2:07 a.m. May 5, 2011

Essays on representations of p-adic groups

Smooth representations

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In this essay I'll define smooth and admissible representations of locally profinite groups and prove their basic properties. I shall generally take the coefficient ring to be an arbitrary commutative Noetherian ring $\mathcal{R}$, assumed to contain $\mathbb{Q}$. The point of allowing representations with coefficients in a ring like this is to allow dealing with families of representations in a reasonable way.

Throughout, let G be a locally profinite group.

Contents

1. Introduction

A **smooth** G module over $\mathcal{R}$ is a representation (π, V) of G on an $\mathcal{R}$-module V such that each v in V is fixed by an open subgroup of G. A smooth representation (π, V) is said to be **admissible** if for each open subgroup K in G the subspace V^K of vectors fixed by elements of K is finitely generated over $\mathcal{R}$. Usually $\mathcal{R}$ will be a field (necessarily of characteristic 0) in which case this means just that V^K has finite dimension.

The subspace of smooth vectors in any representation of G is stable under G, since if v is fixed by K then $\pi(g)v$ is fixed by gKg^{-1}.

Important examples of smooth representations of G are the right- and left-regular representations on the space $C_c^\infty(G, \mathcal{R})$:

$$R_g f(x) = f(xg), \quad L_g f(x) = f(g^{-1}x)\,.$$

or the right- and left-regular representations on the space of uniformly smooth functions on G.

Suppose (π, V) to be a smooth representation of G over $\mathcal{R}$. If D is a smooth distribution of compact support with values in $\mathbb{Q}$, there is a canonical operator $\pi(D)$ on V associated to it. Fix for the moment a right-invariant Haar measure dx on G. Recall that given the choice of dx, smooth $\mathbb{Q}$-valued functions may be identified with smooth distributions:

$$\varphi \longmapsto D_\varphi = \varphi(x)\,dx$$

Suppose φ to be the smooth function of compact support on G such that $D = D_\varphi$. Then for v in V we define

$$\pi(D)v = \int_G \varphi(x)\pi(x)v\,dx\,.$$

If v is fixed by the compact open group K and φ is fixed by K with respect to the right regular representation, this is also

$$\mathrm{meas}(K)\sum_{G/K}\varphi(x)\pi(x)v\,.$$

This definition is independent of the choice of measure dx. We can in fact characterize, if not define, $\pi(D)$ solely in terms of D as a distribution. If F is a linear function on V, then $\Phi(x) = F\big(\pi(x)v\big)$ is a locally constant function on G, and we may apply D to it. Then

$$F\big(\pi(D)v\big) = \int_G \varphi(x)F\big(\pi(x)v\big)\,dx = \langle D, \Phi\rangle\,.$$

Suppose that D_1 and D_2 are two smooth distributions of compact support, corresponding to smooth function φ_1 and φ_2. Then

$$\begin{aligned}\pi(D_1)\pi(D_2)v &= \int_G \varphi_1(x)\pi(x)\,dx \int_G \varphi_2(y)\pi(y)v\,dy \\ &= \int_{G\times G}\varphi_1(x)\varphi(y)\pi(xy)v\,dx\,dy \\ &= \int_G \varphi(z)\pi(z)v\,dz \\ &= \pi(D_\varphi)v \\ \text{where}\quad \varphi(z) &= \int_G \varphi_1(zy^{-1})\varphi_2(y)\,dy \\ &= \int_G \varphi_1(y)\varphi_2(y^{-1}z)\,dy\,.\end{aligned}$$

The distribution D_φ, also smooth and of compact support, is called the **convolution** $D_1 * D_2$ of the two operators D_1 and D_2. The **Hecke algebra** $\mathcal{H}$ is the ring of all smooth locally constant distributions with values in $\mathbb{Q}$. It does not have a multiplicative unit.

If K is a compact open subgroup of G and g in G, the double coset KgK defines an element $\mu_{KgK/K}$ of $\mathcal{H}$:

$$\langle \mu_{KgK/K}, f\rangle = \frac{1}{\mathrm{meas}(K)}\int_{KgK} f(x)\,dx\,.$$

Let $\mathcal{H}(G/\!/K)$ be the ring with $\mathbb{Z}$-basis the distributions $\mu_{KgK/K}$. Its unit is $\mu_{K/K}$, which amounts to integration over K. The operator $\pi(\mu_{K/K})$ is projection from V onto its subspace V^K of vectors fixed by K.

