Introduction to admissible representations of p-adic groups

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Chapter II. Smooth representations

In this chapter I'll define smooth and admissible representations of locally profinite groups and prove their basic properties. I shall generally take the coefficient ring to be a commutative Noetherian ring \( R \), assumed to contain \( \mathbb{Q} \). The point of allowing representations with coefficients in a ring like this is to allow dealing with families of representations in a reasonable way. Occasionally I shall assume \( R \) to be an algebraically closed field \( D \) of characteristic 0, and sometimes even \( \mathbb{C} \).

Throughout this chapter, \( G \) will be an arbitrary locally profinite group. For convenience, I assume that it possesses a countable basis of neighbourhoods of the identity—i.e. that it is what I call elsewhere a König group. Thus this chapter will present results valid without significant assumptions about the structure of \( G \). This necessarily excludes anything really interesting, but includes many basic items.

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SUMMARY. The theory of representations of p-adic groups started off by looking at unitary representations on Hilbert spaces. After [Jacquet-Langlands:1979] introduced the category of admissible representations, the subject almost completely lost its analytical flavour and became algebraic—in many aspects not very different from the theory of representations of finite groups.

1. The Hecke algebra

A smooth \( G \) module over \( R \) is a representation \((\pi, V)\) of \( G \) on an \( R \)-module \( V \) such that each \( v \) in \( V \) is fixed by an open subgroup of \( G \). A smooth representation \((\pi, V)\) is said to be admissible if for each open subgroup \( K \) in \( G \) the subspace \( V^K \) of vectors fixed by elements of \( K \) is finitely generated over \( R \). If \( R \) is a field, this just means that \( V^K \) has finite dimension.

The subspace of smooth vectors in any representation of \( G \) is stable under \( G \), since if \( v \) lies in \( V^K \) then \( \pi(g)v \) lies in \( V^K \).

The group \( G \) acts on functions on \( G \) by the left and right regular representations:

\[
R_g f(x) = f(xg), \quad L_g f(x) = f(g^{-1}x).
\]
I’ll call a locally constant function \( f \) on \( G \) uniformly locally constant if there exists a compact open subgroup \( K \) of \( G \) such that \( R_k f = L_k f \) for all \( k \) in \( K \). The left and right actions of \( G \) commute, hence give rise to a smooth representation of \( G \times G \) on the space of all uniformly locally constant functions.

The **Hecke algebra** \( \mathcal{H}(G) = \mathcal{H}_\mathbb{Q}(G) \) is the ring of smooth rational distributions of compact support on \( G \), in which multiplication is convolution. If \( R \) is any ring containing \( \mathbb{Q} \), then \( \mathcal{H}_R = R \otimes \mathcal{H}_\mathbb{Q} \).

Recall from Proposition I.6.6 that given a choice of rational Haar measure \( dx \), smooth \( \mathbb{Q} \)-valued functions may be identified with smooth distributions:

\[
\varphi \mapsto D_\varphi = \varphi(x) \, dx.
\]

There exists for every smooth representation \((\pi, V)\) of \( G \) a canonical homomorphism from \( \mathcal{H}(G) \) to \( \text{End}_R(V) \). Fix for the moment a right-invariant Haar measure \( dx \) on \( G \). Suppose \( D \) to be \( D_\varphi \). For \( v \) in \( V \) define

\[
\pi(D)v = \int_G \varphi(x)\pi(x)v \, dx.
\]

If \( v \) is fixed by the compact open group \( K \) and \( \varphi \) is fixed by \( K \) with respect to the right regular representation, this is also

\[
\text{meas}(K) \sum_{G/K} \varphi(x)\pi(x)v.
\]

This definition depends only on \( D_\varphi \). We can in fact characterize, if not define, \( \pi(D) \) solely in terms of \( D \) as a distribution. If \( F \) is a linear function on \( V \), then \( \Phi(x) = F(\pi(x)v) \) is a locally constant function on \( G \), and we may apply \( D \) to it. Then

\[
F(\pi(D)v) = \int_G \varphi(x)F(\pi(x)v) \, dx = (D, \varphi).
\]

Suppose that \( D_1 \) and \( D_2 \) are two smooth distributions of compact support, corresponding to smooth function \( \varphi_1 \) and \( \varphi_2 \). Then

\[
\pi(D_1)\pi(D_2)v = \int_G \varphi_1(x)\pi(x) \, dx \int_G \varphi_2(y)\pi(y)v \, dy
= \int_{G \times G} \varphi_1(x)\varphi(y)\pi(xy)v \, dx \, dy
= \int_G \varphi(z)\pi(z)v \, dz
= \pi(D_\varphi)v
\]

where

\[
\varphi(z) = \int_G \varphi_1(zy^{-1})\varphi_2(y) \, dy
= \int_G \varphi_1(y)\varphi_2(y^{-1}z) \, dy.
\]

The distribution \( D_\varphi \) is also smooth and of compact support. It is the **convolution** \( D_1 * D_2 \).

The Hecke algebra \( \mathcal{H} \) does not have a multiplicative unit unless \( G \) is compact.
If $K$ is a compact open subgroup of $G$, the Hecke algebra contains the subalgebra $\mathcal{H}(G/K)$ of distributions invariant under left and right multiplication by elements of $K$. This algebra has as basis the measures $\mu_{KgK/K}$ of $H$:

$$\langle \mu_{KgK/K}, f \rangle = \frac{1}{\text{meas}(K)} \int_{KgK} f(x) \, dx.$$ 

as $g$ varies over $K\backslash G/K$. If $\pi$ is any smooth representation of $G$ then every element of $\mathcal{H}(G/K)$ takes $V^K$ into itself, and

$$\pi(\mu_{KgK/K}) | V^K = \sum_{KgK/K} \pi(x).$$

In practice, a more explicit formula is often used. The map from $K/K \cap gKg^{-1}$ to $KgK/K$, taking $k$ to $kg$, is a bijection. Therefore

(II.1.1) $$\pi(\mu_{KgK/K}) | V^K = \sum_{K/K \cap gKg^{-1}} \pi(kg).$$

The unit of $\mathcal{H}(G/K)$ is $\mu_{K/K}$, which amounts to projection from $V$ onto the subspace $V^K$ of vectors fixed by $K$.

