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Essays on Coxeter groups

Reflections

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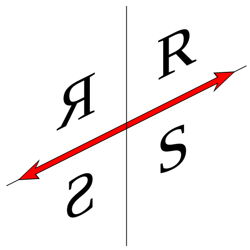


A **linear reflection** of a real vector space V is a linear transformation fixing a hyperplane passing through the origin and acting as multiplication by -1 on a transverse line through the origin. More explicitly, a reflection is a linear map of the form

$$s\colon\ v \longmapsto v - \langle \alpha_s, v\rangle \alpha_s^\vee$$

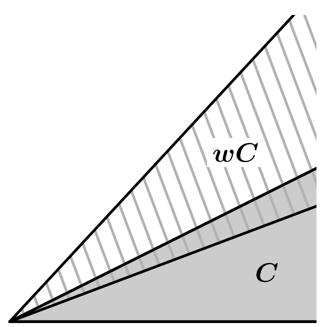
where $\langle \alpha_s, \alpha_s^\vee\rangle = 2$. The pair α_s, $\alpha_s^\vee$ are determined uniquely up to multiplication by c, c^{-1}.

As we shall see in a moment, all reflections look essentially the same, in the sense that they are all conjugate in $\mathrm{GL}(V)$. This essay is concerned essentially with classifying pairs of reflections up to conjugacy, which is a more interesting problem.

The question I really want to answer about pairs of reflections is more specific. If s and t are reflections in distinct hyperplanes, there are four connected components of the complement of these hyperplanes. I call a choice of one of them an **orientation** of the pair, since it is determined by a choice of sign for the corresponding α_s and α_t. Suppose we reflect C by s to get $s(C)$, by t to get $t(C)$, then again to get $st(C)$, etc. In other words, we consider the configuration we get by transforming C by elements of the group W generated by s and t.

Fix an oriented pair s, t, C. I call this oriented pair a **Vinberg pair** if this condition is satisfied: *whenever* $C \cap wC \neq \emptyset$ *then* $w = I$.

That is to say, what doesn't happen is something like this:



The question to be answered is this:

Under what conditions on s, t, *and* C *do they form a oriented Vinberg pair?*

In fact, I shall do more than answer this question, and produce also the complete classification of pairs of linear reflections up to conjugacy, picking out as we go those which give rise to an oriented Vinberg pair. This essay will be a somewhat leisurely exploration of matters raised in §2 of [Vinberg:1971] and will, I hope, illuminate the conditions imposed on a Cartan matrix.

There are other kinds of reflections in common usage besides linear ones, but they can be interpreted as linear ones and will be discussed at the end of this essay.

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1. Single reflections

Suppose s to be a reflection, $\alpha = \alpha_s = 0$ the equation of the hyperplane H_s in which it reflects. There exists a unique eigenvector $\alpha^\vee = \alpha_s^\vee$ of s with $\langle \alpha, \alpha^\vee \rangle = 2$, and then

$$s(v) = v - \langle \alpha, v \rangle \alpha^\vee .$$

If α is replaced by $c\alpha$ ($c \neq 0$), $\alpha^\vee$ is replaced by $c^{-1}\alpha^\vee$.

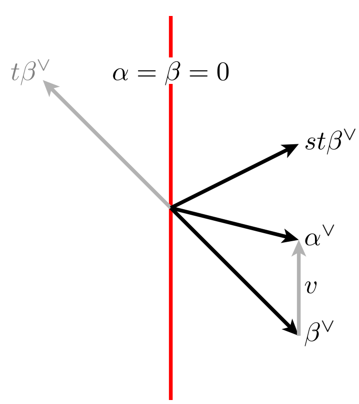
Often a reflection will be orthogonal with respect to some inner product. This happens when the line of -1 eigenvectors is non-isotropic and orthogonal to the hyperplane of reflection $\langle \alpha, u \rangle = 0$. In these circumstances the reflection takes v to

$$v - 2\left(\frac{\alpha^\vee \bullet v}{\alpha^\vee \bullet \alpha^\vee}\right)\alpha^\vee .$$

Here, the inner product induces a linear map from V to its linear dual space $V^\vee$, and α is the image of $2\alpha^\vee/(\alpha^\vee \bullet \alpha^\vee)$.

In a moment we'll need this observation:

[ref-shear] **Lemma 1.1.** *If s and t are reflections in the same hyperplane, then st is a shear parallel to that hyperplane.*



When interpreting the figures involving reflections, keep in mind that locating $\alpha^\vee$ tells you on which side of the line $\alpha = 0$ the linear function α is positive.

Proof. We may assume $\alpha = \beta$, so $\langle \alpha, \beta^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2$ and $v = \alpha^\vee - \beta^\vee$ is parallel to the hyperplane of reflection. But also

$$\begin{aligned} s\beta^\vee &= \beta^\vee - \langle \alpha, \beta^\vee \rangle \alpha^\vee \\ t\beta^\vee &= -\beta^\vee \\ st\beta^\vee &= -\beta^\vee + 2\alpha^\vee \\ &= \beta^\vee + 2v \,. \end{aligned}$$

The group $\mathrm{GL}(V)$ acts on $V^\vee$ according to the prescription $\langle gv^\vee, u \rangle = \langle v^\vee, g^{-1}u \rangle$. If $V = \mathbb{R}^n$, incarnated as column vectors, then linear functions are row vectors and the canonical pairing is the matrix product $v \cdot v^\vee$. If g is represented by the matrix M then g takes v to Mu and $v^\vee$ to $v^\vee M^{-1}$.