For every closed subgroup H of G, define $V(H)$ to be the subspace of V generated by the $\pi(h)v - v$ for h in H.

[projection] **Proposition 1.1.** *For any compact open subgroup K and smooth representation V, we have an equality*

$$V(K) = \{v \in V \mid \pi(\mu_{K/K})v = 0\}$$

and a direct sum decomposition

$$V = V(K) \oplus V^K\,.$$

Proof. If v is fixed by K_* then

$$\frac{1}{[K\colon K_*]} \sum_{K/K_*} \pi(k)v = \pi(\mu_{K/K})v$$

and of course trivially

$$\frac{1}{[K\colon K_*]} \sum_{K/K_*} v = v\ .$$

If we subtract the second from the first, we get

$$v - \pi(\mu_{K/K})v = \frac{-1}{[K\colon K_*]} \sum_{K/K_*} \big(\pi(k)v - v\big) \ \square$$

[vkexactness] **Corollary 1.2.** *The functor* $V \rightsquigarrow V^K$ *is exact for every compact open subgroup* K *of* G*.*

[abelian] **Corollary 1.3.** *Suppose*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

to be an exact sequence of G*-representations. If* V *is smooth, so are* U *and* W*. The representation on* V *is admissible if and only if both* U *and* W *are.*

Thus the categories of smooth and admissible representations are abelian categories.

[restriction-to-K] **Proposition 1.4.** *Suppose* K *to be a fixed compact open subgroup of* G*,* $\mathcal{R}$ *a field. A smooth representation is admissible if and only if its restriction to* K *is the direct sum of irreducible smooth representations of* K*, each with finite multiplicity.*

Proof. Choose a sequence of compact open subgroups K_n normal in K and with $\{1\}$ as limit. Then $V = V(K_n) \oplus V^{K_n}$. The representation of K/K_n decomposes into a finite sum of irreducible representations of K. $\square$

One has to be a bit careful since unless $\mathcal{R}$ is an algebraically closed field, irreducibility is not the same as absolute irreducibility. Things do not behave here very differently from how they do for finite groups.

2. The centre of G

Assume in this section that $\mathcal{R}$ is an algebraically closed field.

If (π, V) is an admissible representation of G then each space V^K is stable under the centre Z_G of G. The subgroup $Z_G \cap K$ acts trivially on it.

If C is a commuting set of linear operators acting on a vector space V of dimension n and γ a map from C to $\mathcal{R}$, let

$$V_{[\gamma]} = \big\{ v \in V \,\big|\, \big(c - \gamma(c)\big)^n v = 0 \big\}$$

Different γ give rise to complementary subspaces, since if $\alpha \ne \beta$ we can find polynomials $a(x)$, $b(x)$ with

$$1 = a(x)(x-\alpha)^n + b(x)(x-\beta)^n\ .$$

[commuting] **Lemma 2.1.** *Suppose* C *to be any commuting set of linear operators acting on the finite dimensional vector space* V *over* $\mathcal{R}$*. There exists a direct sum decomposition of* V *into non-zero spaces* $V_{[\gamma]}$*.*

Proof. The technical problem is that I make no assumption on the size of C, although in subsequent applications C will be finite.

If C is finite the result is familiar and easy to prove by induction. if $C_1 \subseteq C_2$ the decomposition for C_2 refines that for C_1. Any linearly ordered collection of finite subsets of C is countable, and the decompositions corresponding C_i are successive refinements which must eventually stabilize. This allows us to apply Zorn's Lemma to conclude. □

From this follows immediately:

[centre] **Proposition 2.2.** *The Z_G-module V^K decomposes into a direct sum of primary components $V^K_{[\omega]}$, where the ω vary over a finite set of homomorphisms from Z_G to $\mathcal{R}^\times$.*

The characters ω occurring in this decomposition are called the **central characters** of π.

If π is irreducible there is just one component and the centre must act as scalar multiplication by a single character. In general, I call an admissible representation **centrally simple** if this occurs. If Z_G acts through the character ω then π is called an ω-representation. For any central character ω with values in $\mathcal{R}^\times$ the Hecke algebra $\mathcal{H}_{\mathcal{R},\omega}$ is that of uniformly smooth functions on G compactly supported modulo Z_G such that

$$f(zg) = \omega(z)f(g)\,.$$

If π is centrally simple with central character ω it becomes a module over the Hecke algebra $\mathcal{H}_{\omega^{-1}}$:

$$\pi(f)v = \int_{G/Z_G} f(x)\pi(x)v\,dx\,,$$

which is well defined since $f(zx)\pi(zx) = f(x)\pi(x)$.

