Introduction to admissible representations of p-adic groups

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Chapter I. Analysis on profinite groups

All p-adic spaces have a topology in which every point has a countable basis of compact open neighbourhoods. Analysis on such spaces is essentially algebra, and in particular the theory of Haar measures on p-adic groups is elementary. In order to emphasize this, I shall introduce such spaces in purely combinatorial terms.

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SUMMARY. I define an LP (for Locally Profinite) space to be a locally compact Hausdorff topological space in which every point has a neighbourhood basis of compact open subsets. The principal result of the first part is that any locally closed subset of the set of $k$-rational points on an algebraic variety defined over a p-adic field $k$ is an LP space. This discussion is self-contained if mildly unorthodox in its avoidance of the strong form of the Axiom of Choice.

The second part discusses the Schwartz space $C_c^\infty(X)$ of an LP space $X$. One important if easy result is the excision theorem (Proposition I.5.3): If $Y$ is a closed subset of the LP space $X$, then the natural maps make an exact sequence

$$0 \to C_c^\infty(X - Y, V) \to C_c^\infty(X, V) \to C_c^\infty(Y, V) \to 0.$$ 

Integration of Schwartz functions on an LP group against a Haar measure reduces directly to summation. On homogeneous spaces the situation is a bit more complicated. If $G$ is an arbitrary locally compact group and $H$ a closed subgroup, there is in general no $G$-invariant measure on $H\backslash G$, which is to say there is no linear functional from $C_c^\infty(H\backslash G)$ to $\mathbb{C}$ invariant under right translation by elements of $G$. Instead of functions one integrates one-densities of compact support, which are linear functions on the space of smooth functions on $H\backslash G$. What happens here ought to be familiar, since on oriented real manifolds one integrates differential forms, not functions. If $\delta_H, \delta_G$ are the modulus characters of $H, G$ then continuous one-densities on $H\backslash G$ may be identified, but not canonically, with continuous functions $f$ on $G$ such that $f(hg) = \delta_G^{-1}(h)\delta_H(h)f(g)$.

The standard reference for this is [Weil:1965], but in the case of locally profinite groups this material is more elementary, and I shall explain it here. The most useful consequence for representation theory (Theorem
1.9.3) is that if $G$ is an algebraic group defined over $k$ and $H$ a closed subgroup, then the space of smooth one-densities on $H \setminus G$ may be identified with the space of smooth $\mathbb{C}$-valued functions on $G$ such that

$$f(hg) = |\det \Ad_{h \setminus g}(h)|^{-1} f(g)$$

for all $h$ in $H$, $g$ in $G$. We also have in these circumstances, upon choosing right-invariant measures on $H$ and $G$, the integral formula:

$$\int_G f(g) \, d_r g = \int_{H \setminus G} \left( \int_H |\det \Ad_{h \setminus g}(h)| f(hx) \, d_r h \right) \, d_r \varpi .$$

**Part I. Topology**

**1. The $p$-adic numbers**

For an integer $m \neq 0$, let $\ord_p m$ be the exponent of $p$ in its prime factorization and $|m|_p$ be the inverse of the $p$-factor itself. Thus $|p^m|_p = p^{-m}$ and has limit 0 as $m$ goes to $\infty$. Set $|0|_p = 0$. For rational numbers define the $p$-norm

$$|m|_p = |m|_p/|n|_p$$

This norm is multiplicative and satisfies the additive inequality

$$|x + y|_p \leq \sup |x|_p, |y|_p .$$

The field $\mathbb{Q}_p$ of $p$-adic numbers is the completion of $\mathbb{Q}$ with respect to this norm. That is to say, the $p$-adic numbers are defined to be the set of Cauchy sequences $(x_n)$ of rational numbers modulo a certain equivalence condition. A Cauchy sequence $(x_n)$ is one satisfying the condition

for any $\varepsilon > 0$ there exists $N$ such that $|x_n - x_m|_p < \varepsilon$ for all $m, n > N$.

and two sequences $(x_n), (y_n)$ are equivalent if

for any $\varepsilon > 0$ there exists $N$ such that $|x_n - y_m|_p < \varepsilon$ for all $m, n > N$.

The ring of $p$-adic integers $\mathbb{Z}_p$ is defined to be the closure of $\mathbb{Z}$. Every non-trivial ideal in it is $(p^r)$ for some $r$, so $\mathbb{Z}_p$ is a principal ideal domain.

**I.1.1. Proposition.** The image in $\mathbb{Q}_p$ of the rational number $m/n$ ($m, n$ relatively prime) lies in $\mathbb{Z}_p$ if and only if $\gcd(n, p) = 1$.

Proof. If $\gcd(n, p) = 1$, we can find an integer $n_*$ such that $mn_* \equiv 1$ modulo $p$. Then $m/n = mn_*/nn_*$ and it suffices to prove the lemma when $n \equiv 1$ modulo $p$. But if $n = 1 + \ell p$ then in the ring of $p$-adic integers

$$\frac{1}{1+\ell p} = 1 - \ell p + \ell^2 p^2 - \cdots$$

This proves one half the Lemma.

On the other hand, suppose $m/n$ to lie in $\mathbb{Z}_p$. Suppose that $p$ divides $n$. Since $\gcd(m, n) = 1$, $m$ is then relatively prime to $p$. The assumption implies that we can find some integer $q$ such that $|m/n - q|_p > 0$. But

$$\frac{m}{n} - q = \frac{m - nq}{n} .$$

But $m - nq$ must be relatively prime to $p$, a contradiction.

The map from $\mathbb{Z}$ to $\mathbb{Z}/p^r$ induces a homomorphism from $\mathbb{Z}_p$ to $\mathbb{Z}/p^r$ as well. This leads to:
I.1.2. Proposition. The ring $\mathbb{Z}_p$ may be identified with the projective limit of the finite rings $\mathbb{Z}/p^n$.

Proof. Any sequence of integers $(x_n)$ such that $x_{n+1} \equiv x_n \mod p^n$ is a Cauchy sequence in the $p$-adic norm, and the equivalence class of the sequence depends only on the $x_n$ modulo $p^n$.