For every closed subgroup $H$ of $G$, define $V(H)$ to be the subspace of $V$ generated by the $\pi(h)v - v$ for $h$ in $H$. This subspace of $V$ is characterized by the property that every $H$-equivariant linear map from $V$ to a vector space on which $H$ acts trivially factors through $V/V(H)$. If $H$ is compact, something better happens:

II.1.2. Proposition. For any compact open subgroup $K$ and smooth representation $V$, we have an equality

$$V(K) = \{ v \in V \mid \pi(\mu_{K/K})v = 0 \}$$

and a direct sum decomposition

$$V = V(K) \oplus V^K.$$

Proof. If $v$ is fixed by $K_*$ then

$$\pi(\mu_{K/K})v = \frac{1}{|K:K_*|} \sum_{K/K_*} \pi(k)v$$

and

$$v = \frac{1}{|K:K_*|} \sum_{K/K_*} v.$$ 

If we subtract the first from the second, we get

$$v - \pi(\mu_{K/K})v = -\frac{1}{|K:K_*|} \sum_{K/K_*} (\pi(k)v - v).$$

II.1.3. Proposition. Suppose

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

to be an exact sequence of $G$-representations. If the representation on $V$ is smooth so is that on $U$ and $W$, and for every compact open subgroup $K$ the sequence

$$0 \rightarrow U^K \rightarrow V^K \rightarrow W^K \rightarrow 0$$

is also exact.
II.1.4. Corollary. If $V$ is smooth, it is admissible if and only if that on $U$ and $W$ are.

Thus the categories of smooth and admissible representations are abelian.

II.1.5. Proposition. Suppose $K$ to be a fixed compact open subgroup of $G$. A smooth representation defined over $D$ is admissible if and only if the restriction of $\pi$ to $K$ is a direct sum of irreducible representations, each occurring with finite multiplicity.

One has to be a bit careful, since unless $R$ is an algebraically closed field irreducibility is not the same as absolute irreducibility.

Proof. Choose a sequence of compact open subgroups $K_n$ normal in $K$ and with $\{1\}$ as limit. Then $V = V(K_n) \oplus V(K_n)$. But every smooth finite-dimensional representation of $K$ factors through some $K_n$.

As I remarked earlier, in many ways things do not behave here very differently from how they do for finite groups.

2. The centre of G

In this section, assume the coefficient ring to be $D$.

If $(\pi, V)$ is an admissible representation of $G$ then each space $V^K$ is stable under the centre $Z_G$ of $G$. The subgroup $Z_G \cap K$ acts trivially on it.

II.2.1. Proposition. If $(\pi, V)$ is irreducible the centre acts as scalar multiplication by a single character.

Proof. Suppose $V^K \neq \{0\}$. Since $V^K$ has finite dimension and $D$ is algebraically closed, there exists a character $\omega$ of $Z_G$ with values in $D^\times$ such that the subspace

$$V^K(\omega) = \{v \in V^K \mid \pi(z)v = \omega(z)v \text{ for all } z \in Z_G\} \neq \{0\}.$$

If $U$ is the subspace of all $v$ in $V$ satisfying this equation, it is both non-trivial and stable under $G$, hence equal to all of $V$.

In general, I call an admissible representation centrally simple if this occurs. If $Z_G$ acts through the character $\omega$ then $\pi$ is called an $\omega$-representation. For any central character $\omega$ with values in $D^\times$ the Hecke algebra $H_{D,\omega}$ is that of uniformly smooth functions on $G$ compactly supported modulo $Z_G$ such that

$$f(zg) = \omega(z)f(g).$$

If $\pi$ is centrally simple with central character $\omega$ it becomes a module over the Hecke algebra $H_{\omega^{-1}}$:

$$\pi(f)v = \int_{G/Z_G} f(x)\pi(x)v \, dx,$$

which is well defined since $f(zx)\pi(zx) = f(x)\pi(x)$.

But now I generalize this somewhat.

If $C$ is a commuting set of linear operators acting on a vector space $U$ of dimension $n$ and $\gamma$ a map from $C$ to $D^\times$, let

$$U(\gamma) = \{v \in U \mid (c - \gamma(c))^nv = 0 \text{ for some } n \text{ and all } c \in C\}.$$

II.2.2. Lemma. Suppose $C$ to be any commuting set of linear operators acting on the finite dimensional vector space $U$ over $D$. There exists a direct sum decomposition of $U$ into non-zero spaces $U(\gamma)$.

It is called the primary decomposition of $U$.

Proof. The slight technical problem is that no assumption is made on the size of $C$. 
Suppose at first that $C$ is made up of a single element $c$. Let $P(x) = \prod (x - \gamma_i)^{m_i}$ be the characteristic polynomial of $c$. For $m$ larger than all $m_i$

$$\prod_i (c - \gamma_i)^m = 0.$$ 

Since there exist polynomials $a(x)$ and $b(x)$ such that

$$1 = a(x)(x - \gamma_1)^m + b(x)\prod_{i \neq 1} (x - \gamma_i)^m,$$

a simple inductive argument implies that $U$ is the direct sum of primary eigenspaces with respect to $c$.