3. The contragredient

If (π, V) is an admissible representation of G, the smooth vectors in its linear dual $\mathrm{Hom}_{\mathcal{R}}(V, \mathcal{R})$ define its **contragredient** representation $(\widetilde{\pi}, \widetilde{V})$. If K is a compact open subgroup of G then because $V = V^K \oplus V(K)$ the subspace of K-fixed vectors in $\widetilde{V}$ is equal to

$$\mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R})\ .$$

From the exact sequence of R-modules

$$\mathcal{R}^n \longrightarrow V^K \longrightarrow 0$$

we deduce

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R}) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^n, \mathcal{R}) \cong \mathcal{R}^n\ .$$

Therefore $\widetilde{V}^K$ is finitely generated over $\mathcal{R}$, and $\widetilde{\pi}$ is again admissible. If $\mathcal{R}$ is a field, which is often the only case in which contragredients are significant, the assignment of $\widetilde{\pi}$ to π is exact, and the canonical map from V into the contragredient of its contragredient will be an isomorphism.

[contraexact] **Proposition 3.1.** *Suppose $\mathcal{R}$ to be a field. If*

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of admissible representations, then so isIf

$$0 \to \widetilde{W} \to \widetilde{V} \to \widetilde{U} \to 0$$

4. More about Hecke algebras

What is the relationship between smooth G-representations and the associated representation of its Hecke algebra $\mathcal{H}$?

[hecke-same] **Proposition 4.1.** *Suppose (π_i, V_i) are two smooth representations of G. Then*

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}_{\mathcal{H}}(V_1, V_2)\,.$$

Proof. Any G-homomorphism is clearly a homomorphism of modules over the Hecke algebra as well. So suppose now that one is given a map F of modules over the Hecke algebra. Suppose v in V_1, g in G, and choose a compact open subgroup K fixing v, $\pi_1(g)v$, $F(v)$, and $\pi_2(g)F(v)$. Then

$$\begin{aligned} F\big(\pi_1(g)v\big) &= \frac{F\big(\pi_1(\mu_{KgK})v\big)}{|KgK/K|} \\ &= \frac{\pi_2(\mu_{KgK})F(v))}{|KgK/K|} \\ &= \pi_2(g)F(v)\,. \end{aligned}$$ ∎

A smooth representation is said to be **co-generated** by a subspace U if every non-zero G-stable subspace of V intersects U non-trivially. This is dual to the condition of generation, in the following sense:

[co-generation] **Lemma 4.2.** *Suppose K to be a compact open subgroup of G. The admissible representation (π, V) is generated by V^K if and only if its smooth contragredient is co-generated by $\widetilde{V}^K$.*

Proof. Suppose that V is generated by V^K, and suppose U to be a G-stable subspace of $\widetilde{V}$ with $U \cap \widetilde{V}^K = U^K = 0$. If $U^\perp$ is the annihilator of U in $\widetilde{\widetilde{V}} = V$, then $\big(V/U^\perp\big)^K = \widetilde{U}^K = 0$. Thus $V^K = (U^\perp)^K$, and since V^K generates V, $V = U^\perp$ and $U = 0$. The converse argument is similar. ∎

[two-hecke] **Proposition 4.3.** *Suppose that (π_i, V_i) are two smooth representations of G, and that K is a compact open subgroup of G. If*

- *(a) the space V_1 is generated as a G-space by V_1^K;*
- *(b) the space V_2 is co-generated as a G-space by V_2^K.*

Then

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}_{\mathcal{H}(G/\!/K)}(V_1^K, V_2^K)\,.$$

These conditions are satisfied if $V_1 = V_2$ is irreducible, for example.