There is a very concrete way to represent $p$-adic numbers—every $p$-adic rational can be expressed uniquely as an infinite reduced sum

$$
\sum c_i p^i
$$

where only a finite number of the $c_i$ with $i < 0$ are non-zero, and $0 \leq c_i < p$ for all $i$. Indeed, this is perhaps the most straightforward way to define $p$-adic numbers. A key step in this definition is the reduction of any series $\sum c_i p^i$ with $c_i$ in $\mathbb{Z}$ to one with $0 \leq c_i < p$ for all $i$.

And, finally, there is yet one more way to define the $p$-adic numbers—in terms of Witt vectors. This is a very interesting matter, but it won’t be relevant to what will be needed in these essays, and I won’t say more about it here.

Another example of a topological group with a similar structure is the ring $A_f$ of finite adèles. It is the restricted product of the fields $\mathbb{Q}_p$—i.e. the subset of all $(x_p)$ in $\prod_p \mathbb{Q}_p$ for which all but a finite number of the $x_p$ lie in $\mathbb{Z}_p$. This has as basis of neighbourhoods of 0 the products $\prod K_p$ where each $K_p$ is open in $\mathbb{Q}_p$ and $K_p = \mathbb{Z}_p$ for all but a finite number of $p$. The additive subgroup $\prod \mathbb{Z}_p$ is the projective limit of finite quotients by open subgroups.

2. Trees and topologies

A $p$-adic integer can be identified with a sequence $(x_n)$ of compatible integers in the finite rings $\mathbb{Z}/p^n$. We can make a graph with directed edges out of these data: there is one node in the graph for each pair $(n, x)$, and an edge from $(n, x)$ to $(n - 1, y)$ if $n \geq 2$ and $y \equiv x \mod p^{n-1}$. A $p$-adic integer then amounts to a one-way path of nodes in this graph, coming from infinity and terminating at one of the initial nodes in $\mathbb{Z}/p$. These observations should motivate the following discussion.

A rooted tree consists of (1) a set of nodes, (2) a designated root node, and (3) for every node other than the root an assignment of immediate successor node $\text{succ}(x)$, satisfying the condition that from any node $x$ there exists a unique sequence $x_n = x, x_{n-1}, \ldots, x_0$ with each $x_{i-1}$ the successor of $x_i$ and $x_0$ the root node. A successor is defined inductively by the condition that it be either an immediate successor or a successor of an immediate successor.

A node $x$ is an immediate predecessor of another node $y$ if $y = \text{succ}(x)$, and a predecessor if linked by a chain of immediate predecessors.
A rooted tree is said to be **locally finite** if the number of immediate predecessors of every node is finite. A **chain** in the tree rooted at the node $x$ is a finite sequence of nodes $x_0 = x, x_1, \ldots, x_n$ where each $x_{i+1}$ is an immediate predecessor of $x_i$. It is said to have length $n$. A **branch** of a rooted tree is a sequence (finite or infinite) of nodes $(x_i)$ where $x_0$ is the root and each $x_{i+1}$ is an immediate predecessor of $x_i$, satisfying the condition that the sequence stops only at a node with no predecessors.

Locally finite rooted trees possess a recursive structure, since if $T$ is any locally finite rooted tree and $x$ is a node of $T$ then the set of all predecessors of $x$ in $T$, together with $x$ itself, make up a locally finite tree with root $x$. Any subset of $T$ that contains along with a node its successor will also be a locally finite tree with the same root as $T$.

**I.2.1. Lemma.** In any locally finite tree, the number of chains of length $n$ rooted at a given node is finite.

**Proof.** That this is true for chains of length 1 is the definition of locally finite. The proof proceeds by induction on chain length.

**I.2.2. Lemma.** (König’s Lemma) In every locally finite rooted tree with an infinite number of nodes there exists an infinite branch.

**Proof.** Let $\rho$ be the root of the given tree, assumed to possess an infinite number of nodes. It follows from the previous result that there exist chains rooted at $\rho$ of arbitrary length, and in particular that the set $C_0$ of all chains rooted at $x_0 = \rho$ is infinite. Since the number of immediate predecessors of $x_0$ is finite, the subset $C_1$ of chains in $C_0$ passing through some one of them, say $x_1$, is infinite. Similarly there must exist an infinite number among the chains in $C_1$ whose third nodes agree. By induction, we obtain for each $n$ a sequence of sets of chains

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots.$$
where all the chains in $C_n$ agree with each other in their first $n + 1$ nodes, and agree in their first $n$ nodes with the chains in $C_{n-1}$. By choosing $x_n$ to be the common $n$-th node of the chains in $C_n$ we assemble an infinite branch.

The reasoning here, although plausible, is specious or at least highly subtle, since disguised in it is a weak form of the Axiom of Choice. König’s Lemma is one of those results that hovers on the edge of obviousness, like a dim star one cannot quite focus on directly. The degree to which it is not obvious becomes more apparent when one sees (in, for example, the enlightening discussion to be found in §2.3.4.3 of [Knuth:1973]) some of its immediate consequences.

Suppose $T$ to be a locally finite rooted tree with root $\rho$. For every node $x$ in $T$ define $\Omega_x$ to be the set of all branches passing through $x$. König’s Lemma guarantees that this is never empty. Let $\Omega_T$ be the set of all branches of $T$, which is the same as $\Omega_\rho$. We can make a topological space out of $\Omega_T$ by defining as basis of open sets the sets $\Omega_x$—if $\omega$ is any branch in the tree then the sets $\Omega_x$ for each of its nodes $x$ define a basis of neighbourhoods of $\omega$. Two distinct branches must eventually diverge, and therefore it is easy to see that this topology is Hausdorff. The next result shows that this topology is otherwise somewhat special.

**I.2.3. Proposition.** If $T$ is a locally finite rooted tree and $x$ a node in $T$ then $\Omega_{x}$ is closed as well as open in $\Omega_T$.

In other words, any point of $\Omega_T$ has a basis of neighbourhoods that are both open and closed.

**Proof.** Let $x_0 = \rho, x_1, \ldots, x_n = x$ be the chain from $\rho$ to $x$, and let $X = \{x_i\}$. Let $Y$ be the union of all of the immediate predecessors of the $x_i$ for $i < n$, except for $x_{i+1}$. In other words, $y$ is in $Y$ if it is an immediate predecessor of some $x_i$ and is not in $X$. Since $T$ is locally finite, $Y$ is finite. Every branch in $T$ that does not pass through $x$ has to branch off at one of the $x_i$ with $i < n$, and therefore the complement of $\Omega_x$ is the union of the $\Omega_y$ for $y$ in $Y$.

