \mathfrak{g} \) in \( \text{GL}_2(\mathbb{F}_q) \) transforming it to the line through \((1, 0)\), and if \( g \) in \( \text{GL}_2(\mathfrak{o}) \) has image \( \mathfrak{g} \), then \( g L_1 \) is \([\varpi, 1]\), corresponding to the node \( \nu_1 \).

The first part of the Proposition will now follow from this:
2.2. Proposition. Any simple chain that starts out $\nu_0 \rightarrow \nu_{-1}$ may be transformed to the standard chain with the same initial edge by an element of the form

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

with $x$ in $p$. Any simple chain that starts out $\nu_{-1} \rightarrow \nu_0$ may be transformed to the standard chain with the same initial edge by an element of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with $x$ in $o$.

The two cases are equivalent, since

$$\begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix}$$

swaps $\nu_0$ and $\nu_{-1}$. If the chain is finite, this will follow by induction from:

2.3. Lemma. Suppose given a chain $(L_i) (0 \leq i \leq n + 1)$ with $L_i = [\omega^i, 1]$ for $1 \leq i \leq n$. There exists $x \in p^n$ such that

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

takes every $L_i$ to $[\omega^i, 1]$. Implicit in this statement is that when $x$ lies in $p^n$ this matrix takes $[\omega^i, 1]$ to itself for every $i \leq n$.

Proof of the Lemma. We know that the lattices linked to $\nu_0$ are $\langle \langle 1, \omega \rangle \rangle$ and the $\langle \langle \omega u, xu + v \rangle \rangle$ as $x$ ranges over $o/p$. Translating by $\alpha^{m}$, we see that the lattices linked to $\nu_m$ are those with normal forms

$$\begin{bmatrix} \omega^m & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \omega^m \\ \omega^{m+1} & 0 \end{bmatrix}$$

as $x$ ranges over $p^m/p^{m+1}$. The first one amounts to a backtrack. For the last set, the Lemma is clear.

To conclude the proof of Proposition 2.1: since $\mathfrak{t}$ is complete, the product of the matrices

$$\begin{bmatrix} 1 & x \nu \\ 0 & 1 \end{bmatrix}$$

found inductively will then converge.

2.4. Corollary. The distance $|x:y|$ between two nodes $x$ and $y$ is the pair invariant $\text{inv}(x:y)$.

Only a short additional argument is necessary to prove:

2.5. Corollary. The graph $\mathfrak{X}$ is a connected tree.

Proof. If $M$ is any lattice, we may find a basis $(e, f)$ of $L = \mathfrak{d}^2$ such that some $(\omega^m e, \omega^m f)$ is a basis of $M$. Replacing $M$ by some multiple of itself, then we may assume $m = 0, n \geq 0$. That means that there exists a chain of lattices $[u_0, x^k u_0]$ from $L$ to $M$. This proves that the graph is connected. That it is a tree follows from the preceding Proposition, since no standard chain has a loop.

The node $\nu_0$ may be chosen as root. The structure of $\mathfrak{X}$ is completely determined by the properties: (a) it is connected; (b) it is a tree; (c) every node has $q + 1$ neighbours. For example, when $q = 2$ it looks like this:
3. The action of \( K \)

The group \( K = \text{GL}_2(\mathfrak{o}) \) fixes the point \( \nu_0 \). How does it act on the tree?

The nodes at distance 1 from \( \nu_0 \) may be identified with the non-backtracking paths in the tree of length \( n \), or equivalently with the 'lines' of \( (\mathfrak{o}/p^n)^2 \) isomorphic to \( \mathfrak{o}/p^n \). These are the 'lines' spanned by elements \( \lambda = (x, y) \) in which at least one of them is a unit. They may be partitioned into two pieces, one for which \( y \) is a unit and those for which \( y \) lies in \( p \) but \( x \) is a unit. These are in bijection with \( (x, 1) \) with \( x \) in \( p/p^n \) and \( (1, y) \) with \( y \) in \( p \).

To such a pair \( \lambda \) corresponds the lattice \( L_\lambda = \mathfrak{o} \lambda + \omega^m \mathfrak{o}^2 \), and then in turn the node \( \langle \langle L_\lambda \rangle \rangle \).

3.1. Proposition. The group \( K \) acts transitively on \( \mathbb{P}^1(\mathfrak{o}/p^n) \).

Proof. This follows from Proposition 1.5.

3.2. Corollary. The map taking \( \lambda \) to \( \langle \langle L_\lambda \rangle \rangle \) is a \( K \)-equivariant bijection of \( \mathbb{P}^1(\mathfrak{o}/p^n) \) with the nodes of \( X \) at distance \( m \) from \( \nu_0 \).

The distance between \( \nu_0 \) and \( \nu_m \) is \( |m| \). The Cartan decomposition implies that \( K = \text{GL}_2(\mathfrak{o}) \) acts transitively on the \( q^{m-1}(q + 1) \) nodes at distance \( m \) from \( \nu_0 \). This is true for \( \text{SL}_2(\mathfrak{o}) \) as well.

The fixed points of the congruence group

\[
K_m = \{ g \in \text{GL}_2(\mathfrak{o}) \mid g \equiv I \pmod{p^m} \}
\]

are those at distance \( \leq m \) from \( \nu_0 \). Any other node may be transformed by \( k \) in \( K_0 \) to some \( \nu_n \) with \( n > m \). The path from \( \nu_n \) to \( \nu_0 \) intersects the fixed points at node \( \nu_m \).

According to Lemma 2.3:
3.3. Proposition. For \( m \leq n \), the \( K_m \)-orbit of \( \nu_n \) is the set of all nodes at distance \( n \) from \( \nu_0 \) and \( n - m \) from \( \nu_m \).

That is to say, at distance \( n - m \) from \( \nu_m \) and on the outside of the disk fixed by \( K_m \).

The group \( K = SL_2(\mathfrak{o}) \) fixes \( \nu_0 \), representing the lattice \( \mathfrak{o}^2 \), while its twin \( K^* = \alpha K \alpha^{-1} \), with \( \alpha = \begin{bmatrix} 1 & 0 \\ 0 & \infty \end{bmatrix} \), fixes its neighbour \( \alpha(\nu_0) = \nu_1 \). Since every compact subgroup fixes some lattice, these two subgroups of \( SL_2(\mathfrak{t}) \) are maximal compact. They are not conjugate to each other.

4. Apartments

A branch from a node is an infinite simple chain starting at that node. One branch is the chain \( B_0 \) made up of the nodes \( \nu_m \) for \( m \leq 0 \), and another is the chain \( B_\infty \) of the \( \nu_m \) with \( m \geq 0 \). Proposition 2.1 says that any branch can be transformed to \( B_0 \) by an element of \( GL_2(k) \), and hence that \( GL_2(k) \) then acts transitively on branches.

An apartment is the union of two branches from one node with no common edge. One apartment is

\[ A = B_0 \cup B_\infty = \{ \nu_m \mid m \in \mathbb{Z} \} \]

THE ACTION OF G. Elements of \( G \) take apartments to apartments.