Proof. If F lies in $\mathrm{Hom}_G(V_1, V_2)$ then for any f in $\mathcal{H}$ we have

$$F\big(\pi_1(f)v\big) = \pi_2(f)F(v)$$

for every f in $\mathcal{H}$ and v in V_1^K. Conversely, if we are given F in $\mathrm{Hom}_{\mathcal{H}(G/\!/K)}(V_1^K, V_2^K)$ then since V_1^K generates V_1 this formula will serve to define a G-map from V_1 to V_2 once we know that

> *if v lies in V_1^K, f in $\mathcal{H}$, and $\pi_1(f)v = 0$ then $\pi_2(f)F(v) = 0$.*

But if $\pi_1(f)v = 0$ then for every h in $\mathcal{H}$

$$\pi_1(\mu_{K/K} * h)\pi_1(f)v = \pi_1(\mu_{K/K} * h * f * \mu_{K/K})v = 0\,.$$

Since F is assumed to be $\mathcal{H}(G//K)$-covariant,

$$\pi_2(\mu_{K/K} * h * f * \mu_{K/K})F(v) = \pi_2(\mu_{K/K} * h)\pi_2(f)F(v) = 0$$

for every h in $\mathcal{H}$. This means that the G-space generated by $\pi_2(f)F(v)$ has no non-zero K-invariant vectors, which means by assumption that it is 0. □

[hk-irr] **Proposition 4.4.** *Suppose (π, V) to be a smooth representation of G.*

(a) If π is irreducible then V^K is an irreducible module over $\mathcal{H}(G//K)$ for all K.

♣ [two-hecke] *(b) If V satisfies conditions (a) and (b) of Proposition 4.3 and V^K is an irreducible module over $\mathcal{H}(G//K)$ then π is irreducible.*

Proof. Suppose (π, V) to be irreducible, and let U be any non-trivial $\mathcal{H}(G//K)$-stable subspace of V^K. Since V is irreducible, U must generate V as a G-space, so every v in V is of the form $\sum c_i\pi)g_iu_i$ with u_i in U. But then for v in V^K

$$v = \pi(\mu_{K/K})v = \sum c_i\pi(\mu_{K/K})\pi(g_i)u_i = \textit{constant}\, c_i\pi(Kg_iK)u_i$$

which lies in U since U is assumed to be stable under $\mathcal{H}(G//K)$. So $V^K \subseteq U$.

♣ [two-hecke] Conversely, assume conditions (a) and (b) of Proposition 4.3 to hold for V, and assume V^K irreducible. If U is any non-zero G-stable subspace of V then by (b) $U^K \neq 0$ must be a submodule of V^K, but must equal it because of irreducibility. But (a) implies that then $U = V$. □

Does every finite-dimensional module over $\mathcal{H}(G//K)$ arise as the space V^K for some admissible V?
♣ [two-hecke] And more particularly one satisfying the conditions (a) and (b) of Proposition 4.3?

The answer is motivated by a simple observation. Let V be an admissible representation of G, $U = V^K$. To each v in V we can assign the function

$$F_v\colon G \longrightarrow U, \quad g \longmapsto \pi(\mu_{K/K})\pi(g)v$$

Then $f * F_v = \pi(f)F_v$ for every f in $\mathcal{H}(G//K)$, and the map from V to $C^\infty(G, U)$ is covariant with respect to the right regular action of G.

Conversely, if U is a finite-dimensional representation of $\mathcal{H}(G//K)$, define I_U to be the space of all functions $F\colon G \to U$ such that $f * F = \pi(f)F$ for all f in the Hecke algebra. There is a canonical embedding of U itself into this, and let V be the subspace of I_U generated by this copy. It is not hard to verify that $V^K = U$, and that V is also co-generated by U.

5. Characters

If (π, V) is admissible then the trace of every f in $\mathcal{H}(G)$ is well defined since it may be identified with an operator on some V^K, which is finite-dimensional. This defines the character of π as linear functional on the Hecke algebra.

[character] **Proposition 5.1.** *If the (π_i, V_i) are inequivalent irreducible admissible representations of G then their characters are linearly independent.*

♣ [hk-irr] *Proof.* Choose K so small that the $V_i^K \neq 0$ for all i. They then form, according to Proposition 4.4 and
♣ [two-hecke] Proposition 4.3, inequivalent modules over $\mathcal{H}(G//K)$. Because of irreducibility, the image of the Hecke algebra in $\mathrm{End}(U)$ is all of it. Because the π_i are all distinct as well as irreducible, the map from the Hecke algebra into $\prod \mathrm{End}(U_i)$ is surjective. Suppose now that

$$\sum c_i \mathrm{Tr}_i = 0\,,$$

which means that

$$\sum c_i \mathrm{Tr}\big(\pi_i(f)\big) = 0$$

for all f in the Hecke algebra. But then we can choose f in the Hecke algebra such that $\pi_i(f) = I$ but all the other $\pi_j(f) = 0$, which implies that $c_i = 0$. ∎

The following is trivial:

[char-exact] **Proposition 5.2.** *If*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence of admissible G-spaces, then the character of V is the sum of the characters of U and V.