If $C$ is finite a similar inductive argument will imply the Lemma. For each finite subset of $C$ the primary decomposition of $U$ assigns to that subset a partition of $n = \dim(U)$, that determined by the dimensions of its subspaces $U([\gamma])$. There are only a finite number of these partitions; choose one of greatest length. Suppose it belongs to the subset $S$ in $C$. Any larger subset must determine a refinement of the decomposition for $S$, and hence must actually be the same. The decomposition for $S$ is therefore one for all of $C$.

From this follows immediately:

**II.2.3. Proposition.** If $(\pi, V)$ is a finitely generated admissible representation of $G$, the restriction of $\pi$ to $Z_G$ is a direct sum of primary components $V([\omega])$, where the $\omega$ vary over a finite set of homomorphisms from $Z_G$ to $\mathbb{D}^\times$.

The characters $\omega$ occurring in this decomposition are called the **central characters** of $\pi$.

**3. The contragredient**

Suppose $(\pi, V)$ to be a smooth representation of $G$. Let

$$\tilde{V} = \text{Hom}_R(V, R).$$

The group $G$ acts on $\tilde{V}$ according to the recipe

$$\langle \tilde{\pi}(g)\tilde{v}, v \rangle = \langle \tilde{v}, \pi(g^{-1})v \rangle.$$

The point is that the canonical pairing is $G$-invariant. The **contragredient** representation $(\tilde{\pi}, \tilde{V})$ is that on the smooth vectors in $\tilde{V}$.

**II.3.1. Proposition.** Suppose $(\pi, V)$ to be a smooth representation of $G$ and $K$ a compact open subgroup of $G$. Restriction of $f$ to $V^K$ is an isomorphism of $\tilde{V}^K$ with $\text{Hom}_R(V^K, \mathcal{R})$.

**Proof.** Because $V = V^K \oplus V(K)$ and the functions in $\tilde{V}^K$ are precisely those annihilating $V(K)$.

From the exact sequence of $R$-modules

$$\mathcal{R}^n \rightarrow V^K \rightarrow 0$$

we deduce

$$0 \rightarrow \text{Hom}_R(V^K, \mathcal{R}) \rightarrow \text{Hom}_R(\mathcal{R}^n, \mathcal{R}) \cong \mathcal{R}^n.$$

Therefore $V^K$ is finitely generated over $\mathcal{R}$, and $\tilde{\pi}$ is again admissible.

In general, $\tilde{V}$ may be very small. If each $V^K$ is free over $\mathcal{R}$, the canonical map from $V$ into the contragredient of its contragredient will be an isomorphism. Furthermore:

**II.3.2. Corollary.** Suppose $\mathcal{R}$ to be a field. If

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$
is a short exact sequence of admissible representations, then so is
\[ 0 \to \widetilde{W} \to \widetilde{V} \to \widetilde{U} \to 0. \]

Suppose \( U \) and \( V \) both smooth representations. Given a \( G \)-equivariant map from \( U \) to \( V \), we get by duality one from \( \widetilde{V} \) to \( \widetilde{U} \).

**II.3.3. Proposition.** Suppose \( U \) smooth, \( V \) admissible, \( \mathcal{R} \) a field. The canonical map defined above:
\[ \text{Hom}_G(U, V) \to \text{Hom}_G(\widetilde{V}, \widetilde{U}) \]
is an isomorphism.

**Proof.** We can define an inverse. Dualizing once again we get an inverse map
\[ \text{Hom}_G(\widetilde{V}, \widetilde{U}) \to \text{Hom}_G(U, V). \]

But \( U \) is canonically embedded in \( \widetilde{U} \) and since \( V \) is admissible we know that \( V \) may be identified with \( \widetilde{V} \), so we get an inverse map
\[ \text{Hom}_G(\widetilde{V}, \widetilde{U}) \to \text{Hom}_G(U, V). \]

4. Representations of a group and of its Hecke algebra

What is the relationship between smooth \( G \)-representations and the associated representation of its Hecke algebra \( \mathcal{H} \)?

**II.4.1. Proposition.** Suppose \((\pi_i, V_i)\) are two smooth representations of \( G \). Then
\[ \text{Hom}_G(V_1, V_2) = \text{Hom}_\mathcal{H}(V_1, V_2). \]

A mild technical problem here is that \( \mathcal{H} \) has no multiplicative identity.

**Proof.** Any \( G \)-homomorphism is clearly a homomorphism of modules over the Hecke algebra as well. So suppose now that one is given a map \( F \) of modules over the Hecke algebra. Suppose \( v \) in \( V_1 \), \( g \) in \( G \), and choose a compact open subgroup \( K \) fixing \( v \), \( \pi_1(g)v \), \( F(v) \), and \( \pi_2(g)F(v) \). Then
\[
F(\pi_1(g)v) = \frac{F(\pi_1(\mu_{KgK}))v}{|KgK/K|} = \frac{\pi_2(\mu_{KgK})F(\pi_1(g)v)}{|KgK/K|} = \pi_2(g)F(v).
\]

A smooth representation is said to be **co-generated** by a subspace \( U \) if every non-zero \( G \)-stable subspace of \( V \) intersects \( U \) non-trivially. This is dual to the condition of generation, in the following sense:

**II.4.2. Lemma.** Suppose \( \mathcal{R} \) to be a field and \( K \) to be a compact open subgroup of \( G \). The admissible representation \((\pi, V)\) is generated by \( V^K \) if and only if its smooth contragredient is co-generated by \( V^K \).