[single-conjugate] **Proposition 1.2.** *Reflections make up a single conjugacy class in* $\mathrm{GL}(V)$.

Proof. Suppose s, t two reflections corresponding to $(\alpha, \alpha^\vee)$, $(\beta, \beta^\vee)$. The group $\mathrm{GL}(U)$ acts transitively on linear functions so we find g taking α to β. So now we may assume $\alpha = \beta$.

The Proposition follows from the Lemma since the subgroup of $\mathrm{GL}(V)$ stabilizing the hyperplane $\alpha = 0$ acts transitively on vectors $\alpha^\vee$ with $\langle \alpha, \alpha^\vee \rangle = 2$.

2. Pairs of reflections

In contrast to a single reflection, pairs of reflections turn out to make up essentially a one-parameter family of conjugacy classes.

Suppose s and t to be a pair of reflections, corresponding to α, $\alpha^\vee$ and $\beta, \beta^\vee$. Define associated real constants

$$c_{s,t} = \langle \alpha, \beta^\vee \rangle, \quad c_{t,s} = \langle \beta, \alpha^\vee \rangle \,.$$

If (s_*, t_*) is another pair of reflections conjugate to s, t under the linear transformation g, then

$$\begin{aligned} g\alpha = c_\alpha \alpha_*, &\quad g\alpha^\vee = c_\alpha^{-1} \alpha_*^\vee, \\ g\beta = c_\beta \beta_*, &\quad g\beta^\vee = c_\beta^{-1} \beta_*^\vee \end{aligned}$$

for some non-zero constants c_α and c_β, and consequently

$$c_{s_*,t_*} = (c_\beta / c_\alpha) c_{s,t}, \quad c_{t_*,s_*} = (c_\alpha / c_\beta) c_{t,s} \,.$$

The constants $c_{s,t}$ and $c_{t,s}$ are not conjugation-invariant, but the product $n_{s,t} = c_{s,t} c_{t,s}$ is. Also invariant is whether $c_{s,t}$ or $c_{t,s}$ is 0. For generic pairs $n_{s,t}$ will turn out to possess a familiar interpretation.

Here is the first main result of this essay:

[conjugacy-pairs] **Theorem 2.1.** *For $n \neq 0, 4$ there exists exactly one conjugacy class of reflection pairs s, t with $n_{s,t} = n$. In the exceptional cases:*

- $n = 0$: *there exist three conjugacy classes of pairs: (a) $c_{s,t} = c_{t,s} = 0$, in which case s and t commute, generating a group of order 4; (b) one of $c_{s,t}$ and $c_{t,s} = 0$ and the other is positive; (c) one of $c_{s,t}$ and $c_{t,s} = 0$ and the other is negative;*
- $n = 4$: *there exist three classes in dimension 2 and four in dimensions 3 or more: (a) $s = t$; (b) α and β are linearly dependent, but $\alpha^\vee$ and $\beta^\vee$ are not; (c) $\alpha^\vee$ and $\beta^\vee$ are linearly dependent, but α and β are not; (d) (only in dimensions 3 or more) both α, β and $\alpha^\vee$, $\beta^\vee$ are linearly independent.*

Details about these possibilities will appear in the course of the proof.

How do we classify Vinberg pairs up to conjugation? First of all, every Vinberg pair determines a pair of reflections, so one step is to determine which reflection pairs are in the image. Then we have to specify which choices of C are possible, and which among these are conjugate.

[vinberg-pairs] **Theorem 2.2.** *The reflection pair s, t arises from an oriented Vinberg pair in precisely the following circumstances:*

(a) $n_{s,t} = c_{s,t} = c_{t,s} = 0$*;*
(b) $0 < n_{s,t} < 4$ *with* $n_{s,t} = 4\cos^2(\pi/m)$ *for some integer* $m \geq 3$*;*
(c) $n_{s,t} = 4$ *and* α_s *and* α_t *are linearly independent;*
(d) $4 < n_{s,t}$.

For each pair s, t there is a unique conjugacy class of acceptable orientations, charactrized by the conditions $c_{s,t} \leq 0$, $c_{t,s} \leq 0$.

Of course if $n_{s,t} > 0$ then $c_{s,t} < 0$ if and only if $c_{t,s} < 0$. We shall see that α and β are automatically linearly independent as long as $n_{s,t} \neq 4$.

The proof of the first Theorem will take most of the rest of this essay, and the second will be proved as we go along looking at the different cases. But first we'll make a short digression.

3. Vinberg pairs

Suppose s, t, C make up a Vinberg pair. Let $W = W_{\langle s,t \rangle}$ be the group they generate. It has order at least 2. There is a strong relationship between its geometric and combinatorial properties.