4.1. Proposition. The group \( GL_2(\mathfrak{t}) \) acts transitively on apartments.

Proof. It suffices to prove this when one of the apartments is \( A \). Suppose given some other apartment \( \chi \), say with two branches \( \chi_0 \) and \( \chi_\infty \) running out in opposite directions from the same node. Since \( GL_2(\mathfrak{t}) \) acts transitively on branches, we may transform \( \chi_\infty \) to the branch \( B_\infty \). In effect, we may assume \( \chi_\infty = B_\infty \). By Lemma 2.3 we may now find a matrix

\[
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\]

with \( x \) in \( \mathfrak{o} \) that transforms the other branch \( \chi_0 \) of \( X \) into the other branch \( B_0 \) of \( A \). But these matrices fix the all the nodes on \( B_\infty \), so \( X \) is taken to \( A \).

The limit of the lattices \([u_0, \infty^n v_0]\) as \( n \to \infty \) is the line in \( \mathbb{t}^2 \) through \( u_0 \). It is called the end of the chain \( B_\infty \).

The notation for (say) \( B_\infty \) is motivated by this observation, since by convention this line is expressed as \( \infty \) in \( \mathbb{P}^1(\mathfrak{t}) \). Every point of \( \mathbb{P}^1(\mathfrak{t}) \) is the end of some branch, and every infinite branch ends at a point of \( \mathbb{P}^1(\mathfrak{t}) \). The parallel with what happens for \( SL_2(\mathbb{R}) \) is striking.

Since \( GL_2(\mathfrak{t}) \) acts transitively on apartments, every apartment is stabilized by a single split torus. If its ends are \( \lambda \) and \( \mu \) in \( \mathbb{P}^1(\mathfrak{t}) \), these lines are the eigenspaces of that torus. The apartment can be characterized as containing all the nodes corresponding to lattices that split compatibly with the direct sum \( \lambda \oplus \mu \).

Here is an indication of a graphical rendering of the apartment \( A \):
If \([a, b]\) is an interval in \(\mathbb{Z}\), let \(A[a, b]\) be
\[
\nu_a \ldots \nu_b.
\]

It is a matter of convention which infinite geodesic I choose to be standard, since all are equivalent. The choice I have made is the conventional one, and is convenient for visualization.

Proposition 4.1 also implies:

**4.2. Proposition.** Given two apartments and an oriented edge in each, there exists \(g\) in \(\text{GL}_2(k)\) inducing an isometry of one with the other mapping one oriented edge to the other.

The stabilizer in \(K\) of the node \(g \nu_0\) is \(gKg^{-1} \cap K\). This is the same as
\[
\begin{bmatrix} a & b \\ \omega^n c & d \end{bmatrix} \quad \text{if} \quad g = \begin{bmatrix} 1 & 0 \\ 0 & \omega^n \end{bmatrix} \quad (n \geq 0)
\]
with \(a, b, c, d\) in \(\circ\). As \(n\) gets larger and larger, this has as limit the group \(K \cap P\), and this brings out again that asymptotically the building is isomorphic to \(K/K \cap P\) or \(G/P\). More precisely, we can see that the points at distance \(m\) from \(\nu_0\) correspond naturally to the points of \(K_m = \text{PGL}_2(p^m) \setminus \mathbb{P}^1(k)\).

It turns out that \(\text{PGL}_2(\mathbb{F})\) is the group of all isometries of \(\mathfrak{X}\). (Hint: recall that \(\text{GL}_2(k)\) acts transitively on \(\mathbb{P}^1(\mathbb{F})\).)

**COMPACTIFICATION.** One can compactify the tree by adding all points in \(\mathbb{P}^1(\mathbb{F})\). **What is the topology of the union?** For points in the tree, a basis of neighbourhoods is just a basis in the tree itself (the interior of the enlarged space). Now suppose \(u\) spans a line \(\ell_u\) in \(\mathbb{F}^2\). There is a unique branch \(\chi\) ending at \(\ell_u\) whose initial node is \(\nu_0\). If \(x\) is any node of that branch, define \(\Sigma_x\) to be the set of all branches starting at \(x\) and not having any points in common with a point of the geodesic going back from \(x\) to \(\nu_0\). The union of all the points on
branches in $\Sigma_x$ together with their ends make up a neighbourhood of $\ell_u$, and as $x$ varies along $\chi$ towards its end we get a neighbourhood basis.

The action of $\text{GL}_2(\mathfrak{f})$ on the tree extends to one on the compactification, compatible with that on $\mathbb{P}^1(\mathfrak{f})$.

**THE STRUCTURE OF AN APARTMENT.** Let

$$A = \text{ the group of diagonal matrices in } \text{GL}_2(\mathfrak{f}).$$

Elements of $A$ act as translations on $\mathcal{A}$. The compact subgroup $A(\sigma)$ acts trivially on it, so the action factors through $A/A(\sigma)$. The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$$

translates $\nu_m$ to $\nu_{m-1}$. Since

$$\begin{bmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & \omega^{-2m} \end{bmatrix} \text{ modulo scalar matrices},$$

the subgroup $A_1 = A \cap \text{SL}_2(\mathfrak{f})$ shifts by an even number of nodes. The element

$$\sigma = \begin{bmatrix} \omega & -1 \\ 1 & \circ \end{bmatrix}$$

also takes $\mathcal{A}$ to itself, reflecting $\nu_m$ to $\nu_{-m}$. The group generated by $A_1/A(\sigma)$ and $\sigma$ is $N_G(A)/A(\sigma)$. Its quotient by $A(\sigma)$ is the affine Weyl group $W_{\text{aff}}$ of the root system of $\text{SL}_2$. It is a Coxeter group with elementary reflections $\sigma$ and

$$\begin{bmatrix} 1 & \circ \\ \circ & \omega \end{bmatrix}.$$ 

It contains all reflections in the nodes $\nu_m$ of even parity. The segment $\nu_0 - \nu_1$ is a strict fundamental domain for the action of $W_{\text{aff}}$ on $\mathcal{A}$.

It is not hard to see that the group generated by $A$ and $\sigma$ is precisely the stabilizer of $\mathcal{A}$.

We shall find the following useful later on:

**4.3. Proposition.** Any two branches running from $\nu_0$ but not containing any edges in $\mathcal{A}$ are taken into each other by some element of $A(\sigma)$.

**Proof.** The nodes at distance $n$ from $\nu_0$ and at distance $n$ from $\mathcal{A}$ correspond to points $(x, 1)$ in $\sigma/p^n$ with $x$ not in $\wp$. Given this, the proof should be clear.

The analogue for $\text{SL}_2(\mathfrak{f})$ is not true, as is already easy to see for nodes at distance 1 from $\mathcal{A}$ when $p$ is odd.