It implies easily one half of this refinement:

[jh] **Proposition 5.3.** *Two admissible representations of finite G-length have the same Jordan-Hölder factors if and only if they have the same characters.*

Proof. It remains to be seen that if U and V have the same characters then they have the same Jordan-Hölder factors. But for this, by the previous result, it suffices to see that the semi-simplifications of U and V are isomorphic. But this follows from Proposition 5.1 and an induction argument. ∎ ♣ [character]

6. Products

In this section, I assume $\mathcal{R}$ to be an algebraically closed fiedl.

Let G_1, G_2 be two locally profinite groups, and let $G = G_1 \times G_2$. It is also locally profinite.

$H(G//K) \cong H(G_1//K_1) \otimes H(G_2//K_2)$

Théorème 2, p. 87 of [Bourbaki:1958].

7. Matrix coefficients

In this section I assume $\mathcal{R}$ to be an algebraically closed field F.

If (π, V) is an admissible representation the **matrix coefficient** associated to the pair v in V, $\tilde{v}$ in $\widetilde{V}$ is the function

$$c_{v,\tilde{v}} = \langle \pi(g)v, \tilde{v} \rangle \,,$$

which is uniformly smooth. Let $\mathcal{A}(\pi)$ be the space of smooth functions spanned by the matrix coefficienst of π. It is a smooth representation of $G \times G$ (one factor acting on the left, one on the right), and the map from $V \otimes \widetilde{V}$ to $A(G)$ is $G \times G$-covariant.

Let $\mathcal{A}(G)$ be the space of smooth functions on G contained in a $G \times G$-stable admissible subrepresentation of $C^\infty(G)$.

The following is due to Harish-Chandra (Lemme I.6.1 of [Waldspurger:2003]).

[mc] **Proposition 7.1.** *Suppose F to be a smooth function on G. The following are equivalent:*

(a) The function F is contained in some $\mathcal{A}(\pi)$ with π admissible;
(b) the space $A_L(F)$ spanned by all $L_g F$ is an admissible L_G-representation;
item(c) the space $A_R(F)$ spanned by all $R_g F$ is an admissible R_G-representation;
item(d) the function F lies in $\mathcal{A}(G)$.

Proof. What will be shown is that if $A_R(F)$ is admissible, then F is the matrix coefficient of an admissible representation. The argument may be motivated by looking at a matrix coefficient $\langle \pi(g)v, \tilde{v} \rangle$.

Suppose that $V = A_R(F)$ is an admissible representation of L_G. It must be shown that it is contained in some $\mathcal{A}(\pi)$. For every g in G, define

$$\varphi_g\colon V \longrightarrow F, \quad v \longmapsto v(g)\,.$$

We shall need in a minute the equation

$$\begin{aligned}\langle R_h\varphi_g, v\rangle &= \langle \varphi_g, R_{h^{-1}}v\rangle \\ &= (R_{h^{-1}}v)(f) \\ &= v(gh^{-1}) = v(gh^{-1}g^{-1}\cdot g) \\ &= \langle \varphi_{gh^{-1}}, v\rangle\,.\end{aligned}$$

(a) *The function φ_g lies in $\widetilde{V}$.* If F is fixed on the left by K then so is every R_gF, hence all of V. But then from the equation above it follows that if h lies in $g^{-1}Kg$ then $R_h varphi_g = \varphi_g$.

(b) *The space V_* is stable under R_G.* The equation above tells us that $R_h\varphi_g = \varphi_{gh^{-1}}$.

(c) *The space spanned by all φ_g is all of $\widetilde{V}$.*

According to [contraexact:,] it suffices to show that V embeds into the dual of V_*. This is immediate.

(d) For v in V,

$$v(g) = \langle \varphi_g, v\rangle = \langle \varphi_1, R_{g^{-1}}v\rangle\,,$$

so V is contained in a space of matrix coefficients. □

8. Unitary representations

In this section I take $\mathcal{R}$ to be $\mathbb{C}$.