**I.2.4. Proposition.** If $T$ is a locally finite rooted tree then the topological space $\Omega_T$ is compact.

**Proof.** This amounts to the following assertion:

Suppose $X$ to be a set of nodes of $T$ such that the sets $\Omega_x$ ($x$ in $X$) cover $\Omega_T$. Then there exists a finite set of $\Omega_x$ ($x$ in $X$) covering $\Omega_T$.

The assumption means that every branch in $\Omega_T$ lies in some $\Omega_x$ with $x$ in $X$, or equivalently that every branch in $\Omega_T$ has a node in $X$. If $X$ contains the root node, we are immediately through.

Otherwise, let $X_{\text{min}}$ be the set of nodes in $X$ that are minimal—that is to say, $x$ lies in $X_{\text{min}}$ if it lies in $X$ and in the path from the root to $x$ there are no elements of $X$ other than $x$. It is clear that that the $\Omega_x$ for $x$ in $X_{\text{min}}$ cover $\Omega_T$, since every branch has to have a first element in $X$.

It suffices to show that $X_{\text{min}}$ is finite, since a path from $x$ in $X$ to the root must pass through a node in $X_{\text{min}}$. Let $Y$ be the set of nodes in $T$ that are not in $X$ and none of whose successors are in $X$. In particular, $Y$ contains the root node of $T$. Any successor of a node in $Y$ will also be in $Y$, so that $Y$ itself is a rooted tree and we can apply König’s Lemma to it. It is not possible for $Y$ to contain an infinite branch, since any infinite branch in $Y$ would also be a branch in $T$, and by assumption every branch in $T$ must contain a node in $X$. König’s Lemma tells us that $Y$ must be finite. But since $X$ does not contain the root node, every element of $X_{\text{min}}$ is the predecessor of some node in $Y$, and since the tree is locally finite $X_{\text{min}}$ must be finite.

Applying this to each of the rooted trees $\Omega_x$:

**I.2.5. Proposition.** Every point in the topological space $\Omega_T$ possesses a countable basis of compact open neighbourhoods.
3. Locally profinite spaces

Suppose we are given a sequence of finite sets $X_0, X_1, \ldots$ and for each $n > 0$ a surjection $\pi_n : X_n \to X_{n-1}$. We can make a finite union of trees from these data by taking the nodes of our graph to be the points of the $X_n$ and defining the successor of $x$ in $X_n$ to be $\pi_n(x)$. The branches of this tree are the infinite sequences $(x_n)$ where $\pi_n(x_n) = x_{n-1}$, and for a given point $x$ in $X_n$ the set $\Omega_x$ consists of all sequences with $x_n = x$. Let $X$ be the set of such sequences, and let $\Pi_n$ be the canonical surjection $(x_n) \mapsto x_n$ from $X$ to $X_n$. The topological space $X$ is the projective limit of the given sequence of finite sets. According to Proposition I.2.4 it possesses a natural Hausdorff topology with respect to which it is compact. The point $x = (x_n)$ has as a basis of neighbourhoods the sets $\Omega_x = \Pi_{n+1}^{-1}(\Pi_n(x))$.

More generally, suppose $\Sigma$ to be a directed set: $\Sigma$ is ordered, and given any two $\alpha, \beta$ in $\Sigma$ there exists $\gamma \geq \alpha, \beta$. Suppose given (a) for each $\alpha$ in $\Sigma$ a finite set $X_\alpha$ and (b) for each $\alpha > \beta$ a map $p_{\beta,\alpha} : X_\alpha \to X_\beta$. The maps are required to be (a) surjective and (b) consistent, in the sense that $p_{\gamma,\beta} p_{\beta,\alpha} = p_{\gamma,\alpha}$ whenever $\alpha > \beta > \gamma$. The projective limit of this system is defined to be the subset of all $(x_\alpha)$ in $\prod X_\alpha$ such that $p_{\beta,\alpha} x_\alpha = x_\beta$. The product has a natural topology with respect to which it is compact, and the projective limit is closed in it, hence also compact. Proving compactness of the product (Tychonov’s Theorem) requires the full Axiom of Choice, so that the sequentially profinite systems I have examined are definitely simpler.

A topological space $X$ is said to be profinite if it is a projective limit of finite sets. It is said to be totally disconnected if every point has a basis of neighbourhoods that are both open and closed.

I.3.1. Theorem. Suppose $X$ to be a topological space. The following are equivalent:

(a) it is profinite;
(b) it is Hausdorff, compact, and totally disconnected.

I’ll not prove this result, because it will not be used subsequently, except to justify terminology.

I.3.2. Corollary. If $X$ is a Hausdorff topological space, the following are equivalent:

(a) it is locally profinite;
(b) every point has a basis of compact, open sets.

The equivalence justifies applying the convenient term ‘locally profinite’ to spaces satisfying these conditions. I shall call them more succinctly LP spaces. A profinite space is the same as a compact LP space, and is also just a compact Hausdorff space in which every point has a basis of compact open sets. The spaces dealt with in earlier sections are those in which every point possesses a countable basis of compact open neighbourhoods. I’ll call these König spaces.

I.3.3. Lemma. Every covering of a compact LP space by compact open subsets possesses a disjoint refinement by compact open sets.

Proof. Suppose $X$ to be a profinite space and suppose given a covering by compact open subsets $U_i$, which we may assume to be a finite covering. The union of any of the $U_i$ is compact, hence closed in $X$, so that the sets

$$U_{*,i} = U_i - \bigcup_{j < i} U_j$$

are also open, and also cover $X$.

I.3.4. Proposition. If $(X_n)$ is a sequence of König profinite spaces then the product $X = \prod X_n$ is also a König space.
Proof. I recall that open sets in the product topology are of the form \( \prod U_n \) with each \( U_n \) open in \( X_n \) and all but a finite number of \( U_n = X_n \).