Let $S = \{s, t\}$. Choose α_s, α_t so that α_s and α_t are both positive on C. Let $\Delta = \{\alpha_s, \alpha_t\}$. For $T \subseteq S$ let

$$\begin{aligned} C_T &= \{v \in \overline{C} \mid \langle \alpha_r, v \rangle = 0 \text{ for } r \in T\} \\ W_T &= \text{ the subgroup of } W \text{ generated by } r \text{ in } T \ . \end{aligned}$$

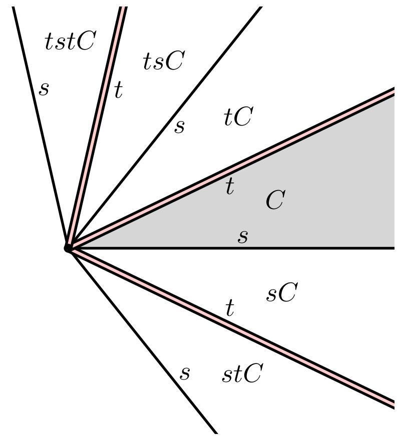
The C_T are the open boundary components of C. Of course the groups W_T other than W itself are rather small. The elements of W_T fix the points of C_T. There is a strong converse:

[fixT] **Proposition 3.1.** *If $wx \in \overline{C}$ for $x \in \overline{C}$ then $w \in W_T$ where $x \in C_T$ (and $wx = x$).*

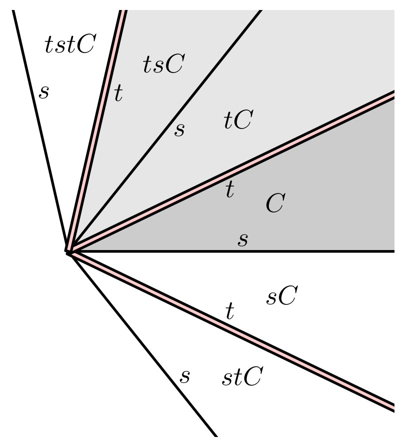
Proof. If $T = S$ there is nothing to be proven. If $T = \emptyset$ then $C_T = C$. If U is a small open disk around x then $wU \cap C$ will not be empty, so there exists y in B with wy in B. By the definition of Vinberg pairs $w = I$. So now suppose T is $\{r\}$ and $wx \in \overline{C}$. Again choose a small disc around x and let B be its intersection with C. Then wB meets either C or rC for some r in S. In the first case $w = I$. In the second case we replace w by rw and wind up in the first case, so $w = r$. ▯

A **chamber** of W is one of the transforms wC. Let $\mathcal{C}$ be the union of all the closed chambers $w\overline{C}$. The definition of a Vinberg pair means that the chambers and their open boundary components partition $\mathcal{C}$. The open chambers wC are the connected components of the complement in $\mathcal{C}$ of the W-transforms of
♣ [fixT] the boundary of C. Proposition 3.1 means that $\overline{C}$ is a strong fundamental domain for the action of W on $\mathcal{C}$.

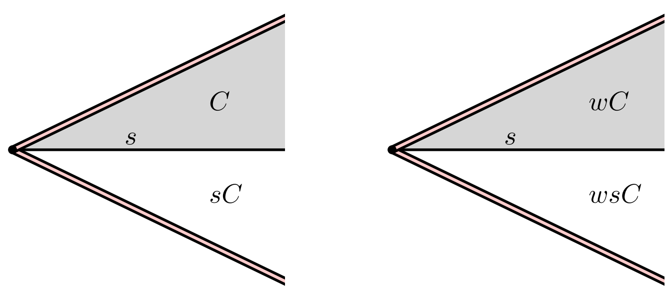
Given a chamber D, there exists a unique w in W with $wD = C$. Thus each edge is mapped uniquely to either s or t. The edges of the chambers wC can thus be *labelled* by s and t, as in the following figure:



A **gallery** is a finite sequence of neighbouring chambers starting at C. In the following figure, we are looking at the gallery C, tC, tsC.



Galleries are labelled by non-repeating words in s and t—the boundaries crossed from C match the generators in the word, left to right. The following figures demonstrate why this is true.



The length $\ell(w)$ of w is the length of the shortest word in s and t representing it. The following is the basic link between combinatorics and geometry:

[swa] **Proposition 3.2.** *Suppose w in W, r in S. Then*

$$\ell(rw) < \ell(w) \text{ if and only if } \alpha_r < 0 \text{ on } wC$$

Proof. The length of w is the length of the shortest gallery from C to wC. If $\langle \alpha_r, wC \rangle < 0$, then any gallery G from C to wC must cross $\alpha_s = 0$ right at the start—it starts out $C_0 = C$, $C_1 = sC$, $C_2, \ldots$ The gallery $C = rC_1, rC_2, \ldots$ is shorter and leads from C to rwC. $\blacksquare$

If we apply this to rw instead of w we deduce the converse. $\blacksquare$

Define a **root** to be a transform $w\alpha$ of some α in Δ. A root is positive if it is positive on C, negative if it is negative on C. The definition of a Vinberg pair means that every root is either positive or negative. Let Σ be the set of all roots, $\Sigma^{\pm}$ the positive or negative ones.

The group W acts on the set of roots:

$$\langle w\lambda, v \rangle = \langle \lambda, w^{-1}v \rangle \,.$$

For each w in W, define

$$\mathcal{L}_w = \{\lambda > 0 \,|\, w^{-1}\lambda < 0\} = \{\lambda \,|\, \langle \lambda, C \rangle > 0, \langle \lambda, wC \rangle > 0\} \,.$$

If $\lambda = w\alpha$ for α in Δ, then $ws_\alpha w^{-1}$ is a reflection in W with the formula

$$ws_\alpha w^{-1}(v) = w(w^{-1}v - \langle \alpha, w^{-1}v \rangle \alpha^\vee = v - \langle w\alpha, v \rangle w\alpha^\vee$$

♣ **[fixT]** If $\lambda = x\alpha = y\beta$ then $y^{-1}x\alpha = \beta$. By Proposition 3.1, $\alpha = \beta$ and $y^{-1}x$ lies in $W_{\{\alpha\}}$, so the definitions

$$\lambda^\vee = w\alpha^\vee, s_\lambda = xs_\alpha x^{-1}$$

are valid.