A **unitary** representation of G is one with a positive definite G-invariant Hermitian inner product. Unitary representations are important because they are the ones that appear in orthogonal decompositions of arithmetic quotients, and this has arithmetic consequences. In one classic example, unitarity is related to Ramanujan's conjecture.

We start with a very simple result, which is trivial to prove.

[unitary-sum] **Proposition 8.1.** *Every admissible unitary representation is a countable direct sum of irreducible unitary representations, each occurring with finite multiplicity.*

It is easy to see that the matrix coefficients of a unitary representation are bounded. A much stronger condition on matrix coefficients is fundamental. Suppose π to be a representation with central character ω. It is said to be **square-integrable modulo the centre Z_G of G** if $\omega| = 1$ and every matrix coefficient is is square-integrable on G/Z_G.

[irr-sqint] **Proposition 8.2.** *If π is an irreducible admissible representation of G, then it is square-integrable if and only if a single matrix coefficient is square-integrable.*

Since an irreducible square-integrable representation may be embedded into $\mathrm{L}^2(G)$, it is unitary. More precisely:

[sqint-unitary] **Proposition 8.3.** *Suppose (π, V) to be an irreducible square-integrable representation. For $\tilde{u}_0 \neq 0$, $\tilde{v}_0 \neq 0$ in $\widetilde{V}$ the pairing*

$$u \bullet v = \int_{G/Z_G} \langle \pi(g)u, \tilde{u}_0\rangle \overline{\langle \pi(g^{-1})v, \tilde{v}_0\rangle}\, dg\,.$$

defines a G-invariant positive definite inner product on V.

The matrix coefficients of a representation are intrinsic, in the sense that isomorphic representations have the same matrix coefficients. In fact, matrix coefficients distinguish a representation. For square-integrable representations, there is a strong form of this assertion.

[schur-orthogonality] **Proposition 8.4.** *Let (π, U) and (ρ, V) be irreducible square-integrable admissible representations with the same central character. For u in U, v in V, $\tilde{u}$ in $\widetilde{U}$, $\tilde{v}$ in $\widetilde{V}$ consider the integral*

$$I = \int_{G/Z_G} \langle \pi(g)u, \tilde{u}\rangle \langle \rho(g^{-1})v, \tilde{v}\rangle \, dg \, .$$

- *(a) If π and ρ are isomorphic then $I = c_\pi \langle u, \tilde{u}\rangle\langle v, \tilde{v}\rangle$ for some constant $c_\pi > 0$;*
- *(b) if they are not isomorphic, $I = 0$.*

If G is finite and the measure normalized so all of G has measure 1, then $c_\pi = 1/d_\pi$, where d_π is the dimension of π. In general, $1/c_\pi$ is called its **formal degree**.

Proof. The pairing taking u in U, $\tilde{v}$ in $\widetilde{V}$ to

$$I = \int_{G/Z_G} \langle \pi(g)u, \tilde{u}\rangle \langle \rho(g^{-1})v, \tilde{v}\rangle \, dg$$

defines a G-invariant pairing of U and $\widetilde{V}$, or equivalently a map from U to the contragredient of $\tilde{\rho}$, which is ρ itself. If π and ρ are not isomorphic then it must consequelently be 0. If π and ρ are isomorphic, it must be a scalar multiple of the canonical pairing. We may as well assume $U = V$, and the integral is equal to $c_{\tilde{u},v}\langle u, \tilde{v}\rangle$. But then it can be seen that $c_{\tilde{u},v}$ is equal to $c_\pi\langle v, \tilde{u}\rangle$ for some c_π.

Fix a G-invariant positive-definite Hermitian inner product on V. Fix $\tilde{v}$ for the moment, and let then choose v_0 in V such that

$$v \bullet v_0 = \langle v, \tilde{v}\rangle$$

for all v in V. Then

$$\begin{aligned} c_\pi (v \bullet v_0)(v \bullet v_0) &= \int_{G/Z_G} \big(\pi(g)v \bullet v_0\big)\big(\pi(g^{-1}v \bullet v_0\big)\, dg \\ &= \int_{G/Z_G} \big(\pi(g)v \bullet v_0\big)\overline{\big(v_0 \bullet \pi(g^{-1}v\big)}\, dg \\ &= \int_{G/Z_G} \big(\pi(g)v \bullet v_0\big)\overline{\big(\pi(g)v_0 \bullet v\big)}\, dg \, . \end{aligned}$$