Let \( \Pi_{n,m} \) be the canonical surjection from \( X_n \) to the finite set \( X_{n,m} \). As a basis of open sets in the topology of \( X \) we therefore have the finite products \( \prod_{n \leq N} \Pi_{n,m}(x_{n,m}) \) with \( x_{n,m} \) in \( X_{n,m} \). In order to prove the Proposition we need to find a cofinal sequence of finite sets onto which \( X \) surjects. This can be the sequence

\[
X_{1,1}, X_{1,2} \times X_{2,1}, X_{1,3} \times X_{2,2} \times X_{3,1}, \ldots
\]

4. \( p \)-adic spaces

The ring of \( p \)-adic integers \( \mathbb{Z}_p \) is the projective limit of the finite rings \( \mathbb{Z}/p^n \). The ordinary integers \( \mathbb{Z} \) may be embedded in \( \mathbb{Z}_p \), since \( m \) may be identified with the sequence \( (m \mod p^n) \). If \( q \) is an integer relatively prime to \( p \) then for every \( n > 0 \) there exists a multiplicative inverse of \( q \) modulo \( p^n \), so that all rational numbers \( m/q \) with \( q \) prime to \( p \) may also be identified with elements of \( \mathbb{Z}_p \). More generally, the \( p \)-adic integers with multiplicative inverses in \( \mathbb{Z}_p \) are precisely those whose image in \( \mathbb{Z}/p \) does not vanish. A \( p \)-adic rational number other than 0 can be identified with a unique expression \( m/p^k \) where \( m \) is a unit in \( \mathbb{Z}_p \). These make up the field \( \mathbb{Q}_p \).

Generalizing this construction, I define \textit{\( p \)-adic field} to be a field \( \mathfrak{q} \) containing a ring \( \mathfrak{o} \) and an ideal \( \mathfrak{p} \) of \( \mathfrak{o} \) satisfying these conditions:

(a) \( \mathfrak{o}/\mathfrak{p} \) is a finite field, say of \( q \) elements;
(b) the canonical projections from \( \mathfrak{o} \) to \( \mathfrak{o}/\mathfrak{p} \) and from \( \mathfrak{o}/\mathfrak{p}^{n+1} \) to \( \mathfrak{o}/\mathfrak{p}^{n} \) identify the ring \( \mathfrak{o} \) with the projective limit of the quotients \( \mathfrak{o}/\mathfrak{p}^{n} \);
(c) if \( \varpi \) lies in \( \mathfrak{o} - \mathfrak{p}^2 \), then every non-zero element of \( \mathfrak{q} \) may be expressed uniquely as \( u\varpi^n \) where \( u \) lies in \( \mathfrak{o} - \mathfrak{p} \).

If \( x \) is an element of \( \mathfrak{o} - \mathfrak{p} \) there exists an element \( y \) of \( \mathfrak{o} - \mathfrak{p} \) such that \( xy \equiv 1 \) modulo \( \mathfrak{p} \). If then \( m = xy - 1 \) the series

\[
u = 1 - m + m^2 - m^3 + \cdots
\]

converges to an element of \( \mathfrak{o} \) because of condition (b), since modulo any power of \( \mathfrak{p} \) the series terminates. The limit will be a multiplicative inverse of \( xy \), so that \( yu \) will be a multiplicative inverse of \( x \). Hence every element of \( \mathfrak{o} - \mathfrak{p} \) is a unit of \( \mathfrak{o} \). The ideal \( \mathfrak{p} \) is the only prime ideal of \( \mathfrak{o} \) other than \( (0) \), and every non-zero ideal of \( \mathfrak{o} \) is a power of \( \mathfrak{p} \). If \( \varpi \) lies in \( \mathfrak{o} - \mathfrak{p}^2 \) then multiplication by \( \varpi^n \) induces a bijection of \( \mathfrak{o}/\mathfrak{p} \) with \( \mathfrak{o}/\mathfrak{p}^{n+1} \), so that \( \mathfrak{o}/\mathfrak{p}^{n} \) is a finite ring of cardinality \( q^n \).

The field \( \mathbb{Q}_p \) of \( p \)-adic integers is a \( p \)-adic field with \( \mathfrak{o} = \mathbb{Z}_p \) and \( \mathfrak{p} = (p) \). Every finite algebraic extension of a \( p \)-adic field is a \( p \)-adic field. The completion of any algebraic number field of finite degree with respect to any non-prime ideal of its ring of algebraic integers is a \( p \)-adic field, and is a finite extension of \( \mathbb{Q}_p \). The field of power series in \( x \) with coefficients in \( \mathbb{F}_q \) and a finite number of negative powers of \( x \)

\[
e^{-n}x^{-n} + e^{-(n-1)}x^{-(n-1)} + \cdots
\]

make up the quotient field \( \mathbb{F}_q((x)) \) of the ring of formal power series \( \mathbb{F}_q[[x]] \). It is a \( p \)-adic field with \( \mathfrak{p} = (x) \).

The completion of any \( \mathbb{F}_q \)-rational local ring on a non-singular algebraic curve over \( \mathbb{F}_q \) is isomorphic to it. Conversely, any \( p \)-adic field is either some \( \mathbb{F}_q((x)) \) or a finite algebraic extension of \( \mathbb{Q}_p \).

Throughout this book we will work with a fixed \( p \)-adic field \( (\mathfrak{q}, \mathfrak{o}, \mathfrak{p}) \), where \( \mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q \). If \( x = u\varpi^n \) with \( u \) a unit in \( \mathfrak{o} \) then its norm \( |x| = |x|_p \) is defined to be

\[
|x| = q^{-n}.
\]

Thus when \( n \geq 0 \) the index of the ideal \( (x) \) in \( \mathfrak{o} \) is \( |x|^{-1} \).
The ring $\mathfrak{o}$ is a profinite space. The ideals $p^n$ form a basis of neighbourhoods of 0, and any locally closed subspace of a finite dimensional vector space over $\mathfrak{t}$ will be a König space. Hence:

I.4.1. Theorem. Any locally closed subset of the set of $\mathfrak{t}$-rational points on an algebraic variety defined over $\mathfrak{t}$ is a König space.

Now let $F$ be a global field. The ring $\mathcal{A} = \mathcal{A}_{F, f}$ of finite adèles is the restricted product of the non-Archimedean completions $F_v$—the subset of $(x_v)$ in $\prod F_v$ for which all but a finite number of the $x_v$ lie in the integer ring $\mathfrak{o}_v$. The product $\prod \mathfrak{o}_v$ is an open neighbourhood of 0 in $\mathcal{A}$, and according to Proposition I.3.4:

I.4.2. Theorem. The ring of finite adèles of $F$ is a König space.

Part II. Analysis

5. Smooth functions on an LP space

Fix a coefficient field $\mathcal{D}$ of characteristic 0. Much later it will be assumed to be algebraically closed, and even more specifically $\mathcal{C}$, but for the moment it might even be just $\mathbb{Q}$.