One feature of the apartment $\mathcal{A}$ that becomes more significant for groups of higher rank is that its structure mirrors that of the unipotent subgroup of upper triangular matrices. This group is filtered by subgroups

$$\begin{bmatrix} 1 & \wp^n \\ 0 & 1 \end{bmatrix},$$

and the set of points on $\mathcal{A}$ fixed by this subgroup consists of all those on the branch

$$\nu_{-n} - \nu_{-n+1} - \nu_{-n+2} - \cdots .$$

It is an interesting exercise to describe all the orbits of $A$ on the tree, and of $A \cap \text{SL}_2(\mathfrak{f})$. Draw a few of the latter on the picture of the tree.
5. Iwahori subgroups

5.1. Proposition. Suppose

\[ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \]

The following are equivalent:

(a) \( c \equiv 0 \mod p \);
(b) \( g \) fixes the lattice flag \([u_0, \infty v_0] \subset [u_0, v_0] ; \)
(c) \( g \) fixes all points on the edge \( \nu_0 - \nu_1 \);
(d) \( g \) lies in the intersection \( K \cap \alpha K \alpha^{-1} \).

Let \( I \) be the group of all such matrices. The conjugates of \( I \) are called \textbf{Iwahori subgroups}. Each acts trivially on exactly one edge of \( X \), but rotates branches running away from each end of that edge.

Proposition 1.5 asserts that \( K \) is the disjoint union of \( I \) and \( P(\mathfrak{o})wN(\mathfrak{o}) \).

The following is a consequence of Proposition 2.2:

5.2. Proposition. The orbits of \( I \) among the nodes of the building are the points at a fixed oriented distance from the edge \( \nu_0 - \nu_1 \).

In particular, each \( I \)-orbit among the nodes of \( \mathfrak{X} \) intersects \( A \) in exactly one node. This establishes a bijection between the orbits of \( I \) and the nodes of \( A \).

As one consequence of the description of \( I \) orbits:

5.3. Proposition. Given an apartment \( A \) and a chamber \( C \) in it, there exists a unique map \( \rho = \rho_{A,C} \) from the tree onto \( A \) with these properties:

(a) \( \rho \) is the identity on \( A \);
(b) it is an isometry on every apartment containing \( C \).

If \( I_C \) is the Iwahori subgroup fixing \( C \), then \( \rho(bx) = \rho(x) \) for all \( b \) in \( I_C \), \( x \) in the tree.

Proof. The proof is geometric. Choose a point \( y \) in the middle of \( C \). If \( x \) is an arbitrary point in the building, there exists a unique geodesic from \( x \) to \( y \). But there also exists a unique geodesic of the same length in \( A \) that agrees with the first for points inside \( C \). Map \( x \) to its endpoint.

It is interesting to figure out how to compute the retraction \( \rho \) for the standard apartment \( \mathfrak{cal} A_0 \) and chamber \( e_0 \). That is to say, given \( g \) in \( \text{PGL}_2(\mathfrak{t}) \), the retraction of \( \rho(ge_0) \) will be an edge in \( A_0 \). Which?

I’ll answer this in a strong sense by orienting the edges, with \( e_0 \) going from \( \nu_0 \) to \( \nu_{-1} \). Here the answer is given by a result that is important in representation theory. First of all, any oriented edge of \( A_0 \) is the transform of an element in the subgroup \( \Omega \) of matrices of the form

\[ \begin{bmatrix} \omega^m & 0 \\ 0 & \omega^n \end{bmatrix}, \begin{bmatrix} 0 & \omega^n \\ \omega^m & 0 \end{bmatrix} \]

which is unique up to a scalar multiple.

If \( g = b_1 \omega b_2 \) with the \( b_i \) in \( I \) and \( \omega \) in \( \Omega \), then \( \rho g e_0 \) is equal to \( \omega e_0 \). The proof of the following result will tell us how to find \( \omega \).

5.4. Proposition. Every \( g \) in \( \text{PGL}_2(\mathfrak{t}) \) may be expressed as a product \( b_1 \omega b_2 \) with the \( b_i \) in \( I \), \( \omega \) in \( \Omega \).

Proof. To go with this claim is an algorithm involving \textbf{elementary Iwahori operations} on columns:
• Add to a column $d$ a multiple $xc$ of a previous column $c$ by some $x$ in $\sigma$;
• add to a column $c$ a multiple $xd$ of a subsequent column by $x$ in $\pi$;
• multiply a column by a unit in $\sigma$;

and also on rows:
• Add to a row $c$ a multiple $xd$ of a subsequent row $d$ with $x$ in $\sigma$;
• add to a row $d$ a multiple $xc$ of a previous row with $x$ in $\pi$;
• multiply a row by a unit in $\sigma$;

Each of these column (row) operations amounts to right (resp. left) multiplication by what I’ll call an Iwahori matrix.

Here are some examples:

\[
\begin{bmatrix}
u & v \\
1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\
0 & 1 \end{bmatrix} = \begin{bmatrix} u & xu + v \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix}
u & v \\
1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\
x & 1 \end{bmatrix} = \begin{bmatrix} u + xv & v \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & x \\
0 & 1 \end{bmatrix} \begin{bmatrix} u & v \\
0 & 1 \end{bmatrix} = \begin{bmatrix} u + xv & v \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 \\
x & 1 \end{bmatrix} \begin{bmatrix} u & v \\
0 & 1 \end{bmatrix} = \begin{bmatrix} u & xv + v \\
0 & 1 \end{bmatrix}
\]

5.5. Proposition. *iwahori-partition* Any invertible $2 \times 2$ matrix can be reduced by elementary Iwahori row and column operations to a unique matrix in $\Omega$.

Let $m$ be the given matrix. First of all, it is easy to apply elementary row operations to obtain a matrix of the form

\[
\begin{bmatrix} \sigma^k & * \sigma^n \\
\sigma^\ell & \sigma^k & * \sigma^n \end{bmatrix}.
\]

Let’s look at the two cases separately.

(1)

\[g = \begin{bmatrix} \sigma^k & u\sigma^n \\
0 & \sigma^\ell \end{bmatrix}.\]

If $k \leq n$ we can subtract $u\sigma^{n-k}$ times the first column from the second to get

\[
\begin{bmatrix} \sigma^k & 0 \\
0 & \sigma^\ell \end{bmatrix}.
\]

If $\ell \leq n$ we can subtract $u\sigma^{n-\ell}$ times the second row from the first to get the same matrix.

So now we may assume $k > n$ and $\ell > n$. Subtract from the second row $\sigma^{\ell-n}/u$ times the first. This gives

\[
\begin{bmatrix} \sigma^k & u\sigma^n \\
-\sigma^{k+\ell-n}/u & 0 \end{bmatrix}.
\]

Divide the second column by $u$, multiply the second row by $-u$:

\[
\begin{bmatrix} \sigma^k & \sigma^n \\
\sigma^{k+\ell-n} & 0 \end{bmatrix}.
\]
Subtract $w^{k-n}$ times the second column from the first to get

$$\begin{bmatrix} 0 & w^n \\ w^{k+\ell-n} & 0 \end{bmatrix},$$

which is Iwahori reduced.