**Proof.** Suppose that \( V \) is generated by \( V^K \), and suppose \( U \) to be a \( G \)-stable subspace of \( \widetilde{V} \) with \( U \cap \widetilde{V}^K = U^K = 0 \). If \( U^\perp \) is the annihilator of \( U \) in \( \widetilde{V} = V \), then \( (V/U^\perp)^K = U^K = 0 \). Thus \( V^K = (U^\perp)^K \), and since \( V^K \) generates \( V \), \( V = U^\perp \) and \( U = 0 \). The converse argument is similar.

**II.4.3. Proposition.** Suppose that \((\pi_i, V_i)\) are two smooth representations of \( G \) and that \( K \) is a compact open subgroup of \( G \). If
proof. If \( F \) lies in \( \text{Hom}_G(V_1, V_2) \) then for any \( f \) in \( \mathcal{H} \) we have
\[
F(\pi_1(f)v) = \pi_2(f)F(v)
\]
for every \( f \) in \( \mathcal{H} \) and \( v \) in \( V_1^K \). Conversely, if we are given \( F \) in \( \text{Hom}_{\mathcal{H}(G//K)}(V_1^K, V_2^K) \) then since \( V_1^K \) generates \( V_1 \) this formula will serve to define a \( G \)-map from \( V_1 \) to \( V_2 \) once we know that
\[
\text{if } v \text{ lies in } V_1^K, f \text{ in } \mathcal{H}, \text{ and } \pi_1(f)v = 0 \text{ then } \pi_2(f)F(v) = 0.
\]
But if \( \pi_1(f)v = 0 \) then for every \( h \) in \( \mathcal{H} \)
\[
\pi_1(\mu_{K/K}h)\pi_1(f)v = \pi_1(\mu_{K/K}h \ast f \ast \mu_{K/K})v = 0.
\]
Since \( F \) is assumed to be \( \mathcal{H}(G//K) \)-equivariant,
\[
\pi_2(\mu_{K/K}h \ast f \ast \mu_{K/K})F(v) = \pi_2(\mu_{K/K}h)\pi_2(f)F(v) = 0
\]
for every \( h \) in \( \mathcal{H} \). This means that the \( G \)-space generated by \( \pi_2(f)F(v) \) has no non-zero \( K \)-invariant vectors, which means by assumption that it is 0.

II.4.4. Proposition. Suppose \( R \) to be \( \mathbb{D} \) and \( (\pi, V) \) to be a smooth representation of \( G \). Then
(a) if \( \pi \) is irreducible then \( V^K \) is an irreducible module over \( \mathcal{H}(G//K) \) for all \( K \);
(b) if \( V \) satisfies conditions (a) and (b) of Proposition II.4.3 and \( V^K \) is an irreducible module over \( \mathcal{H}(G//K) \) then \( \pi \) is irreducible.

proof. Suppose \( (\pi, V) \) to be irreducible, and let \( U \) be any non-trivial \( \mathcal{H}(G//K) \)-stable subspace of \( V^K \). Since \( V \) is irreducible, \( U \) must generate \( V \) as a \( G \)-space, so every \( v \) in \( V \) is of the form \( \sum c_i\pi(g_i)u_i \) with \( u_i \) in \( U \). But then for \( v \) in \( V^K \)
\[
v = \pi(\mu_{K/K})v = \sum c_i\pi(\mu_{K/K})\pi(g_i)u_i = \text{constant } c_i\pi(Kg_iK)u_i,
\]
which lies in \( U \) since \( U \) is assumed to be stable under \( \mathcal{H}(G//K) \). So \( V^K \subseteq U \).
Conversely, assume conditions (a) and (b) of Proposition II.4.3 to hold for \( V \), and assume \( V^K \) irreducible. If \( U \) is any non-zero \( G \)-stable subspace of \( V \) then by (b) \( U^K \neq 0 \) must be a submodule of \( V^K \), but will equal it because of irreducibility. But (a) implies that then \( U = V \).

II.4.5. Proposition. Suppose \( R \) to be \( \mathbb{D} \) and \( (\pi, V) \) to be an irreducible admissible representation of \( G, K \) compact and open in \( \hat{G} \). The homomorphism from \( \mathcal{H}(G//K) \) to \( \text{End}_G(V^K) \) is surjective.

Proof. This follows from Theorem II.10.1.

Does every finite-dimensional module over \( \mathcal{H}(G//K) \) arise as the space \( V^K \) for some admissible \( V \)? And more particularly one satisfying the conditions (a) and (b) of Proposition II.4.3? We can obtain a partial answer to these questions. It is motivated by a simple observation. Let \( V \) be an admissible representation of \( G, U = V^K \). To each \( v \) in \( V \) we can assign the function
\[
F_v: G \to U, \quad g \mapsto \pi(\mu_{K/K})\pi(g)v
\]
Then \( f \ast F_v = \pi(f)F_v \) for every \( f \) in \( \mathcal{H}(G//K) \), and the map from \( V \) to \( C^\infty(G, U) \) is equivariant with respect to the right regular action of \( G \).

Conversely, if \( U \) is a finite-dimensional representation of \( \mathcal{H}(G//K) \), define \( I_U \) to be the space of all functions \( F: G \to U \) such that \( f \ast F = \pi(f)F \) for all \( f \) in the Hecke algebra. There is a canonical embedding of \( U \) itself into this, and let \( V \) be the subspace of \( I_U \) generated by this copy. It is not hard to verify that \( V^K = U \), and that \( V \) is also co-generated by \( U \). But whether this representation of \( G \) is admissible seems to depend on \( G \).
5. Characters

If \((\pi, V)\) is admissible then for every \(f\) in \(\mathcal{H}(G)\) the trace of \(\pi(f)\) is well defined since it may be identified with an operator on some \(V^K\), which is finite-dimensional. This defines the character of \(\pi\) as a linear functional on the Hecke algebra.