Suppose $X$ to be a locally profinite space and $V$ a vector space over $\mathcal{D}$. Define

\[
C(X, V) = \text{space of continuous functions on } X \text{ with values in } V \\
C_c(X, V) = \text{subspace of those with compact support} \\
C^\infty(X, V) = \text{space of locally constant functions with values in } V \\
C^\infty_c(X, V) = \text{subspace of those with compact support} \\
\mathcal{D}(X) = \text{linear dual of } C^\infty_c(X) = C^\infty_c(X, \mathcal{D}).
\]

The functions in $C^\infty$ are called smooth, and they are continuous. The elements of $\mathcal{D}$ are distributions.

For any open set $U$ of $X$ let $\mathfrak{char}_U$ be its characteristic function.

I.5.1. Lemma. If $Y$ is a closed subset of the locally profinite space $X$ and $f$ in $C^\infty_c(Y, V)$, then there exists a finite sum $\sum \mathfrak{char}_U \cdot v_U$ whose restriction to $Y$ is $f$.

Proof. Suppose $f$ to be in $C^\infty_c(Y)$. Because of Proposition I.2.5, we can find a covering $\{U_i\}$ of the support of $f$ by compact open subsets $U_i$ of $X$ with the property that $f$ is constant on each $U_i \cap Y$. Apply Lemma I.3.3 to get a refinement by disjoint sets $U_{s,i}$.

On each $U_{s,i} \cap Y$ the function $f$ takes a constant value $v_i$. The linear combination $\sum \mathfrak{char}_{U_{s,i}} \cdot v_i$ then lies in $C^\infty_c(X, V)$ and has image $f$ in $C^\infty_c(Y, V)$.

If $Y = X$:

I.5.2. Corollary. Every function in $C^\infty_c(X, V)$ is a finite sum of functions $\mathfrak{char}_U \cdot v$ with $v$ in $V$.

I.5.3. Proposition. (Excision Lemma) If $Y$ is a closed subset of the locally profinite space $X$, then the natural maps make an exact sequence

\[
0 \rightarrow C^\infty_c(X - Y, V) \rightarrow C^\infty_c(X, V) \rightarrow C^\infty_c(Y, V) \rightarrow 0.
\]

Proof. The only interesting point is the final surjectivity, which follows from Lemma I.5.1.
6. Locally profinite groups

I define an LP group to be a Hausdorff topological group possessing a basis of neighbourhoods of 1 that are compact open subgroups.

In $\text{GL}_n(\mathfrak{f})$ the subgroup $\text{GL}_n(\mathfrak{o})$ of invertible matrices with coefficients in $\mathfrak{o}$ is a compact open subgroup, with the congruence subgroups

$$\text{GL}_n(\mathfrak{p}^m) = \{ g \in \text{GL}_n(\mathfrak{o}) \mid g \equiv I \pmod{\mathfrak{p}^m} \}$$

forming a basis of neighbourhoods of the identity. The group $\text{GL}_n(\mathfrak{o})$ may be identified with the projective limit of the groups $\text{GL}_n(\mathfrak{o}/\mathfrak{p}^m)$. Hence the group of $\mathfrak{f}$-rational points on any closed subgroup of $\text{GL}_n(\mathfrak{f})$, and in particular the group of $\mathfrak{f}$-rational points on any affine algebraic group defined over $\mathfrak{f}$, is a locally profinite group.

**Smooth functions on an LP group.**

I.6.1. Proposition. Suppose $X$ to be any locally profinite space on which the locally profinite group $G$ acts continuously. Then for every compact open set $\Omega$ in $X$ there exists a compact open subgroup $K$ of $G$ such that $K\Omega = \Omega$.

**Proof.** Because the action of $G$ is continuous, left multiplication in $G$ is continuous. For any point $x$ of $\Omega$ there exists a compact subgroup $K$ and a neighbourhood $U$ of $x$ such that $KU \subseteq \Omega$. Since $\Omega$ is compact, $\Omega$ will be covered by a finite number of these, say by the $K_iU_i$. Then $\Omega$ will be stable with respect to the intersection of the $K_i$.

If $G$ is a locally profinite group, it acts by the right- and left-regular representations on $C^\infty(G), C^\infty_c(G)$, and $\mathcal{D}(G)$ according to the recipes

$$L_g f(x) = f(g^{-1}x)$$
$$R_g f(x) = f(xg)$$
$$\langle L_g \Psi, f \rangle = \langle \Psi, L_g^{-1} f \rangle$$
$$\langle R_g \Psi, f \rangle = \langle \Psi, R_g^{-1} f \rangle .$$

We have

$$L_{g_1g_2} = L_{g_1}L_{g_2}, \quad R_{g_1g_2} = R_{g_1}R_{g_2} .$$

Define the subspace $C^\infty_u(G)$ to be that of smooth functions on $G$ with values in $\mathbb{Q}$ that are left-invariant under multiplication by elements of some compact open subgroup. Because of Proposition I.6.1, it contains $C^\infty_c(G)$. A smooth distribution is one that is locally right-invariant under some open subgroup of $G$. Uniform smoothness for distribution means global right-invariance under some fixed open subgroup.

I.6.2. Lemma. If $G$ is a locally profinite group and $f$ a function on $G$ with values in $V$ with compact support, the following are equivalent:

(a) the function $f$ lies in $C^\infty_c(G, V)$;
(b) there exists a compact open subgroup $K$ of $G$ such that $L_k f = f$ for all $k$ in $K$;
(c) there exists a compact open subgroup $K$ of $G$ such that $R_k f = f$ for all $k$ in $K$.

This is straightforward.

**Haar measures.** The theory of Haar measures on locally profinite groups is very simple. In practice, as will be explained in a moment, integrals are nearly always sums—even though occasionally infinite.

I.6.3. Proposition. Let $G$ be an LP group. Given a compact open subgroup $K$ and a constant $c_K$ in $\mathbb{Q}^\times$, there exists a unique right $G$-invariant measure $\mu$ on $G$ such that

$$\langle \mu, \text{char}_K \rangle = c_K .$$
Chapter I. Analysis on profinite groups

Proof. First I assign measures to compact open subsets of $G$.