[lw] Proposition 3.3. *We have*

(a) for r in S, $\mathcal{L}_r = \{\alpha_r\}$;
(b) if $w^{-1}\alpha_r > 0$ then $\mathcal{L}_{rw}$ is the disjoint union of $r\mathcal{L}_w$ and $\{\alpha_r\}$.

Proof. The set $\mathcal{L}_r$ certainly contains α_r. If it contains some root λ, then $\langle \lambda, C \rangle > 0$ while $\langle \lambda, rC \rangle < 0$. This implies that the half-plane $\lambda = 0$ coincides with $\alpha = 0$. But then $s_\lambda = s_\alpha$ and since $\lambda > 0$ we have $\lambda = \alpha$. $\blacksquare$

I'll leave the next few results as exercises.

[length-roots] Corollary 3.4. *The length of w is the cardinality of $\mathcal{L}_w$.*

[separating] Proposition 3.5. *The length of w is the number of root hyperplanes separating C from wC.*

[calc-separating] Proposition 3.6. *A point of V lies in $\mathcal{C}$ if and only if the number of root hyperplanes separating C from v is finite.*

[calc-convex] Corollary 3.7. *The region $\mathcal{C}$ is convex.*

The possible domains $\mathcal{C}$ can be broadly classified.

[calC] Lemma 3.8. *There are three possibilities for a convex open cone in a two-dimensional real vector space V:*

(a) It is all of V;
(b) it is an open half-plane;
(c) it is an acute wedge.

Proof. Let $\mathcal{C}$ be the cone. Place a circle around the origin. If all of the circle is contained in $\mathcal{C}$, then $\mathcal{C}$ is all of U. Otherwise, there exists a point on the circle not in $\mathcal{C}$. The intersection of $\mathcal{C}$ with the circle may then be defined as a proper subinterval of $(0, 2\pi)$. If the arc has length more than π, then $\mathcal{C}$ must contain the origin, a contradiction. So it must be of length at most π. If it is π, then $\mathcal{C}$ is a half-plane. If not, it is an acute wedge. ▯

We shall see that the three cases are those where W is finite, equivalent to a group of affine reflections, and that where W is a group of hyperbolic orthogonal transformations.

We can summarize this discussion. I define a **Cartan matrix** $(c_{s,t})$ indexed by a finite set Δ to be any real matrix satisfying these conditions:

(a) $c_{\alpha,\alpha} = 2$ for all s;
(b) $c_{\alpha,\beta} = 0$ if and only if $c_{\beta,\alpha} = 0$;
(c) $c_{\alpha,\beta} \le 0$ for all $\alpha \ne \beta$.

We have been discussing here the case in which Δ has two elements.

[vinberg-2] **Theorem 3.9.** *Suppose given a set Δ of two linearly independent linear functions on V, and a map $\alpha \mapsto \alpha^\vee$ from Δ into V. The group generated by the reflections $s_{\alpha,\alpha^\vee}$ for α in Δ, together with the region*

$$C = \{\alpha > 0 \textit{ for all } \alpha \in \Delta\}$$

form a Vinberg pair if and only if the matrix $(\langle \alpha, \beta^\vee \rangle)$ is a Cartan matrix.

4. When $n = 4$

One way to understand conjugation-invariant phenomena is to consider geometrical configurations—for example, how the various hyperplanes and lines associated to reflections relate qualitatively to each other. As we shall see, this works to distinguish $n = 0$ or 4 from the other cases.

The **Cartan matrix** of the pair is

$$\begin{bmatrix} \langle \alpha, \alpha^\vee \rangle & \langle \alpha, \beta^\vee \rangle \\ \langle \beta, \alpha^\vee \rangle & \langle \beta, \beta^\vee \rangle \end{bmatrix} = \begin{bmatrix} c_{s,s} & c_{s,t} \\ c_{t,s} & c_{t,t} \end{bmatrix} = \begin{bmatrix} 2 & c_{s,t} \\ c_{t,s} & 2 \end{bmatrix} .$$

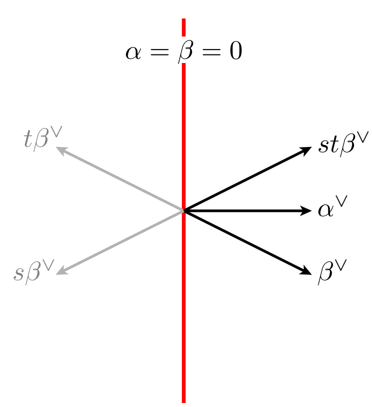
If either α, β or $\alpha^\vee$, $\beta^\vee$ are linearly dependent, the first form of this matrix shows that the Cartan matrix must be singular, hence:

[lin4] **Lemma 4.1.** *If either α, β or $\alpha^\vee$, $\beta^\vee$ are linearly dependent then $n_{s,t} = 4$.*

In analyzing the pairs of reflections s, t with $n = 4$ there are *a priori* four possibilities to be considered: (a) both α, β and $\alpha^\vee$, $\beta^\vee$ are dependent; (b) α, β are dependent but $\alpha^\vee$, $\beta^\vee$ are not; (c) α, β are independent but $\alpha^\vee$, $\beta^\vee$ are dependent; (d) both α, β and $\alpha^\vee$, $\beta^\vee$ are independent.

In case (a) we must have $s = t$.

In case (b), the reflection hyperplanes of s and t are the same. We may take $\alpha = \beta$.