II.5.1. Proposition. If the \((\pi_i, V_i)\) make up a finite set of inequivalent irreducible admissible representations of \(G\) then their characters are linearly independent.

Proof. Choose \(K\) so small that \(V_i^K \neq 0\) for all \(i\). They then form, according to Proposition II.4.4 and Proposition II.4.3, inequivalent modules over \(\mathcal{H}(G//K)\). Because of irreducibility, the image of the Hecke algebra in \(\text{End}(U_i)\) is all of it. Because the \(\pi_i\) are all distinct as well as irreducible, Theorem II.10.1 implies that the map from the Hecke algebra into \(\prod \text{End}(U_i)\) is surjective. Suppose now that

\[ \sum c_i \text{tr} \pi_i = 0, \]

which means that

\[ \sum c_i \text{tr} (\pi_i(f)) = 0 \]

for all \(f\) in the Hecke algebra. But then we can choose \(f\) in the Hecke algebra such that \(\pi_i(f) = I\) but all the other \(\pi_j(f) = 0\), which implies that \(c_i = 0\). 

The following is trivial:

II.5.2. Proposition. If

\[ 0 \to U \to V \to W \to 0 \]

is an exact sequence of admissible \(G\)-spaces, then the character of \(V\) is the sum of the characters of \(U\) and \(W\).

It implies easily one half of this refinement:

II.5.3. Corollary. Two admissible representations of finite \(G\)-length have the same Jordan-Hölder factors if and only if they have the same characters.

Proof. It remains to be seen that if \(U\) and \(V\) have the same characters then they have the same Jordan-Hölder factors. But for this, by the previous result, it suffices to see that the semi-simplifications of \(U\) and \(V\) are isomorphic. But this follows from Proposition II.5.1 and an induction argument.

6. Matrix coefficients

If \((\pi, V)\) is an admissible representation the matrix coefficient associated to the pair \(v\) in \(V\), \(\tilde{v}\) in \(\tilde{V}\) is the \(\mathcal{R}\)-valued function

\[ F_{\tilde{v}, v} = \langle \tilde{v}, \pi(g)v \rangle, \]

which is uniformly smooth. Let \(\mathcal{A}(\pi)\) be the space of smooth functions spanned by the matrix coefficients of \(\pi\). It is a smooth representation of \(G \times G\) (one factor acting on the left, one on the right), and the map from \(\tilde{V} \otimes V\) to \(C^\infty(G)\) is \(G \times G\)-equivariant. In particular, for any fixed \(\tilde{v}\) the map

\[ F_{\tilde{v}}: v \mapsto F_{\tilde{v}, v} \]

is equivariant from \(V\) to \(C^\infty(G)\), with respect to the right regular action.

Now suppose that \(\mathcal{R} = \mathbb{D}\). Let \(\mathcal{A}(G) = \mathcal{A}\mathbb{D}(G)\) be the space of smooth functions on \(G\) contained in a \(G \times G\)-stable admissible subrepresentation of \(C^\infty(G)\). This is what Harish-Chandra called the space of automorphic forms on \(G\).

II.6.1. Proposition. Suppose \(F\) to be a smooth function on \(G\) fixed on left and right by elements of some compact open subgroup. The following are equivalent:
(a) the function $F$ is contained in some $A(\pi)$ with $\pi$ admissible;
(b) the space $A_L(F)$ spanned by all $L_gF$ is an admissible $L_G$-representation;
(c) the space $A_R(F)$ spanned by all $R_gF$ is an admissible $R_G$-representation;
(d) the function $F$ lies in $A(G)$.

Proof. Suppose that $V = A_R(F)$ is an admissible representation of $G$. We want to find to see $\tilde{v}$ in $\tilde{V}$ and $v$ in $V$ such that

$$F(g) = \langle \tilde{v}, R_g v \rangle.$$  

It is natural to take $v = F$, and which case this equation can be written as

$$[R_g F](1) = \langle \tilde{v}, R_g F \rangle.$$  

This suggests defining

$$\langle \tilde{v}, f \rangle = f(1)$$  

for all $f$ in $V$. Then

$$(R_k \tilde{v}, f) = \langle \tilde{v}, R_{k^{-1}} f \rangle = f(k^{-1}) = [L_k f](1)$$  

for all $f$ in $V$. But the left action of $G$ commutes with the right action, so if $L_k F = F$ for all $k$ in the compact open subgroup $K$ then $L_k v = v$ for all $v$ in $A_R F$, and $R_k \tilde{v} = v$, so that $\tilde{v}$ lies in $\tilde{V}$. 

Suppose $(\pi, V)$ to be an admissible representation. Abusing slightly commonly used terminology, I’ll call it cuspidal if its matrix coefficients are of compact support modulo $Z_G$. It is easy to see that this happens if and only if one of its non-zero matrix coefficients satisfies this condition. (For Harish-Chandra, these were ‘super-cuspidal’. The term ‘cuspidal’ for him refers to a certain category of representations that includes all representations whose matrix coefficients are square-integrable. This terminology seems obsolescent.)

II.6.2. Proposition. An irreducible cuspidal representation defined over $\mathbb{D}$ is projective and injective in the category of smooth representation of $G$ with the same central character.

Proof. Postponed to a later section.

7. Unitary representations

In this section I take $\mathcal{R}$ to be $\mathbb{C}$.