(2)

$$g = \begin{bmatrix} 0 & w^\ell \\ w^k & u w^n \end{bmatrix},$$

where $u$ is a unit.

If $n > \ell$ or $n \geq k$ this can be reduced to

$$\begin{bmatrix} 0 & w^\ell \\ w^k & 0 \end{bmatrix}.$$

So now we assume $n \leq \ell$, $n < k$. Subtract $w^{k-n}/u$ times the second column from the first:

$$\begin{bmatrix} -w^{k+\ell-n}/u & w^\ell \\ w^{k+\ell-n}/u & u w^n \end{bmatrix}.$$

Multiply and divide by $u$:

$$\begin{bmatrix} w^{k+\ell-n} & w^\ell \\ 0 & w^n \end{bmatrix}.$$

Subtract $w^{\ell-n}$ times the second row from the first:

$$\begin{bmatrix} w^{k+\ell-n} & 0 \\ 0 & w^n \end{bmatrix}.$$

This retraction has one important feature. If $x$ and $y$ are two points on the tree, the geodesic between them retracts onto a polygonal line on $A$, so that

$$|\rho(x) : \rho(y)| \leq |x : y|.$$

6. Orbits of $N$ and the Iwasawa factorization

As $m \to \infty$ the lattice $[1, w^m]$ passes off to the line through $(1, 0)$ in $\mathbb{P}^1(\mathbb{F})$. In other words, the group $N$ of all upper triangular unipotent matrices fixes the end of the branch $\{ \nu_m \mid m \geq 0 \}$, which amounts to $\infty$ in $\mathbb{P}^1(\mathbb{F})$. There is a finite approximation of this phenomenon. Let $N(p^m)$ be the subgroup of

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with $x \in p^m$. The following is elementary, but useful to refer to.

6.1. Proposition. (a) Elements in $N(p^m)$ fix all nodes $\nu_k$ with $k \geq m$. (b) The $N(p^{-m})$-orbit of $\nu_n$ for $n < m$ are the points $x$ at distance $m-n$ from $\nu_m$ other than those on a path starting back to $\nu_{m+1}$.

For $m \leq 0$ these are also the orbits of the Iwahori subgroup $I$.

As $m \to \infty$ the group $N(p^{-m})$ expands, consistently with what happens at the end point. Claim (a) is trivial. For (b) look at Lemma 2.3.
There is an important relation between the Cartan and Iwasawa factorizations. I recall first what happens for \( G = \text{SL}_2(\mathbb{R}) \). Let \( K = \text{SO}(2) \), \( N \) be the group of unipotent upper triangular matrices, \( A \) be the group of diagonal matrices, and \( P = AN \).

The **Cartan factorization** asserts that \( G = KAK \). Geometrically things are simple. We first represent \( G \) by Möbius transformations of the unit disk, conjugating the more familiar action on the upper half plane by the Cayley transform. If \( g = k_1ak_2 \) then it is also \( k_1a^{-1}k_2 \). Choose \( a \) so that \( r = a(O) \) lies in the interval \((0,1)\). Then \( g(P) = k_1a(O) \) will lie at angle \(-2\theta\) on the circle of radius \( r \) around \( O \) if

\[
k_1 = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]

The **Iwasawa factorization** asserts that \( G = NAK \). There is a simple geometric description here, too. If \( g = nak \) then \( g(O) = na(O) \), which is on the \( N \)-orbit of \( r \). The \( N \)-orbits are the circles inside the unit disk and tangent to 1. So we find the circle of this sort which passes through \( g(O) \), and then find where it intersects the real line.

**How do these different factorizations compare?** The answer is that if \( g(O) \) lies on the circle of radius \( r \) around \( O \) and on the \( N \)-orbit through \( \rho \) then \(-r \leq \rho \leq r\).
Here is the generalization of this for $\text{SL}_2(\mathbb{k})$:

**6.2. Proposition.** Suppose $g$ in $\text{GL}_2(\mathbb{k})$. Then

(a) if $g = nak$ is its Iwasawa factorization, then it has Cartan factorization $g = k_1 dk_2$ with $a(\nu_0)$ in the convex hull in $\mathcal{A}$ of (a.k.a. line segment between) $d(\nu_0)$ and $d^{-1}(\nu_0)$;

(b) if $\nu$ lies in $\mathcal{A}$ then the intersection of its $K$-orbit and $N$-orbit is just $\nu$ itself.

The proof of the Proposition follows an argument suggested by this picture. It uses the retraction constructed in the last section.

**Proof.** Suppose $g = nak$. Then $n$ will fix some ray of points on the apartment $\mathcal{A}$, since $N$ fixes $\infty$. Suppose it fixes a chamber $C$. Let $\rho$ be the retraction $\rho_{\mathcal{A},C}$. Then $a$ is $\rho(x)$. The matrix $d$ is determined by the geodesic from $\nu_0$ to $x$, and the image of this path under $\rho$ has length at least that of $\rho$. But this means exactly what the Proposition asserts.

I leave claim (b) as an exercise.

A generalization of the result for arbitrary real semi-simple groups has been proved in [Kostant:1973], and this in turn has been generalized in [Atiyah:1982]. A first step towards a generalization of this for $p$-adic groups can be found in §4.4 of [Bruhat-Tits:1972] (see also Theorem 2.6.11(3)–(4) of [Macdonald:1971]), and the precise $p$-adic analogue of Kostant’s result can be found in [Hitzelberger:2010].
7. A fixed point theorem

For any two points \( x, y \) in the tree, let \( m_{x,y} \) be the midpoint of the geodesic between them. The following asserts that in some sense the tree has non-positive curvature.

**7.1. Proposition.** (Bruhat-Tits inequality) Given points \( x, y, \) and \( z \) on the tree, let \( m = m_{x,y} \). Then

\[
|z:m|^2 + |m:x|^2 \leq \frac{|z:x|^2 + |z:y|^2}{2}.
\]

Keep in mind that \( |m:x| = |m:y| \).

*Proof.* This is an equality on an apartment, according to a theorem of Pappus. It is an easy vector calculation, or can be proved by applying Pythagoras’ Theorem a few times.

In general, fix an apartment \( \mathfrak{A} \) containing \( x \) and \( y \) and let \( E \) be an edge containing \( m \). If \( \rho \) is the retraction determined by \( \mathfrak{A} \) and \( E \), then

\[
|z:x|^2 + |z:y|^2 \geq |\rho(z):x|^2 + |\rho(z):y|^2 = 2|\rho(z):m|^2 + 2|m:x|^2 = 2|m|^2 + 2|m:x|^2.
\]

Any metric space satisfying this condition is called **semi-hyperbolic**. All Bruhat-Tits buildings and all non-compact real symmetric spaces fall in this category. Any two points on a semi-hyperbolic space have a unique midpoint between them. The sphere, for example, is not semi-hyperbolic.