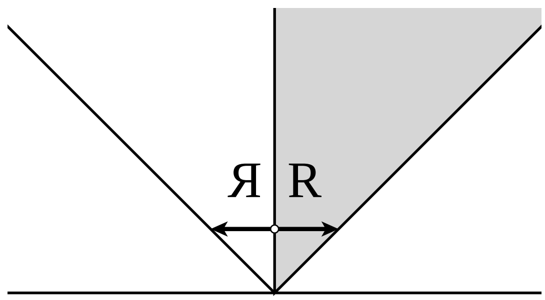


When $\alpha = \beta$

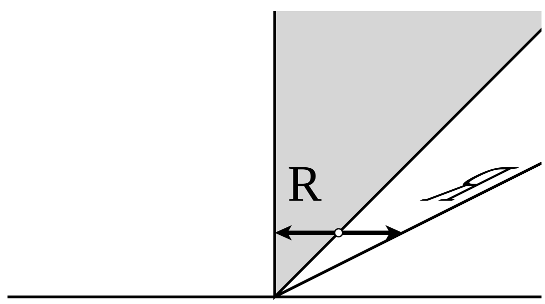
In this case, we can choose our coordinate system so that $\alpha = \beta = x$. By swapping and scaling we may ♣ [ref-shear] assume that $\alpha^\vee = (2, 0)$ and $\beta^\vee = (2, -1)$. As we have already seen in Lemma 1.1, st is a shear parallel to $\alpha = 0$. The region between the lines through b and $st(b)$ where $\alpha > 0$ is a fundamental domain for the action of W on the region $x \neq 0$. In any event, since α and β are linearly dependent, there is no chance for a Vinberg pair.

Cases (c) and (d), where α, β are independent, are the most interesting since they offer candidates for Vinberg pairs.

If L is the intersection of the kernels of α and β, then s and t induce reflections on the two-dimensional quotient U/L. In this plane a and b are linearly dependent. We have $c_{s,t}c_{t,s} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4$, and we can scale α so that in fact $c_{s,t} = c_{t,s}$, both being equal to 2 or -2. In either case, $\alpha^\vee$ and $\beta^\vee$ lie on a single line through the origin, and both s and t transform a vector v in a direction parallel to that line. In particular the products st and ts are shears along that line.

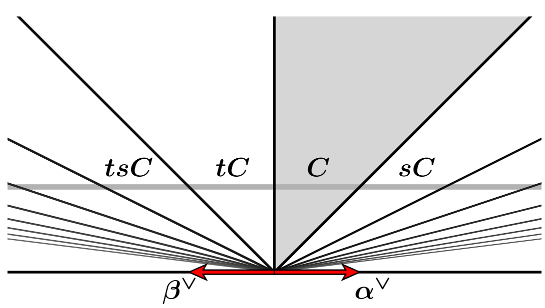


The reflection s



The reflection t

The region C on one side of that line and in between the two lines $\alpha = 0$ and $\beta = 0$ will be a fundamental domain for the group acting on the corresponding open half-plane. If we choose the signs of α and β correctly we may assume that this region is that where $\alpha > 0$ and $\beta > 0$. In that case, with our normalization, $c_{s,t} = c_{t,s} = -2$. In this case the group is infinite, consisting of all products 1, s, t, st, ... In this case we get a Vinberg pair.



We now know what happens in dimension two, hence in U/L in particular. In higher dimensions, this argument shows that $\alpha^\vee - \beta^\vee$ lies in L. There is a further dichotomy, depending on whether $\alpha^\vee = \beta^\vee$ or not. Both cases give rise to Vinberg pairs. The cone $\mathcal{C}$ is the half space containing C.

From now on, suppose that $n_{s,t}$ is not equal to 4. The next result is straightforward to prove.

[nondegenerate] **Proposition 4.2.** *When $n_{s,t}$ is not equal to 4, then α and β are linearly independent, as are $\alpha^\vee$ and $\beta^\vee$. The plane spanned by $\alpha^\vee$ and $\beta^\vee$ is transverse to the linear subspace of codimension 2 where α and β are both equal to 0.*

In effect we may now restrict ourselves to dimension two. In these circumstances, reflections fix lines. And we may make explicit matrix computations.

[reflection-matrices] **Lemma 4.3.** *Take $\alpha^\vee$ and $\beta^\vee$ as a basis of V. The matrices of s, t, and st are then*

$$\begin{bmatrix} -1 & -c_{s,t} \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -c_{t,s} & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 + n_{s,t} & c_{s,t} \\ -c_{t,s} & -1 \end{bmatrix}.$$

As one consequence, $-2 + n_{s,t}$ is the trace of st, hence clearly a conjugation invariant of the pair.

5. When $n = 0$

There remains one more exceptional case to deal with—when $n_{s,t} = 0$. This happens only when either $c_{s,t} = \langle \alpha, \beta^\vee \rangle = 0$ or $c_{t,s} = \langle \beta, \alpha^\vee \rangle = 0$, or both. If both are equal to 0, then s and t are a commuting pair of reflections whose product amounts to multiplication by -1. The group generated by s and t has four elements, and any one of the four quadrants is a fundamental domain. Uniquely in this situation, all choices of sign give Vinberg pairs.