A unitary representation of $G$ is one with a positive definite $G$-invariant Hermitian inner product. Unitary representations are important because they are the ones that appear in orthogonal decompositions of arithmetic quotients, and this has arithmetic consequences. In one classic example, unitarity is related to Ramanujan’s conjecture.

We start with a very simple result, which is trivial to prove.

II.7.1. Proposition. Every admissible unitary representation is a countable direct sum of irreducible unitary representations, each occurring with finite multiplicity.

This requires the assumption that $G$ possesses a countable basis of neighbourhoods of the identity.

It is easy to see that the matrix coefficients of a unitary representation are bounded. A much stronger condition on matrix coefficients is fundamental. Suppose $\pi$ to be an irreducible representation with central character $\omega$. It is said to be square-integrable modulo the centre $Z_G$ of $G$ if $|\omega| = 1$ (i.e. its central character is unitary) and every matrix coefficient is square-integrable on $G/Z_G$. A cuspidal representation with unitary central character is square-integrable, for example.

II.7.2. Proposition. If $\pi$ is an irreducible admissible representation of $G$, then it is square-integrable if and only if a single non-zero matrix coefficient is square-integrable.

Since a square-integrable representation may be embedded into $L^2(G)$, it is unitary. More precisely:
II.7.3. Proposition. Suppose $(\pi, V)$ to be an irreducible square-integrable representation. For $\tilde{v}_0 \neq 0$ in $\tilde{V}$ the pairing

$$u \cdot v = \int_{G/ZG} \langle \pi(g)u, \tilde{v}_0 \rangle \langle \pi(g)v, \tilde{v}_0 \rangle \, dg .$$

defines a $G$-invariant positive definite inner product on $V$.

8. Schur orthogonality

Matrix coefficients of a representation are intrinsic, in the sense that isomorphic representations have the same matrix coefficients. In fact, matrix coefficients distinguish a representation. For certain representations, there is a strong form of this assertion. I’ll call an irreducible representation formally square-integrable if it is either cuspidal or square-integrable. (The distinction is that cuspidal representations do not have to be defined over $\mathbb{C}$.) If $(\pi, V)$ is formally square-integrable, the integral

$$I = I(u, \tilde{u}, v, \tilde{v}) = \int_{G/ZG} \langle \pi(g)u, \tilde{u} \rangle \langle \rho(g^{-1})v, \tilde{v} \rangle \, dg = \int_{G/ZG} \langle \pi(g)u, \tilde{u} \rangle \langle v, \tilde{\rho}(g)v \rangle \, dg$$

(with $u$ in $U$, $v$ in $V$, $\tilde{u}$ in $\tilde{U}$, $\tilde{v}$ in $\tilde{V}$) makes sense.

II.8.1. Proposition. Let $(\pi, U)$ and $(\rho, V)$ be formally square-integrable admissible representations with the same central character. Then

(a) if $\pi$ and $\rho$ are isomorphic then $I = c_\pi \langle u, \tilde{u} \rangle \langle v, \tilde{v} \rangle$ for some constant $c_\pi$;

(b) if they are not isomorphic, $I = 0$.

The constant $c_\pi$ is not zero, and if $D = \mathbb{C}$, $c_\pi > 0$.

If $G$ is compact and the measure normalized so all of $G$ has measure 1, then $c_\pi = 1/d_\pi$, where $d_\pi$ is the dimension of $\pi$. In general, $1/c_\pi$ is called its formal degree.

Proof. For the moment, fix $\tilde{u}, v$. The pairing taking $u$ in $U$, $\tilde{v}$ in $\tilde{V}$ to

$$I = \int_{G/ZG} \langle \pi(g)u, \tilde{u} \rangle \langle \rho(g^{-1})v, \tilde{v} \rangle \, dg$$

defines a $G$-invariant pairing of $U$ and $\tilde{V}$, or equivalently a map from $U$ to the contragredient of $\tilde{\rho}$, which is $\rho$ itself. If $\pi$ and $\rho$ are not isomorphic then it must consequently be 0. If $\pi$ and $\rho$ are isomorphic, it must be a scalar multiple of the canonical pairing. We may as well assume $U = V$, and the integral is equal to $c_{\tilde{u}, v} \langle u, \tilde{v} \rangle$. But then it can be seen that $c_{\tilde{u}, v}$ is equal to $c_\pi \langle v, \tilde{v} \rangle$ for some $c_\pi$.

To see that $c_\pi \neq 0$ requires a discussion of embedding fields of definition into $\mathbb{C}$ that I’ll skip. But suppose now that $D = \mathbb{C}$. Fix a $G$-invariant positive-definite Hermitian inner product on $V$. Fix $\tilde{v}$ for the moment, and let then choose $v_0$ in $V$ such that

$$v \cdot v_0 = \langle v, \tilde{v} \rangle$$

for all $v$ in $V$. Then

$$c_\pi (v \cdot v_0)(v \cdot v_0) = \int_{G/ZG} \langle \pi(g)v \cdot v_0 \rangle \langle \pi(g^{-1})v \cdot v_0 \rangle \, dg$$

$$= \int_{G/ZG} \langle \pi(g)v \cdot v_0 \rangle \langle v_0 \cdot \pi(g^{-1})v \rangle \, dg$$

$$= \int_{G/ZG} \langle \pi(g)v \cdot v_0 \rangle \langle \pi(g)v_0 \cdot v \rangle \, dg .$$

If we set $v = v_0$ we deduce that $c_\pi > 0$.

Now for the proof of Proposition II.6.2. It is motivated by the analogous case of modules over a commutative ring with unit, in proving that a module is projective if and only if it is a summand of a free module. But
in our case there are some minor difficulties because the Hecke algebra is not commutative and does not possess a unit.