If \( X \) is any bounded set in the tree and \( c \) a point in the tree, there exists \( R \geq 0 \) such that \( |c:x| < R \) for all \( x \) in \( X \). Define \( R_c(X) \) to be the least upper bound of all such \( R \), and define the radius \( R_X \) of \( X \) to be the least upper bound of all \( R_c(X) \) as \( c \) varies. A circumcentre for \( X \) is a point \( c \) with the property that \( |c:x| \leq R_X \) covers \( X \). The following is an observation due to Serre.

**7.2. Corollary.** Every bounded subset of the tree has a unique circumcentre.

*Proof.* Choose a sequence \( c_i \) such that \( R_{c_i}(X) \to R(X) \). The semi-hyperbolic inequality implies that it is a Cauchy sequence.

The case we shall be interested in is that in which \( X \) is a finite set. Is there a simple algorithm to find its circumcentre?

We have now a new proof of a result we have seen before. In contrast to the earlier proof, this one can be expanded into one for all buildings.

**7.3. Corollary.** Any compact subgroup of \( \text{SL}_2(t) \) fixes some point on the tree.

*Proof.* Because it fixes the circumcentre of any orbit.

Hence the subgroups fixing nodes of the tree are maximal compact subgroups of \( \text{SL}_2(t) \), and there are two conjugacy classes of them. For \( \text{PGL}_2(t) \) there is just one.
8. Appendix. Using a map to navigate in the tree

I’ll discuss here how to draw the tree for $G = \text{SL}_2(\mathbb{Q}_2)$. This can be done at several levels of sophistication.

First I’ll describe how to draw the basic tree. This is the tree simply as a geometric object, a collection of branches, and no association with an automorphism group. There are a number of parameters that determine it—the dimensions of nodes and edges, how these should shrink with depth, how edges are arrayed around a node, and colour choice. The drawing is then done by recursion, either explicitly or implicitly, with a stack, out to some given depth. Each node is assigned an angle in, as well as location. A node draws itself, and if the specified depth has not been exceeded it then draws edges out to neighbours, and finally draws those neighbours by recursion. I’ll leave details as an exercise.

Still on the purely geometric level of drawing is a procedure for drawing nodes along a path like $\infty LRR$ as indicated in this figure:

So it is relatively simple to draw the tree as a geometric object. But for really useful (i.e. ‘intelligent’) drawings we want to translate back and forth between nodes in the drawing and lattices, or between nodes and elements of $G$. That is to say, suppose $\sigma = \mathbb{Z}(2)$, the localization of $\mathbb{Z}$ at $(2)$. We use the action of $\text{GL}_2(\mathbb{Q})$ on the tree, rather than that of the 2-adic field, because it is computationally feasible. The nodes in the tree are the same for localizations as for completions, and pretty much the only difference between the two groups is that the group over $\mathbb{Q}$ is smaller and does not act transitively on apartments.

In other words, we want to associate to each node in the geometric tree a $2 \times 2$ invertible matrix in $\text{GL}_2(k)$, and vice-versa. This means building a bijection between certain $g$ and paths like $\infty LRR$ as explained above.

The best way is to use the description of Iwahori orbits. We know how to factor every $g$ in $G$ as $b\alpha^m k$ according to $G = IAK$, where $I$ is the Iwahori subgroup. So we first write $g = b\alpha^n k$, with

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix}.$$  

The node $g\nu_n$ will be at distance $|n|$ from $\nu_0$, and in the $I$-orbit of $\alpha^n$. The map $b \mapsto b\alpha^n K$ is a bijection between this orbit and $I/I \cap \alpha^n K\alpha^{-n}$, which is not difficult to parametrize explicitly.
The Bruhat-Tits tree

The cases (1) \( n > 0 \) and (2) \( n \leq 0 \) are different.

(1) For \( n > 0 \) the subgroup \( I \cap \alpha^n K \alpha^{-n} \) is that of all integral matrices

\[
\begin{bmatrix}
a & b \\ c & d
\end{bmatrix}
\]

with \( c \in p^n \).

\[x \mapsto \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\]

from \( p/p^n \) is a bijection with \( I/I \cap \alpha^n K \cap \alpha^{-n} \).

(2) When \( n \leq 0 \), for similar reasons, the map

\[x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\]

defines a bijection of \( o/p^n \) with \( I/I \cap \alpha^n K \cap \alpha^{-n} \). The corresponding maps back from \( I \) are

\[
\begin{bmatrix}
a & b \\ c & d
\end{bmatrix} \mapsto c/a \ (n > 0), \ a/b \ (n < 0).
\]

This makes \( q^n + q^{n-1} \) in all, which is indeed the size of \( \mathbb{P}^1(o/p^n) \).

Now take \( q = 2 \). The nodes of the tree are parametrized by sequences of \( L \) and \( R \), either from the root node if \( n \leq 0 \) or from the infinite node if \( n > 0 \). So we must define the map from the section matrices parametrized by \( x \) modulo \( p^n \) to such a sequence, and vice-versa.

Suppose \( n > 0 \). We are given \( x \) as an even integer \( 2y \) modulo \( 2^n \). We find the bits of \( y \) and read from low order \( i = 0 \) up to order \( i = n - 1 \), translating bit \( i \):

\[
i = 0, 2, 4, \ldots \quad \text{odd} \mapsto L, \quad \text{even} \mapsto R
\]

\[
i = 1, 3, 5, \ldots \quad \text{even} \mapsto L, \quad \text{odd} \mapsto R
\]

Now suppose \( n \leq 0 \). We are given \( x \) as an integer modulo \( 2^{|n|} \). We find the bits of \( x \) and read from low order \( i = 0 \) up to order \( i = |n| - 1 \), translating bit \( i \):

\[
i = 0, 2, 4, \ldots \quad \text{odd} \mapsto L, \quad \text{even} \mapsto R
\]

\[
i = 1, 3, 5, \ldots \quad \text{even} \mapsto L, \quad \text{odd} \mapsto R
\]

In short, the rules are the same! They can be summarized in a table:

<table>
<thead>
<tr>
<th>bit index parity</th>
<th>bit parity</th>
<th>L or R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( R )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( L )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( R )</td>
</tr>
</tbody>
</table>

But now you can see that they can be formulated most succinctly as addition modulo 2, with \( R = 0, L = 1 \).

One final remark—it might seem at first that the map between \( LR \) paths and nodes is somewhat arbitrary. But in fact some labelings are better than others, in the sense that the geometry of the action of \( G \) looks more or less comprehensible. The one I have chosen here seems to be best. One reason for this is that the geometry of the orbits the matrices

\[
\begin{bmatrix}
t & 0 \\ 0 & 1/t
\end{bmatrix}
\]

is simple.
9. Appendix. Centrefold
10. References


Part II. Analysis on the tree

In the case of the non-Euclidean plane eigenspaces of the Laplacian $\Delta$ are important representations of $\mathfrak{sl}_2(\mathbb{R})$. Is there an analogue for the Bruhat-Tits tree? In either the Euclidean or the non-Euclidean plane we have the asymptotic formula

$$\lim_{r \to 0} \left( \frac{2}{r^2} \right) \int_{B(r)} f(x + s) \, ds - f(x) = [\Delta f](x)$$

for every smooth function $f$. This and classical results of Hecke suggest the definitions in this part.