Suppose that the measure $\mu$ is known to exist. If $K_*$ is a compact open subgroup contained in $K$ then

$$\langle \mu, \text{char}_{K_*} \rangle = [K: K_*]^{-1} \epsilon_K$$

since $K$ is the disjoint union of the $K_* x$ as $x$ runs over representatives of $K_* \backslash K$, and $\langle \mu, \text{char}_{K_* x} \rangle = \langle \mu, \text{char}_{K_*} \rangle$. If $K_*$ is an arbitrary compact open subgroup, then

$$\langle \mu, \text{char}_{K_*} \rangle = \frac{[K_*: K \cap K_*]}{[K: K \cap K_*]} \epsilon_K$$

But knowing $\langle \mu, \text{char}_{K_*} \rangle$ for all compact open subgroups $K_*$, together with right $G$-invariance, determines $\langle \mu, f \rangle$ for any smooth function $f$ of compact support, since $f$ is a linear combination of $\text{char}_{K_* x}$ for some one $K_*$ and a finite set of $x$ in $G$. This argument when run backwards gives the recipe for constructing $\mu$ as a distribution.

To define it as a measure, one must evaluate $\langle \mu, f \rangle$ for any continuous function of compact support. But if $f$ is a continuous function of compact support, there exist arbitrarily close functions in $C^\infty_c(G)$, with the same support, and one can define $\langle \mu, f \rangle$ as a limit, exactly as one defines Riemann sums.

I shall call such any $\mathbb{Q}$-distribution on $G$ with positive $\epsilon_K$ that is right $G$-invariant a (rational) right Haar measure on $G$, and if $d_r x$ is one write

$$\text{meas}(U, d_r x) = \langle d_r x, \text{char}_U \rangle$$

$$\int_G f(x) d_r x = \langle d_r x, f \rangle$$

I have said that, in practice, integration on $G$ amounts to summation. Let’s make this explicit.

I.6.4. Lemma. Suppose (a) $V$ is a vector space over $\mathbb{Q}$, (b) $f$ lies in $C^\infty_c(G, V)$, and (c) $K$ is a compact open subgroup of $G$. If $f$ is left-invariant with respect to $K$, then

$$\int_G f(x) d_r x = \text{meas}(K) \sum_{K \backslash G} f(x),$$

and if it is right-invariant then

$$\int_G f(x) d_r x = \text{meas}(K) \sum_{G/K} \delta_G(g) f(x).$$

Proof. Straightforward. For example:

$$\int_G f(x) d_r x = \sum_{K \backslash G} \int_{K_9} f(x) d_r x$$

$$= \text{meas}(K) \sum_{K \backslash G} f(K g).$$

Here is a frequently useful characterization:

I.6.5. Proposition. Suppose $G$ to be an LP group, assigned a right-invariant Haar measure $d_r g$. For $f$ in $C^\infty_c(G)$,

$$\int_G f(y) d_r g = 0$$
if and only if is a linear combination of functions of the form $R_g\varphi - \varphi$, with $\varphi \in C^\infty_c(G)$

Proof. One way is trivial. As for the other, suppose that

$$\int_G f(g) \, d_r g = 0.$$  

If $f$ is left-invariant under $K$, this means that

$$\sum_{K \backslash G} f(g_i) = 0,$$

if the support of $f$ is the disjoint union of the $Kg_i$. Let $\text{char}_{g}$ be the characteristic function of $Kg$. Then

$$\text{char}_{g} = R_g^{-1}) \text{char} \cdot 1$$

and so

$$f = \sum_i f(g_i) \text{char}_{g_i}$$

THE MODULUS CHARACTER. If $d_r x$ is any right Haar measure on $G$ then any left translation $L_g d_r x$ is also a right Haar measure, and must be therefore a scalar multiple of $d_r x$. In other words for each $g$ in $G$ there exists a scalar $\delta_G(g)$ such that

$$\int_{gU} d_r x = \delta_G(g) \int_U d_r x$$

for all compact open subsets $U$ of $G$, or in brief

$$d_r g x = \delta_G(g) \, d_r x.$$  

The constant $\delta_G(g)$ is independent of the right Haar measure chosen, since all others are just scalar multiples of it. It depends multiplicatively on $g$:

$$\delta_G(g_1, g_2) = \delta_G(g_1) \delta_G(g_2).$$

This multiplicative character of $G$ with values in the positive rational numbers is called the modulus character of $G$. If $d_r x$ is a right Haar measure on $G$ then $\delta_G(x)^{-1} d_r x$ is a left Haar measure.

The modulus character also clearly characterizes how conjugation affects measures, since

$$\text{meas}(g U g^{-1}) = \delta_G(g) \text{meas}(U).$$

The group is called unimodular if $\delta_G$ is trivial. If $K$ is a compact subgroup of $G$ then the image of any character of $K$ with values in the positive rational numbers has to be a torsion group, hence trivial. Therefore every compact group is unimodular.

SMOOTH DISTRIBUTIONS. Assume for the moment a right-invariant Haar measure $d_r g$ chosen on $G$. If $\varphi$ is a smooth function on $G$ with values in $V$ then the formula

$$(D_{\varphi}, f) = \int_G \varphi(x) f(x) \, d_r x \quad (f \in C^\infty_c(G, \mathbb{Q}))$$

defines a distribution $D_{\varphi}$ on $G$ with values in $V$, and

$$R_g D_{\varphi} = D_{R_g \varphi}, \quad L_g D_{\varphi} = \delta_G(g)^{-1} D_{L_g \varphi}.$$
If \( \varphi \) is right-invariant under an open group \( K \), then \( D_\varphi \) will also be right-invariant under \( K \).

Conversely, suppose \( D \) to be a smooth distribution, which will be locally right-invariant by some compact open subgroup \( K \). We can associate to it a function value at \( g \) in \( G \) by the formula
\[
\varphi(g) = \langle D, \text{char}_{gK} \rangle \text{meas}(gK).
\]

Local right-invariance of \( D \) implies immediately that
\[
\langle D, \text{char}_{gK} \rangle = \langle D, \text{char}_{gK^*} \rangle \left( \text{meas}(gK) / \text{meas}(gK^*) \right).
\]

for any compact open subgroup \( K^* \) of \( K \), which means that the definition of \( \varphi(g) \) is independent of the choice of \( K \) with respect to which \( D \) is right-invariant. It is also straightforward to see that \( D \) is then the same as \( D_\varphi \). We have proved:

I.6.6. Proposition. Suppose \( V \) to be a vector space over \( \mathbb{Q} \). Given a right-invariant Haar measure on \( G \), the correspondence \( \varphi \mapsto D_\varphi \) is a right-
\( G \)-equivariant isomorphism between the space of smooth functions with values in \( V \) and that of \( V \)-valued smooth distributions on \( G \).