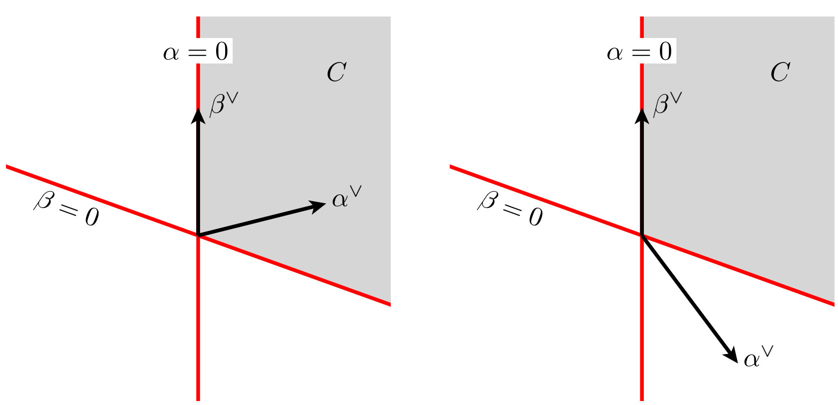
Otherwise, by swapping s and t if necessary (and swapping amounts to conjugation in $\mathrm{GL}(V)$) we may suppose that $\langle \alpha, \beta^\vee \rangle = 0$, but $\langle \beta, \alpha^\vee \rangle \neq 0$. There are two cases, according to the sign of $\langle \beta, \alpha^\vee \rangle$. The matrix is st is

$$\begin{bmatrix} -1 & 0 \\ -c_{t,s} & -1 \end{bmatrix}.$$

Since the two unipotent matrices

$$\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}.$$

are not conjugate in $\mathrm{GL}_2(\mathbb{R})$, the two possible signs give distinct conjugacy classes:



Then the line $\alpha = 0$ is fixed by both s and t. The reflection s interchanges the two sides of this line, but t preserves them. The pair of reflections t and sts now both stabilize each side of the line $\alpha = 0$, and the

n-invariant of this pair is 4, since they share the eigenvector $\beta^\vee$. From what have already learned in this case, the group generated by thes two reflections contains shears along $\beta^\vee$, and no orientation gives a Vinberg pair.

Or, in other words, if $n_{s,t} = 0$ and the region $\alpha > 0$, $\beta > 0$ is a fundamental domain then s and t are a commuting pair of reflections, and $c_{s,t}$ and $c_{t,s}$ both vanish. From now on, we assume $n_{s,t}$ to be neither 0 nor 4.

6. When $n \neq 0, 4$

[AAA] **Proposition 6.1.** *If n is neither 0 nor 4, there is a unique conjugacy class of pairs (s,t) of reflections with $n_{s,t} = n$.*

Proof. Let s be any reflection, say corresponding to α, $\alpha^\vee$. All choices of s are certainly conjugate. We may as well take α to be the function x and the vector $\alpha^\vee$ to be $(2,0)$. The matrices commuting with s are then the diagonal matrices. We now look for β and $\beta^\vee$ with $\langle \beta, \beta^\vee \rangle = 2$, $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = n$. Let β be any linear function independent of α and not vanishing on $\alpha^\vee$. All choices are conjugate by a linear transformation preserving α and a. Suppose $c = \langle \beta, \alpha^\vee \rangle$. It remains to choose $\beta^\vee$ independent of $\alpha^\vee$ with $\langle \alpha, \beta^\vee \rangle = n/c$. This requires that n differ from 4, and then all choices are conjugate by a linear transformation fixing both the line $\alpha = 0$ and that through $\alpha^\vee$.

[CCC] **Proposition 6.2.** *Suppose $n_{s,t}$ to be neither 0 nor 4. Then there exists a non-degenerate quadratic form invariant under both s and t, unique up to scalar multiple. It will be definite if and only if $0 < n_{s,t} < 4$.*

Proof. Suppose $u \bullet v$ to be a non-degenerate inner product on U. It will be invariant under s and t if and only if $\alpha^\vee$ is perpendicular to the line $\alpha = 0$, and similarly for $\beta^\vee$ and β. The space of vectors fixed by s is spanned by $c_{s,t}\alpha^\vee - 2\beta^\vee$, and that fixed by t is spanned by $2\alpha^\vee - c_{t,s}\beta^\vee$. For invariance, therefore, we must have

$$
\begin{aligned}
(c_{s,t}\alpha^\vee - 2\beta^\vee) \bullet \alpha^\vee &= 0 \\
(2\alpha^\vee - c_{t,s}\beta^\vee) \bullet \beta^\vee &= 0 \\
2(\beta^\vee \bullet \alpha^\vee) &= c_{s,t}(\alpha^\vee \bullet \alpha^\vee) \\
c_{t,s}(\beta^\vee \bullet \beta^\vee) &= 2(\alpha^\vee \bullet \beta^\vee)
\end{aligned}
$$

Since $n_{s,t}$ is not 0, neither $c_{s,t}$ nor $c_{t,s}$ vanish. Therefore the single number $\alpha^\vee \bullet \beta^\vee$ determines the inner product on all of V. The matrix of the quadratic form is thus

$$
\begin{bmatrix} \alpha^\vee \bullet \alpha^\vee & \alpha^\vee \bullet \beta^\vee \\ \alpha^\vee \bullet \beta^\vee & \beta^\vee \bullet \beta^\vee \end{bmatrix} = (\alpha^\vee \bullet \beta^\vee) \begin{bmatrix} 2/c_{s,t} & 1 \\ 1 & 2/c_{t,s} \end{bmatrix} .
$$