The principal references for this material are [Cartier:1971/2] and [Cartier:1973].

11. The Hecke algebra

The integral Hecke algebras of $\text{SL}_2(\mathfrak{t})$ and $\text{PGL}_2(\mathfrak{t})$ are rings of ‘algebraic correspondences’ on the tree $X$. The definitions in these terms mimic Hecke’s original definitions of the classical operators $T_n$.

Let $C(X)$ be the space of functions on the nodes of $X$. The group $\text{GL}_2$ acts on it by the left regular representation:

$$[L_g F](x) = F(g^{-1}(x)).$$

This induces representations of both $\text{PGL}_2(\mathfrak{t})$ and $\text{SL}_2(\mathfrak{t})$.

11.1. Proposition. Composition of operators defines the structure of a ring on $\mathcal{H}_Z$. The inclusion of $T$ in $\mathcal{H}_Z$ induces an isomorphism of $\mathbb{Z}[T]$ with it.

To prove this, we must show that $\mathcal{H}_Z$ is generated by $T$ and that the powers of $T$ form a $\mathbb{Z}$-basis of it. These are both an immediate consequence of:

11.2. Lemma. We have

$$T^o T = T_2 + (q + 1) I$$

$$T^o T_m = T_{m+1} + q T_{m-1} \quad (m \geq 2).$$
Proof. As the figure above shows, every node has $q + 1$ neighbours. If $y$ is at distance $m \geq 1$ from $\nu_0$ it has $q$ neighbours at distance $m + 1$ from $\nu_0$ and 1 at distance $m - 1$. Thus, for example:

$$[T \circ T](x) = \sum_{y \sim x, z \sim y} z = \sum_{|z|=2} z + \sum_{y \sim x} x = T_2(x) + (q + 1)I(x).$$

Here $x \sim y$ means $|x:y| = 1$.

**SL(2).** Now let

$$G = \text{SL}_2(\mathfrak{f})$$
$$K = \text{SL}_2(\mathfrak{o}).$$

Since there are two orbits of $\text{SL}_2(\mathfrak{f})$ among the nodes of $X$, the representation of $\text{SL}_2(\mathfrak{f})$ on $\mathbb{C}[X]$ is the direct sum of two components, determined by support.

Let $\mathcal{H}_\mathbb{Z} = \mathcal{H}_\mathbb{Z}(G)$ now be the algebra generated by $S = T_2$ rather than that generated by $T$. For this operator we have relations that can be derived from those above:

11.3. Proposition. If $S = T_2$ then

$$S \circ S = T_4 + (q - 1)T_2 + q(q + 1)I$$
$$S \circ T_{2m} = T_{2m+2} + (q - 1)T_{2m} + q^2T_{2m-2}.$$

They can also be derived more directly from this, which the figure also illustrates:

11.4. Lemma. Suppose $m \geq 2$. Among the nodes at distance 2 from $\nu_m$ are $q^2$ at distance $m + 2$ from $\nu_0$, $q - 1$ at distance from $m$ from $\nu_0$, and 1 at distance $m - 2$ from it.

11.5. Corollary. The Hecke algebra of $\text{SL}_2$ is the polynomial ring $\mathbb{Z}[S]$. 
12. Spherical functions

I recall that an admissible representation \((\pi, V)\) of a \(p\)-adic group is one with these two properties: (1) every vector in \(V\) is fixed by some open subgroup (i.e.

B is smooth); (2) for any open subgroup, the subspace of vectors fixed by all elements in that subgroup is finite-dimensional.

If \(G\) is either \(\text{PGL}_2(\mathbb{F})\) or \(\text{SL}_2(\mathbb{F})\), the action of \(G\) on \(\mathbb{C}(\mathbb{X})\) commutes with all Hecke operators, and in consequence it acts on the space of eigenfunctions of a Hecke operator.

**PGL(2)**. Let \(G = \text{PGL}_2(\mathbb{F})\), \(K = G(\mathbb{O})\). For \(\lambda\) in \(\mathbb{C}\), let \(V(\lambda)\) be the space of functions \(f\) on the nodes of \(\mathbb{X}\) such that \(Tf = \lambda f\). This is essentially the \(p\)-adic analogue of classical conditions on eigenfunctions of the Laplacian in the Euclidean plane, except that the value of a harmonic function at a point \(P\) is the average of its values on the unit circle around \(P\), so that the true analogue of \(\Delta\) is \(I - T/(q + 1)\).

**12.1. Proposition.** The representation of \(G\) on \(V(\lambda)\) is admissible, and \(V(\lambda)^K\) has dimension 1.

The condition on an eigenfunction \(\varphi\) is that

\[
\lambda \varphi(x) = \sum_{y \sim x} \varphi(y).
\]

**Proof.** In several steps.

**Step 1.** For each \(n \geq 0\) let \(K_n\) be the congruence subgroup \(\text{PGL}_2(\mathbb{F}^n)\) (those elements congruent to \(I\) modulo \(\mathbb{F}^n\)). Suppose the eigenfunction \(\varphi\) to be fixed by \(K_n\). Let \(B_n\) be the ‘ball’ of nodes at distance \(\leq n\) from \(\nu_0\), which are all fixed by \(K_n\). I claim that the values of \(\varphi\) at all nodes at distance \(> n\) from \(\nu_0\) are determined by its values on \(B_n\).

Suppose \(x\) to be one of the nodes at distance exactly \(n\) from \(\nu_0\). Call a node \(y\) external to \(x\) if the geodesic to \(\nu_0\) passes through \(x\). This includes \(x\) itself. Then \(K_n\) fixes \(x\) and, if \(m \geq n\), acts transitively on all nodes at distance \(m\) from \(\nu_0\) and external to \(x\). Hence \(\varphi\) takes the same value, say \(\varphi_{x,m}\), at all those nodes. My earlier claim will follow from the new claim that all these \(\varphi_{x,m}\) are determined by the values of \(\varphi\) at the neighbours of \(x\), if any, inside \(B_n\).

**Step 2.** First we look at the value of \(\varphi\) at the external neighbours of \(x\) at distance 1. There are two cases, according to whether \(x = \nu_0\) or not.

Suppose \(x = \nu_0\). This happens only when \(n = 0\) and \(\varphi\) is fixed by \(K\) itself. The function \(\varphi\) takes the same value \(\varphi_m\) at all nodes at distance \(m\) from \(\nu_0\). There are \(q + 1\) neighbours of \(\nu_0\) at distance 1, so

\[
\lambda \varphi_0 = (q + 1) \varphi_1, \quad \varphi_1 = \frac{\lambda}{q + 1} \varphi_0.
\]

If \(n \geq 1\), there is one neighbour \(y\) inside \(B_n\), \(q\) outside, and we must have

\[
q \varphi_{x,n+1} = \lambda \varphi_{x,n} - \varphi(y).
\]

In either case, the values \(\varphi_{x,n+1}\) are determined by the values of \(\varphi\) inside \(B_n\).