If we set $v = v_0$ we deduce that $c_\pi > 0$. ∎

9. Induced representations

If H is a closed subgroup of G and (σ, U) is a smooth representation of H, the **unnormalized** smooth representation $\,|\,(\sigma \,|\, H, G)$ **induced** by σ is the right regular representation of G on the space of all uniformly smooth functions $f\colon G \to U$ such that

$$f(hg) = \sigma(h)f(g)$$

for all h in H, g in G. Let

$$\delta_{H\backslash G} = \delta_G/\delta_H \, .$$

The **normalized** induced representation is

$$\mathrm{Ind}(\sigma \,|\, H, G) = \,|\, \big(\sigma\delta_{H\backslash G}^{-1/2} \,\big|\, H, G\big) \, .$$

♣ **[one-densities]** The normalization is motivated by Corollary 7.5(profinite) , which asserts that $\mathrm{Ind}(\delta_{H\backslash G}^{1/2})$ is the space of smooth functions on $H\backslash G$ and $\mathrm{Ind}(\delta_{H\backslash G}^{-1/2})$ that of smooth one-densities. This will lead to an important duality.

The compactly supported induced representations Ind_c is on the analogous space of functions of compact support on G modulo H.

[induced-admissible] **Proposition 9.1.** *If $H\backslash G$ is compact and (σ, U) admissible then $\mathrm{Ind}(\sigma \,|\, H, G)$ is an admissible representation of G.*

The hypothesis holds when G is a reductive $\mathfrak{p}$-adic group and H a parabolic subgroup.

Proof. If $H\backslash G/K$ is the disjoint union of cosets HxK (for x in a finite set X), then the map

$$f \longmapsto \big(f(x)\big)$$

is a linear isomorphism

$$\mathrm{Ind}(\sigma \,|\, H, G)^K \cong \oplus_{x\in X} U^{H\cap xKx^{-1}} \;\square$$

[also-free] **Corollary 9.2.** *If U is free over $\mathcal{R}$ so are the induced representations.*

This follows from the proof.

Suppose (π, V) to be a smooth representation of G, (σ, U) one of H. The map

$$\Lambda\colon \mathrm{Ind}(\sigma \,|\, H, G) \to U$$

taking f to $f(1)$ is an H-morphism from $\mathrm{Ind}(\sigma)$ to $\sigma\delta_H^{1/2}\delta_G^{-1/2}$. If we are given a G-morphism from V to $\mathrm{Ind}(\sigma \,|\, H, G)$ then composition with Λ induces an H-morphism from V to $\sigma\delta_H^{1/2}\delta_G^{-1/2}$.

[frobenius] **Proposition 9.3.** (Frobenius reciprocity) *If π is a smooth representation of G and σ one of H then evaluation at 1 induces a canonical isomorphism*

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma \,|\, H, G)\big) \to \mathrm{Hom}_H\big(\pi, \sigma\delta_H^{1/2}\delta_G^{-1/2}\big) \,.$$

♣ **[one-densities]** For F in $\mathrm{Ind}(\widetilde{\sigma} \,|\, H, G)$ and f in $\mathrm{Ind}_c(\sigma \,|\, H, G)$ then according to Corollary 7.5(profinite) the function $\langle F(g), f(g)\rangle$ is a left-H-invariant one-density of compact support on $H\backslash G$. If we are given right invariant Haar measures dg on G and dh on H then we can define a canonical pairing between $\mathrm{Ind}(\widetilde{\sigma} \,|\, H, G)$ and $\mathrm{Ind}_c(\sigma \,|\, H, G)$ according to the formula

$$\langle F, f\rangle = \int_{H\backslash G} \langle F(x), f(x)\rangle \, dx$$

Thus there is an essentially canonical G-covariant map from $\mathrm{Ind}(\widetilde{\sigma} \,|\, H, G)$ to the smooth dual of $\mathrm{Ind}_c(\sigma \,|\, H, G)$. In particular, if $\mathcal{R} = \mathbb{C}$ and σ is unitary so is $\mathrm{Ind}(\sigma \,|\, H, G)$.

10. References

1. N. Bourbaki, **Modules et anneaux semi-simples**, Chapter 8 of **Algèbres**. Hermann, 1958.

2. j.-L. Waldspurger, 'La formule de Plancherel pour les groupes p-adiques', *Journal of the Institute of mathematics of Jussieu* **2** (2003), 235-333.