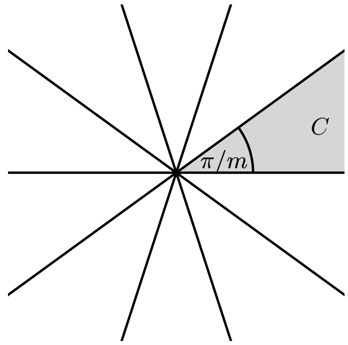
In order for the form to be definite, it is necessary and sufficient that $c_{s,t}$ and $c_{t,s}$ have the same sign, and that the determinant be positive. The first condition occurs if and only if $n_{s,t} > 0$. In these circumstances, the second occurs if and only if $n_{s,t} < 4$. ▯

[DDD] **Proposition 6.3.** *Suppose $0 < n_{s,t} < 4$, the pair s and t to be part of a Vinberg pair, it is necessary and sufficient that one of these hold:*

(a) $s = t$ (when $n = 4$);
(b) s and t commute (when $n = 0$);
(c) $n_{s,t} = 4\cos^2(\pi/m)$ for some positive integers $m \geq 3$ and $0 < p < m$.

In the last case, the region $\alpha > 0$ and $\beta > 0$ is an acceptable choice of C, as is $-C$. This is also the case when $c_{s,t}$ and $c_{t,s}$ are both negative.

The picture accompanying the last assertion is this:



This theorem is not difficult. The cone $\mathcal{C}$ in all cases is all of V.

Suppose now that the invariant quadratic form is indefinite. By choosing coordinates suitably, we may assume it to be $Q(x, y) = y^2 - x^2$, and we may also assume Q to be non-negative on the line $2x = \alpha = 0$. The special orthogonal group in these circumstances is that of Lorentz (or hyperbolic) rotations

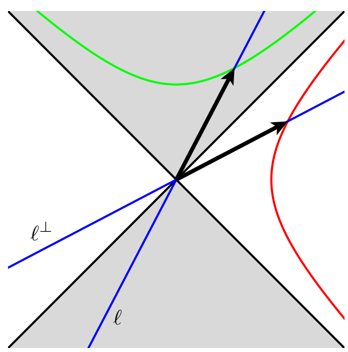
$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} .$$

If $s \neq t$ are any two hyperbolic reflections, then the product st is a hyperbolic rotation, and if ℓ is any line on which $Q \neq 0$, then any neighbourhood of any point of the cone $Q = 0$ will meet an infinite number of lines of the form $(st)^n \ell$.

We must have $n > 4$ or $n < 0$. Let

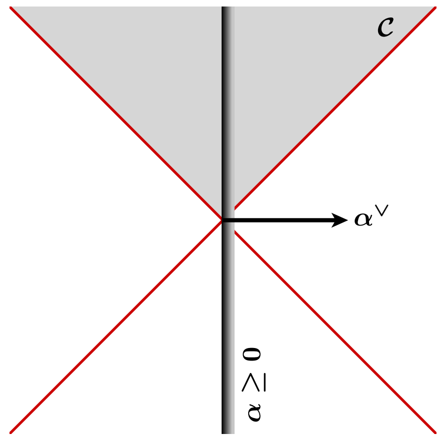
$$w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Then $Q(wv) = -Q(v)$ and since $w^2 = -1$, so $v \perp wv$. The linear transformation w is more or less a reflection in the null cone:



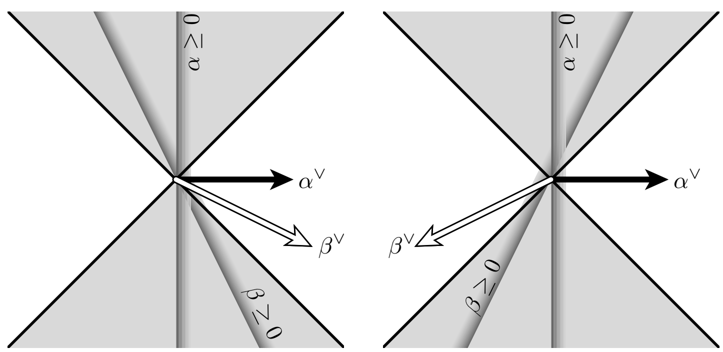
[AAA] **Proposition 6.4.** *When the quadratic form is indefinite, the reflections s and t are part of a Vinberg pair if and only if n is greater than or equal to 4, and both $c_{s,t}$ and $c_{t,s}$ are negative.*

Proof. Applying only transformations in $\mathrm{O}(Q)$, we may assume that $\alpha^\vee = (1, 0)$.



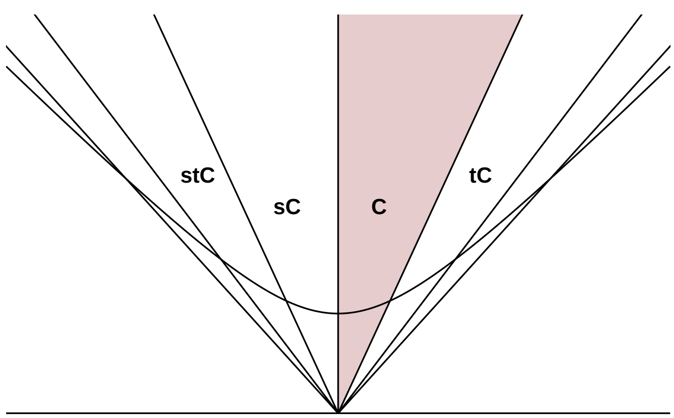
The product $\rho = st$ will be a hyperbolic rotation. If P is any point not on the null cone $y^2 - x^2 = 0$, the images $\rho^n(P)$ as $n \to \pm\infty$ will cluster close to the cone. Therefore, if s and t are part of a Vinberg pair, the region $\alpha \geq 0$, $\beta \geq 0$ cannot contain any part of the null cone. This rules out all cases where $\beta^\vee$ lies in the region $Q > 0$. This eliminates all cases where $n < 0$.