**Step 3.** Suppose now \(m \geq n + 1\), \(y\) at distance \(m\) from \(\nu_0\) and external to \(x\). Then all neighbours of \(y\) are external to \(x\), and by Lemma 11.4

\[
\lambda \varphi_{x,m} = \varphi_{x,m-1} + q \varphi_{x,m+1}.
\]

That is to say, the function \(\varphi_{x,m}\) for \(m \geq n + 1\) satisfies the difference equation

\[
q \varphi_{x,m} - \lambda \varphi_{x,m-1} + \varphi_{x,m-2} = 0.
\]
This is a difference equation of second order. The function \( \varphi \) also satisfies initial conditions determined by its values \( \varphi_{x,n} \) and \( \varphi_{x,n+1} \).

**Step 4.** There is a well known recipe for the solution of a difference equation. Set \( \varphi_{x,m} = \alpha^m \). Plugging into the equation, we see that \( \alpha \) must be a root of

\[
x^2 - \left( \frac{\lambda}{q} \right) x + 1/q = 0.
\]

If this equation has distinct roots \( r_1, r_2 \), then the general solution is of the form \( cr_1^m + dr_2^m \). Set \( r_1 = z/\sqrt{q} \), \( r_2 = z^{-1}/\sqrt{q} \) where now we require that \( z + z^{-1} = \lambda/\sqrt{q} \). This makes the solution

\[
\varphi_{x,m} = q^{-m/2}(cz^m + d z^{-m})
\]

for constants \( c, d \) satisfying the given initial conditions near the boundary of \( B_n \).

**Step 5.** The exceptional case is when the roots are equal, which happens when \( \lambda = \pm 2 \). In this case \( r = \pm 1/\sqrt{q} \) is the unique root of the characteristic equation. The solutions of the difference equation are linear combinations of \( q^{-m/2} z^m \) and \( m q^{-m/2} z^m \), with \( z = \pm 1 \).

**Step 6.** To summarize: if \( n = 0 \) and \( x = \nu_0 \), there is a unique function \( \varphi_m \) satisfying the difference equation with a given value of \( \varphi_0 \), and proportional to \( \varphi_0 \). Therefore \( V^K_\lambda \) has dimension one. Otherwise (\( n > 0 \)), the function \( \varphi \) is uniquely determined by its values on the ball \( B_n \). This proves the Proposition.

One curious consequence of the proof is that restriction to \( B_n \) is an isomorphism of \( V^K_\lambda \) with the subspace of functions on \( B_n \) satisfying \( T \varphi = \lambda \varphi \) internally.

In all cases, \( \varphi \) has a well defined asymptotic behaviour on every branch running out from \( B_n \). (a) If the polynomial

\[
x^2 - \left( \frac{\lambda}{\sqrt{q}} \right) x + 1 = 0
\]

has distinct roots \( z^\pm 1 \) then there exist constants \( c, d \) such that

\[
\varphi(y) = q^{-m/2}(cz^m + d z^{-m})
\]

if \( y \) lies at distance \( m \) from \( B_n \). (b) If it has one root \( z \) then

\[
\varphi_y \sim q^{-m/2}(cz^m + dm z^m).
\]

The constants depend on the branch.

If \( n = 0 \) we can get a completely explicit formula for \( \varphi \) with a bit more work. Suppose \( \varphi_0 = \varphi(\nu_0) = 1 \). I phrase the formula in terms I shall justify later.

**12.4. Proposition.** The unique solution of the difference equation for \( \varphi \) is

\[
\varphi_m = \frac{q^{-m/2}}{1 + 1/q} \left( \frac{1 - q^{-1} z^{-2}}{1 - z^{-2}} \right) z^m + \left( \frac{1 - q^{-1} z^2}{1 - z^2} \right) z^{-m}
\]

as long as \( z \neq \pm 1 \).

Once this equation is at hand, one can verify that it is correct by evaluating it for \( m = 0, 1 \). Finding it in the first place is just a matter of solving a \( 2 \times 2 \) system of linear equations.

One can also find an explicit formula in the case \( z = \pm 1 \).

**SL(2).** Now I take \( G = \text{SL}_2(\mathbb{F}) \). Here we use \( S \) instead of \( T \), and consider the set of nodes with even parity, a single \( \text{SL}_2(\mathbb{F}) \)-orbit. So \( V^K_\lambda \) is the space of functions on this orbit such that \( S \varphi = \lambda \varphi \). The proof of admissibility
is essentially the same, but I am interested in an explicit formula when \( \varphi \) is fixed by \( K \). Because of the Cartan decomposition, \( \varphi \) is determined by its values on \( \nu_{2m} \). Let \( \varphi_{2m} = \varphi(\nu_{2m}) \).

According to Lemma 11.4, the difference equation and initial conditions are now

\[
\varphi_2 = \frac{\lambda}{q(q+1)} \cdot \varphi_0
\]

\[
0 = q^2 \varphi_{2m} - (\lambda - q + 1) \varphi_{2m-2} + \varphi_{2m-4}.
\]

We get solutions

\[
ur_1^m + dr_2^m
\]

where the \( r_i \) are roots of the indicial equation

\[
x^2 - \left( \frac{\mu}{q^2} \right) x + 1/q^2 = 0 \quad (\mu = \lambda - q + 1).
\]

I set \( r_1 = z/q, r_2 = z^{-1}/q \), where now \( z + z^{-1} = \mu/q \).

The final conclusion is:

12.5. Proposition. When \( G = SL_2(t) \), the spherical function is

\[
\varphi_{2m} = \frac{q^{-m}}{1 + 1/q} \left( \frac{1 - q^{-1}z^{-1}}{1 - z^{-1}} \right)^m + \frac{1 - q^{-1}z}{1 - z} z^{-m}
\]

as long as \( z \neq 1 \), and some non-trivial linear combination of \( 1/q^m \) and \( m/q^m \) when \( z = 1 \).

The case \( z = -1 \) will turn out to be especially interesting. Explicitly:

12.6. Corollary. If \( z = -1 \) then

\[
\varphi_{2m} = (-q)^m.
\]

So in both cases, PGL_2 and SL_2, not only is the space of functions fixed by \( K \) of dimension 1, but we know exactly what the functions in the space are. For example, if \( z = q \) in the last formula, the representation is on the space of functions \( \varphi \) such that \( q(q+1)\varphi(\nu) \) is the sum of the values of \( \varphi \) at the nodes at distance 2 from \( \nu \). This contains the constants, and we get \( \varphi_m \equiv 1 \).

13. Deconstruction

In this section I’ll look at these formula more closely, term by term.