There is one large class of distributions we shall often use. Suppose \( H \) to be any compact subgroup of \( G \) (not necessarily open). Then associated to \( H \) is the distribution \( \mu_H \) (not necessarily smooth) defined by the formula
\[
\langle \mu_H, f \rangle = \frac{1}{[H:H \cap K]} \sum_{H/H \cap K} f(h)
\]
if \( f \) in \( \mathcal{C}_c^\infty(G) \) is right-invariant under \( K \). In effect it evaluates the average value of \( f \) on \( H \).

7. Smooth vector bundles

In this section, I’ll explain an analogue of equivariant vector bundles on a quotient \( H \backslash G \), with \( G \) an LP group and \( H \) a closed subgroup. This quotient is also an LP space, which can be covered by translates of the profinite quotients \((H \cap K) \backslash K\), for \( K \) a compact open subgroup of \( G \). The principal application I have in mind will be covered in the next section, and in fact my real purpose in this section, which might appear a bit formal, is only to motivate definitions in the next.

BUNDLES. Let \( \mathbb{D} \) be an arbitrary field of characteristic 0. If \( V \) is a vector space over \( \mathbb{D} \), a smooth representation of \( H \) on \( V \) is a homomorphism \( \pi \) from \( H \) to \( \text{GL}_\mathbb{D}(V) \) with the property that the stabilizer of each \( v \) in \( V \) contains an open subgroup. For example, if \( V \) has finite dimension and \( \mathbb{D} = \mathbb{R} \) or \( \mathbb{C} \), this will be true if and only if \( \pi \) is continuous, since in this case \( \text{GL}_\mathbb{D}(V) \) does not possess arbitrarily small subgroups.

The group \( H \) acts on the product \( V \times G \)—the element \( h \) takes
\[
(v, g) \mapsto (\pi(h)v, hg).
\]

The quotient \( H \backslash (V \times G) \) is the fibre product \( \mathcal{V} = V \times_H G \).

The following might not ever be needed in these notes, but it answers a natural question.

I.7.1. Proposition. There exists a continuous section of the canonical projection \( G \longrightarrow H \backslash G \) over all of \( H \backslash G \).

Consequently, such vector bundles are always topologically trivial.

Proof. Given a compact open subgroup \( K \), the quotient is covered by subsets \( HgK \) as \( g \) ranges over some subset of \( G \). But \( HgK = H \cdot gKg^{-1} \cdot g \), and \( gKg^{-1} \) is again compact and open. It therefore suffices to prove the Lemma when \( G = K \) itself is compact.

I’ll give the proof, in fact, only when \( K \) is sequentially profinite. For the general case, I refer to [Serre:1964].
Let $K_n$ be a shrinking sequence of compact open normal subgroups of $K$, and let $H_n = K_n \cap H$. The first claim is that for each $n$ there exist continuous sections of the canonical projections

$$H_n \backslash K \rightarrow H \backslash K.$$ 

The quotient $H \backslash K$ is the union of some finite number of cosets $HkK_n$. But $K_n$ is normal in $K$, so $HkK_n = HK_nk$. Hence it suffices to prove that there exists a section over $(K_n \cap H) \backslash K_n$. However, $(K_n \cap H) \backslash K_n$ embeds into both $H_n \backslash K$ and $H \backslash K$, so of course there is a section over it.

We now consider a tree whose nodes are continuous sections

$$s_n: H \backslash G \rightarrow H_n \backslash G$$

and where the successor of a section is its composition with the canonical projection from $H_{n+1} \backslash G$ to $H_n \backslash G$. Since $H_{n+1}$ has finite index in $H_n$, such systems form a locally finite tree. By König’s Lemma (Lemma I.2.2) there exists an infinite branch in this tree, hence a sequence of compatible continuous sections $s_n: H \backslash G \rightarrow H_n \backslash G$, hence a map from $H \backslash G$ to the projective limit of the $H_n \backslash G$, which can be canonically identified with $G$.

There is a canonical projection $\Pi$ from $V$ to $X$, making this diagram commutative:

$$
\begin{array}{ccc}
V \times G & \longrightarrow & V \times_H G \\
\downarrow & & \downarrow \Pi \\
G & \longrightarrow & X
\end{array}
$$

The fibre over any point $x$ is the inverse image $\Pi^{-1}(x)$. The fibre containing $g$ is $[g] = Hg$. The embedding

$$v \mapsto \text{the image in } V \text{ of } (v, 1)$$

is an isomorphism of $V$ with the fibre over $[1]$. Let $\eta$ be its inverse, a map from the fibre over $[1]$ to $V$. Any two fibres are isomorphic to each other and to $V$, but there is in general no canonical choice of isomorphism.

The group $G$ acts on both $G$ and $V \times G$ on the right. It commutes with $H$, hence also acts compatibly on both $X$ and $V$. I shall sometimes find it convenient to describe this as a left action:

$$\lambda_g: x \mapsto g\cdot x = x \cdot g^{-1}.$$ 

The map $\lambda_g$ maps the fibre over $y = \Pi(x)$ to that over $y \cdot g^{-1}$. In particular, it is an isomorphism of the fibre over $[g]$ with that over $[1]$. The map $\Lambda = \eta \cdot \lambda_g$ is an isomorphism of the fibre over $[g]$ with $V$. The group $H$ takes the fibre over $[1]$ to itself, and the map $\eta$ is $H$-equivariant.

A section of the bundle is a map $s$ from $X$ back to the fibre product—i.e. it assigns to each element of $X$ an element of its fibre. I’ll call it locally constant if for every $x$ in $X$ there exists a compact open subgroup $K$ such that $s(x \cdot k) = s(x) \cdot k$ for each $k$ in $K$.