Suppose \((\pi, V)\) to be an irreducible cuspidal representation of \(G\) with formal degree \(d_\pi\) and central character \(\omega\). Suppose \((\sigma, U)\) to be an arbitrary smooth representation of \(G\) with central character \(\omega\), and \(F: U \to V\) a \(G\)-equivariant surjection. I need to construct a splitting of \(F\) from \(V\) back to \(U\). What I shall do is embed \(V\) as a summand of \(H_\omega\), and then construct a suitable map from \(H_\omega\) to \(U\).

The proof is relatively straightforward, but will be clearer if I make three succinct observations:

1. The map from \(V\) to \(C^\infty(G)\) taking \(v\) to the matrix coefficient \(\langle \pi(g)v, \ddot{v}\rangle\) is a \(G\)-equivariant map from \((\pi, V)\) to \((R, H_\omega)\). Here \(R\) is the right regular representation of \(G\).

2. Given a vector \(u\) in \(U\), the map taking \(f\) in \(H_\omega\) to \(\sigma(f)u\), where \(\sigma^\vee(g) = f(g^{-1})\), is a \(G\)-equivariant map from \((R, H_\omega)\) to \((\sigma, U)\).

3. Schur orthogonality for a cuspidal representation \(\pi\):

   \[
   \int_{G/Z} \langle \pi(g)u, \ddot{u}\rangle \langle \pi(g)^{-1}v, \ddot{v}\rangle \, dg = \frac{1}{d_\pi} \langle u, \ddot{v}\rangle \langle v, \ddot{v}\rangle.
   \]

**Step 1.** Fix \(\dddot{v}\) in \(\dddot{V}\) and \(v_0\) in \(V\) such that \(\langle \dddot{v}, v\rangle = d_\pi\). For any \(v\) in \(V\) let \(\gamma_v\) be the matrix coefficient \(\langle \pi(g)v, \dddot{v}\rangle\). By Theorem V.1.1 the map \(v \mapsto \gamma_v\) embeds \(V\) into \(H_\omega(G)\).

**Step 2.** We next want to define a \(G\)-equivariant projection \(P\) from \(H_\omega \to V\). To do this, I shall pick a suitable vector \(v_0 \neq 0\) in \(V\) and map \(f\) in \(H_\omega\) to

\[
P f = R_{f^\vee} v_0.
\]

It follows from the remark (2) above that this is a \(G\)-equivariant map from \(H_\omega\) to \(V\).

We now want to choose \(v_0\) so that \(P \gamma_v = v\) for \(v\) in \(V\). Well, we compute

\[
P \gamma_v = R_{\gamma_v^\vee} v_0
\]

\[
= \int_{G/Z} \gamma_v(x) \pi(x) v_0 \, dx
\]

\[
= \int_{G/Z} \langle \pi(x)^{-1}v, \dddot{v}\rangle \pi(x) v_0 \, dx.
\]

This last is an element of \(V\). But according to Schur orthogonality

\[
\left\langle \int_{G/Z} \langle \pi(x)^{-1}v, \dddot{v}\rangle \pi(x) v_0 \, dx, \dddot{v}\right\rangle = \int_{G/Z} \langle \pi(x)^{-1}v, \dddot{v}\rangle \langle \pi(x) v_0, \dddot{v}\rangle \, dx = \frac{1}{d_\pi} \langle v, \dddot{v}\rangle \langle v_0, \dddot{v}\rangle.
\]

so if we choose \(\langle \pi(x) v_0, \dddot{v}\rangle = d_\pi\) then \(P v = v\).

**Step 3.** Go back to our projection \(F: U \to V\). Choose \(u_0 \in U\) with \(F(u_0) = v_0\). Let \(\Pi\) be the map from \(H_\omega\) to \(U\) taking \(f\) to \(\sigma(f)u_0\). Here \(\hat{f}(g) = f(g^{-1})\). This is a \(G\)-morphism. The diagram

\[
\begin{array}{ccc}
H_\omega & \xrightarrow{\Pi} & U \\
P \downarrow & & \downarrow F \\
V & &
\end{array}
\]

is commutative, so

\[
F \circ \Pi = P, \quad F \circ (\Pi \circ \gamma) = (F \circ \Pi) \circ \gamma = P \circ \gamma = I
\]

and \(\Pi \circ \gamma\) splits \(F\).

For injectivity, apply projectivity to duals.
9. Induced representations

If $H$ is a closed subgroup of $G$ and $(\sigma, U)$ is a smooth representation of $H$, the unnormalized smooth representation $\text{Ind}(\sigma | H, G)$ induced by $\sigma$ is the right regular representation of $G$ on the space of all uniformly smooth functions $f: G \to U$ such that

$$f(hg) = \sigma(h)f(g)$$

for all $h$ in $H, g$ in $G$. Let

$$\delta_{H\backslash G} = \delta_H \delta_G .$$

The normalized induced representation is

$$\text{Ind}(\sigma | H, G) = \{(\sigma\delta_H^{-1/2}_G | H, G) .$$

Why the $\delta$-factor? Well, $\text{Ind}(\delta_H^{1/2}_G)$ is the space of smooth functions on $H\backslash G$. The normalization is motivated by Theorem I.7.6, which asserts that $\text{Ind}(\delta_H^{-1/2}_G)$ is isomorphic to that of smooth one-densities. The symmetry between $\delta^{1/2}\delta$ suggests a useful duality.

The compactly supported induced representations $\text{Ind}_c$, is on the analogous space of functions of compact support on $G$ modulo $H$.