MEASURES. Where does the \( 1 + 1/q \) come from? This constant factor is required to make \( \varphi_0 = 1 \). If you follow the trail back, you’ll see that the principal motivation for this was to have \( \varphi \) agree with the matrix coefficient of the pair \( \varphi_{\chi^{-1}}, \varphi_{\chi} \). The pairing of \( I(\chi^{-1}) \) and \( I(\chi) \) is defined by an integral of one-densities on \( P \setminus G \), and this in turn was set equal to an integral over \( K \). The normalization thus comes down to a choice of measure on \( G \) such that \( \text{meas}(K) = 1 \).

A more natural measure to choose on \( G \) is one determined by a \( t \)-rational volume form. When adapted to integrating on \( P \setminus G \), this amounts to the following technique. Suppose \( f \) to be a function in \( \text{Ind}(\delta^{1/2}) \). Because of the Bruhat decomposition, the group \( G \) is the union of two open subsets \( P\sigma N \) and \( P\sigma N^{-1} = PN \). Here \( N \) is the unipotent radical of the Borel subgroup opposite to \( P \). Express \( f \) as the sum of two functions \( f_\sigma \) and \( f_1 \), of compact support on each of these pieces. Then up to a scalar,

\[
\int_{P \setminus G} f = \int_N f(\sigma n) \, dn + \int_{N^e} f(\overline{\sigma n}) \, d\overline{n}.
\]
To make this more explicit, one must fix measures on $N$ and $\overline{N}$. On the first, make $\text{meas}(N(a)) = 1$, and similarly for $\text{meas}(\overline{N}(a))$. With this measure, what is $\int_{P^G} \varphi_{\delta^{1/2}}$? Let $\varphi = \varphi_{\delta^{1/2}}$. Apply the Bruhat decomposition of $G(\mathbb{P}_p)$ to express $K$ as the disjoint union of the inverse image $K_1$ of $P(\mathbb{P}_q)$ and its complement $K_2$. Express $\varphi$ correspondingly as the sum of $\varphi_1$ and $\varphi_2$. Then

$$
\int_{P^G} \varphi = \int_{P^G} \varphi_1 + \varphi_2 = \int_{N(a)} 1 + \int_{N(p)} 1 = 1 + 1/q .
$$

In other words, the term $1 + 1/q$ is the ratio of the two measures we are looking at. This turns out to be an important point for all reductive groups. It plays a role in calculating Tamagawa measures, and similar considerations arise in computing intertwining operators between principal series.

**The Outer Factor.** Where does the factor $q^{-m/2}$ come from? This is $\delta^{-1/2}(\alpha^m)$. It is there for the same reason that the definition of $I(\chi)$ includes a normalization factor. The consequence here is that the rest of the expression is symmetric in $z$ and $z^{-1}$.

**Asymptotic Expansions.** I can explain most simply what is going on by an example. Suppose $\lambda = q + 1 = \sqrt{q}(\sqrt{q} + \sqrt{q}^{-1})$. Then $V_\lambda$ contains the constants, and the terms in the expansion along branches are of the form $c + dq^{-m}$ for constants depending on the branch. As $m \to \infty$ every matrix coefficient has as limit a locally constant function on $\mathbb{P}(t)$.

**Frobenius Reciprocity.** What happens more generally is best explained by a basic tool. Define

$$
\Omega : I(\chi) \longrightarrow C , \quad f \longmapsto f(1) .
$$

It satisfies the equation

$$
\langle \Omega , R_pf \rangle = \chi(p)\delta^{1/2}(p)\langle \Omega , f \rangle ,
$$

and is therefore a $P$-equivariant map to the $P$-module $C_{\chi^{1/2}}$. Composition with $\Omega$ gives us a map

$$
\text{Hom}_G(V, I(\chi)) \longrightarrow \text{Hom}_P(V, C_{\chi^{1/2}}) .
$$

The subgroup $N \subset P$ acts trivially on $C_{\chi^{1/2}}$, and any $P$-equivariant map from $V$ to $C_{\chi^{1/2}}$ factors through the space of coinvariants $V_N = V/V(N)$, with $V(N)$ the subspace spanned by all $\pi(n)v - v$ ($n \in N$). The group $A$ acts on this space.

**13.1. Proposition.** (Frobenius reciprocity) If $V$ is an admissible representation of $G$, then $F \mapsto \Omega \circ F$ induces an isomorphism of $\text{Hom}_G(V, I(\chi))$ with $\text{Hom}_A(V_N, C_{\chi^{1/2}})$.

I leave this as an exercise.

Suppose $V = V_\lambda$ with $\lambda \neq 2\sqrt{q}$. What can we say about $V_N$? If $f$ lies in $V$, then there exists $m \gg 0$ with the property that $f$ restricted to

$$
A[m, \infty) = \nu_m - \nu_{m+1} - \nu_{m+2} - \cdots
$$

is the sum $\varphi_k = q^{-k/2}(cz^k + dz^{-k})$ of two functions. If $n$ lies in $N$, then $nv_k = \nu_k$ for $k \gg 0$. Therefore $f$ and $L_n f$ have the same restriction to $A[k, \infty)$. There exists exactly one $A$-finite function on $A/A(\alpha)$ agreeing with $\varphi_m$ for $m \gg 0$. The $A$-stable space spanned by it has dimension two. The map from $f$ to $\varphi$ is $N$-invariant and is also $A$-equivariant. As an $A$-module, this is the direct sum of two one-dimensional representations, and by Frobenius reciprocity we get $G$-embeddings into the two principal series $I(\chi_\pm^1)$.

**The Dual Group.** The formulas for $\varphi_m$ can be put into a curious form. Take the case of $\text{PGL}_2$. It becomes for $m \geq 1$

$$
\varphi_m = \frac{q^{-m/2}}{1 + 1/q} \left( \frac{1 - q^{-1}z^{-2}}{1 - z^{-2}} \right)^m + \frac{1 - q^{-1}z^{-2}}{1 - z^{-2}} z^{-m} = \frac{q^{-m/2}}{1 + 1/q} \left( \frac{z^{m+1} - z^{-(m+1)}}{z - z^{-1}} \right) - q^{-1} \left( \frac{z^{m-1} - z^{-(m-1)}}{z - z^{-1}} \right) = \frac{1}{1 + 1/q} \left( q^{-m/2}(z^m + z^{m-2} + \cdots + z^{-m}) - q^{-(m-2)/2}(z^{m-2} + \cdots + z^{-(m-2)}) \right)
$$
The expression $z^m + \cdots + z^{-m}$ is the same as the character of the irreducible representation of $\text{SL}_2(\mathbb{C})$ of dimension $m + 1$, evaluated at

$$
\begin{bmatrix}
  z & 0 \\
  0 & 1/z
\end{bmatrix}.
$$

This is not at all an accident. In general, a formula due to Ian Macdonald expresses spherical functions in terms of Weyl characters for the complex group that Langlands calls its dual. The dual of $\text{PGL}_2$ is $\text{SL}_2(\mathbb{C})$ and that of $\text{SL}_2$ is $\text{PGL}_2(\mathbb{C})$.

14. References