The linear transformation taking (x, y) to $(x, -y)$ preserves α and $\alpha^\vee$, but inverts top and bottom half planes, so we may assume $\beta^\vee$ is in the bottom half. This leaves two possibilities:



$c_{s,t} > 0$, $c_{t,s} > 0$ $\qquad$ $c_{s,t} < 0$, $c_{t,s} < 0$

But in the first, the region $\alpha \geq 0$, $\beta \geq 0$ contains some part of the null cone, so the only one that qualifies is that in which $c_{s,t} < 0$, $c_{t,s} < 0$. ▯



What is the significance of n in the indefinite case? The product st will be a hyperbolic rotation, similar to a linear transformation

$$\begin{bmatrix} t^2 & \\ & 1/t^2 \end{bmatrix}.$$

Its trace is $-2+n$, so

$$-2+n = t^2 + 1/t^2, \quad n = t^2 + 2 + 1/t^2 = 4\left(\frac{t+1/t}{2}\right)^2 = 4\cosh^2\theta$$

where θ is now a hyperbolic angle.

7. Affine reflections

I have mentioned that there are reflections of interest that are not linear, but also that they are in fact essentially linear. In this section I'll deal with affine reflections.

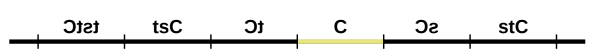
An **affine reflection** reflects points in an affine hyperplane $f(x) = 0$ for some affine function $f(x)$. The function f can be expressed as

$$f(x) = \langle \alpha, x \rangle + c$$

for some linear function α and constant c. The linear function α is the **gradient** ∇f of f, and for any point x in V and vector v we have $f(x+v) = f(x) + \langle \alpha, v \rangle$. The reflection then takes a point x to $x - f(x)\alpha^\vee$ for some vector $\alpha^\vee$ in V with $\langle \alpha, \alpha^\vee \rangle = 2$.

An affine reflection is a special case of a linear one, through the familiar trick of embedding an affine space of dimension n into a vector space U of dimension $n+1$. Thus v in V maps to $(v, 1)$ in U. Let δ be the function on U taking (v, x) to x. The affine reflection on V corresponding to α, $\alpha^\vee$, and c is the restriction to V of the reflection defined by the linear function $\alpha + c\delta$ on U. In the discussion of pairs of reflections with $n = 4$ we have already seen an implicit example of this.

Here is what happens for the group generated by affine reflections in the points at the end of a line segment in one dimension.



It is an infinite Coxeter group with two generators, say s and t, and with $m_{s,t}$ infinite. As we have already seen, the realization by affine reflections is the restriction to a line of a linear Coxeter group in dimension 2.

8. Non-Euclidean reflections

A **non-Euclidean reflection** reflects points of a non-Euclidean space in a non-Euclidean hyperplane. This also can be explained in terms of linear reflections.

The non-Euclidean space H_n of n dimension is the component $Q(x) = x_{n+1}^2 - x_1^2 - \ldots - x_n^2 = 1, x_{n+1} > 0$ of the hyperbolic sphere, which can be parametrized by Euclidean space E_n according to the formula

$$x_{n+1} = \sqrt{x_1^2 + \cdots + x_n^2}\,.$$

It is taken into itself by the connected component of the orthogonal group of Q and there is a unique Riemannian metric on it invariant under this group and restricting to the Euclidean metric at $(0, 0, \ldots, 0, 1)$. There are two models of this in Euclidean space of n dimensions, the familiar Poincaré model and the slightly less familiar Klein model. (There is also, for $n = 2$, the upper half-plane, but that is another story.) Both models identify H_n+ with the interior of the unit ball in E_n centred at the origin.

For the Klein model, this ball is identified with the intersection of the slice $x_{n+1} = 1$ of the region $Q(x) > 0$ (the interior of a homogeneous cone). A point x on H maps to $x_* = x/x_{n+1}$. It happens that a non-Euclidean geodesic line between two points x and y is the intersection with H of the plane through the origin containing x and y, and this intersects the slice in a line between x_* and y_*. Thus in the Klein model. points of H are identified with points of the interior of a unit ball, and geodesics between such points are line segments in the ball.

The Poincaré model is a transformation of the Klein model. A point x in the interior of the unit ball in E_n maps to the point above it on the upper hemisphere in E_{n+1}, then by South pole stereographic projection back onto the equator. In this transformation, geodesics become arcs of circles perpendicular to the boundary of the unit ball.

The point for us is that *non-Euclidean reflections are those induced on the hyperbolic sphere, or either of its models, by linear reflections in* E_{n+1}. For both models, we start with a hyperplane $\alpha = 0$ which intersects the region $Q(x) > 0$ and a vector $\alpha^\vee$ transverse to it. In the Klein model, we take a point x in the slice $Q(x) > 0$, $x_{n+1} = 1$ to $x - \langle \alpha, x\rangle \alpha^\vee$ and then project it back onto the slice. We have seen examples of this also in the discussion above of pairs of linear reflections with $n > 4$. In the Poincaré model we unravel by stereographic projection, reflect, and ravel again.

9. References

1. E. B. Vinberg, 'Discrete linear groups generated by reflections', *Math. USSR Izvestia* **5** (1971), 1083–1119.