II.9.1. Proposition. If $H\backslash G$ is compact and $(\sigma, U)$ admissible then $\text{Ind}(\sigma | H, G)$ is an admissible representation of $G$.

The hypothesis holds when $G$ is a reductive p-adic group and $H$ a parabolic subgroup.

Proof. If $H\backslash G / K$ is the disjoint union of cosets $HxK$ (for $x$ in a finite set $X$), then the map

$$f \mapsto (f(x))$$

is a linear isomorphism

$$(\text{II.9.2} ) \quad \text{Ind}_c(\sigma | H, G)^K = \bigoplus_{x \in X} U^{H\backslash xKx^{-1}}.$$

Suppose $(\pi, V)$ to be a smooth representation of $G, (\sigma, U)$ one of $H$. The map

$$\Lambda: \text{Ind}(\sigma | H, G) \to U$$

taking $f$ to $f(1)$ is an $H$-morphism from $\text{Ind}(\sigma)$ to $\sigma\delta_H^{1/2}_G \delta^{-1/2}_G$. If we are given a $G$-morphism from $V$ to $\text{Ind}(\sigma | H, G)$ then composition with $\Lambda$ induces an $H$-morphism from $V$ to $\sigma\delta_H^{1/2}_G \delta^{-1/2}_G$.

II.9.3. Proposition. (Frobenius reciprocity) If $\pi$ is a smooth representation of $G$ and $\sigma$ one of $H$ then evaluation at $1$ induces a canonical isomorphism

$$\text{Hom}_G(\pi, \text{Ind}(\sigma | H, G)) \to \text{Hom}_H(\pi, \sigma\delta_H^{1/2}_G \delta^{-1/2}_G) .$$

For $F$ in $\text{Ind}(\tilde{\sigma} | H, G)$ and $f$ in $\text{Ind}_c(\sigma | H, G)$ the product $(F(g), f(g))$ lies in $\text{Ind}(\delta_H^{-1/2}_G)$. On this space, fix a positive $G$-invariant integral

$$\int_{P \backslash G} \phi(x) dx .$$

II.9.4. Proposition. The pairing

$$\langle F, f \rangle = \int_{H\backslash G} (F(x), f(x)) dx$$

defines an isomorphism of $\text{Ind}(\tilde{\sigma} | H, G)$ with the smooth dual of $\text{Ind}_c(\sigma | H, G)$.

In particular, if $R = \mathbb{C}$ and $\sigma$ is unitary so is $\text{Ind}(\sigma | H, G)$.
10. Appendix. Semi-simple algebra

In this section I summarize relevant results from [Bourbaki:1958]. In this section, suppose $R$ to be any algebra over $\mathbb{D}$. The main item is this:

II.10.1. Theorem. Suppose given a finite collection of irreducible, finite-dimensional $R$-modules $V_i$ that are pair-wise non-isomorphic. The canonical map from $R$ to $\prod_i \text{End}_\mathbb{D}(V_i)$ is surjective.

This is Corollaire 1 to Proposition 4 of §4.3 in [Bourbaki:2011]. It is (as we shall see) elementary, but it is not easy to extract a succinct account from the literature. The proof I give here proceeds in several steps.

A semi-simple module over $R$ is any direct sum of irreducible modules.

II.10.2. Lemma. A finite-dimensional $R$-module $V$ is semi-simple if and only if every submodule is a summand.

Proof. The only non-trivial thing to prove is that any submodule $U$ is a summand. This will be shown by induction on the codimension of $U$. If it is 0, there is nothing to prove.

Suppose $V$ to be the sum $\bigoplus V_i$ of irreducibles. If all the $V_i$ are contained in $U$ then $V = U$. Otherwise, say $V_1$ is not contained in $U$. Since it is irreducible, we must have $V_1 \cap U = \{0\}$. The projection $V \to V/V_1 = \bigoplus_{i \neq 1} V_i$ is then injective. The codimension of the image of $U$ has smaller codimension than that of $U$ in $V$, so induction $U$ possesses an $R$-stable complement $W$ in $V/V_1$. But $W$ may be identified with a submodule in $V$, and $W \oplus V_1$ is a complement in $V$.

The commutant $R'$ of $R$ in $\text{End}_\mathbb{D}(V)$ is the ring of operators commuting with it. The bicommutant $R''$ is the commutant of the commutant. It contains $R$.

II.10.3. Lemma. Given a finite collection of irreducible $R$-modules $V_i$ of finite dimension that are pair-wise non-isomorphic. If $V = \bigoplus V_i$, the canonical map from the bicommutant of $R$ to $\prod_i \text{End}_\mathbb{D}(V_i)$ is surjective.

Proof. This is immediate from the definition of bicommutant.

Theorem II.10.1 will now follow from:

II.10.4. Lemma. Given any semi-simple $R$-module $V$ of finite dimension, the image of the bicommutant in $\text{End}_\mathbb{D}(V)$ is the same as that of $R$.

Proof. Suppose $(e_i)$ (for $1 \leq i \leq m$) to be a basis of $V$. It must be shown that if $\rho$ lies in the bicommutant, there exists $r$ in $R$ such that $r(e_i) = \rho(e_i)$ for all $i$.

The representation of $R$ on $W = V^m$ is also semisimple. Let $e = (e_i)$ be diagonally embedded. By Lemma II.10.2 the submodule $U = R(e)$ is a direct summand of $W$. The projection from $V$ onto $U$ lies in the commutant $R'$ of $W$, so the bicommutant $R''$ takes $W$ into itself. But this means that $R''(e) = R(e)$, which implies that there for every $\rho$ in $R''$ there exists $r$ in $R$ such that $r(e) = \rho(e)$.

11. References