If $s$ is a section of $V$, define $F = F_s$ from $G$ to $V$:

$$F(g) = \Lambda(s([g])).$$

This make sense—$v = s([g])$ is in the fibre over $[g]$, $\lambda_g(v)$ is in the fibre over $[1]$, and $\eta$ takes this to $V$. Since $\eta$ is $H$-equivariant, this map satisfies the condition

$$(I.7.2) \quad F(hg) = \pi(h)F(g) \text{ for all } h \text{ in } H, g \text{ in } G.$$ 

I.7.3. Lemma. The map defined in this way is a bijection between locally constant sections of $V$ with locally constant functions $F$ from $G$ to $V$ satisfying $(I.7.2)$. \(\Box\)
Proof. There are two things to be verified. The first is that \( F \) is locally constant. The second is that every suitable \( F \) gives rise to a section. I leave these as exercises.

SHEAVES. The vector bundle \( \mathcal{V} \) determines a sheaf \( S = S_\mathcal{V} \) whose sections are sections of \( \mathcal{V} \). For each open \( U \) in \( X \), let
\[
S(U) = \Gamma(U, \mathcal{V}) = \text{locally constant sections of } \mathcal{V} \text{ over } U.
\]

(a) The sheaf \( S \) is \( G \)-equivariant. For every \( g \) in \( G \) and \( U \) open in \( X \) the map \( y \mapsto y \cdot g \) induces a map
\[
R_g : S(U) \rightarrow S(U \cdot g^{-1})
\]
through the formula
\[
[R_g s](x) = s(x \cdot g) \cdot g^{-1}.
\]
In terms of Lemma I.7.3 this becomes
\[
R_g F(x) = F(xg).
\]

(b) Sections are locally constant. If \( U \) is any open subset of \( X \), \( s \) is in \( S(U) \), and \( x \) is in \( U \), then there exists a compact open \( K \) with \( U \cdot K = U \) such that \( R_k(s) = s \) for all \( k \) in \( K \).

This implies that if the image of \( s \) in the fibre at \( x \) vanishes, then there exists a compact open subgroup \( K \) such that \( s(x \cdot K) = 0 \).

(c) Each \( S(U) \) is a module over \( \mathbb{C}^\infty(U, \mathbb{D}) \). This according to the formula
\[
[\varphi \cdot s](x) = \varphi(x) \cdot s(x),
\]
which makes sense because each fibre is a vector space. The key property here is that for a smooth function \( \varphi \) and section \( s \)
\[
R_g(\varphi \cdot s) = R_g(\varphi) \cdot R_g(s).
\]

One important consequence is that fibres of the bundle are determined canonically in terms of the sheaf. Let \( m_x \) be the ideal of \( \mathbb{C}^\infty(X, \mathbb{D}) \) vanishing at \( x \), and let \( M = S(X) \). In these terms, the fibre at \( x \) is the quotient \( S_x = M/m_x M \). In particular, \( V \) is isomorphic to \( M/m_{[1]} M \). The group \( H \) takes \( m_{[1]} \) into itself, and hence the action of \( H \) on \( V \) may also be defined in terms of \( S \).

The stalk of the sheaf \( S \) at \( x \) is the direct limit of the \( S(U) \) as \( U \) shrinks to \( \{x\} \). There is a canonical map from the stalk at \( x \) to the fibre at \( x \). Because sections are locally constant, this is an isomorphism.

The real point of the present discussion is a converse to the conclusion of this discussion:

I.7.4. Theorem. If \( S \) is any sheaf on \( H \backslash G \) satisfying the conditions in the previous theorem then it is that associated to vector bundle associated to the representation of \( H \) on \( S_{[1]} \).

In particular, I’ll eventually define a line bundle in terms of the sheaf whose sections are analogues of smooth differential volume forms.

Proof. Pretty much straightforward.
8. Integration on quotients

If $H$ and $G$ are both unimodular then invariant Haar measures on $G$ and $H$ will determine a $G$-invariant measure on the quotient $H \backslash G$ (as we’ll see in a short while). But we shall be often concerned with a situation in which $G$ is unimodular and $H$ is not. In the general situation more care is required, since there is then no $G$-invariant measure on $H \backslash G$. What one integrates instead are **one-densities**, which are for arbitrary locally compact spaces the analogue of differential forms on oriented manifolds. As one might expect from this example, one-densities are sections of a line bundle.

**THE SHEAF OF ONE-DENSITIES.** A distribution on an open set $U$ is an element of the dual of $C_c^\infty(U, \mathbb{D})$. Define a sheaf $\Omega^\infty$ by the requirement that $\Omega^\infty(U)$ be the vector space of locally constant distributions on $U$.

**I.8.1. Theorem.** The sheaf $\Omega^\infty$ is that associated to a vector bundle over $H \backslash G$. The fibre $\Omega^\infty[H \backslash G]$ is a one-dimensional space on which $H$ acts by the character

$$\delta_{H \backslash G}(h) = \delta_H(h)/\delta_G(h).$$

Intuitively, the character is a **local modulus factor**. We shall see more about this later on.

The proof will be rather long, largely because I want to motivate it carefully.

To define differential forms on $H \backslash G$ when $G$ is a Lie group requires choosing local coordinates $x_i$, specifying a form as $\omega(x) \, dx_1 \ldots dx_n$. This is usually done by defining the form as a ratio of forms on $H$ and $G$. Integration then reduces to integration over a subset of $\mathbb{R}^n$. The analogous procedure here will be to choose right-invariant measures on $H$ and $G$. So I begin by fixing such measures. Let $X = H \backslash G$.

For $f$ in $C_c^\infty(G)$ define

$$\tilde{f}(g) = \int_H f(hg) \, dh.$$  

**I.8.3. Lemma.** The map $f \mapsto \tilde{f}$ is a surjection from $C_c^\infty(G)$ to $C_c^\infty(X)$.

**Proof.** Suppose $f$ to be in $C_c^\infty(X)$, say fixed by elements of $K$. Then it is a finite linear combination of functions constant on sets $HgK$, so we may assume that $f$ is equal to the characteristic function of $HgK$. But then it is, up to scalar, equal to $\tilde{f}$ with $f$ equal to the characteristic function of $gK$. \hfill \Box

If $F$ is a smooth linear functional on $C_c^\infty(H \backslash G)$, then we can define a smooth distribution $D_F$ on $G$ according to the formula

$$\langle D_F, f \rangle = \langle F, \tilde{f} \rangle.$$

The map taking $F$ to $D_F$ is injective, according to the previous Lemma. Since

$$\langle L_h f, g \rangle = \int_H f(h^{-1} x g) \, dx = \delta_H(h) \tilde{f}(g)$$

since

$$\langle L_h D_F, f \rangle = \langle D_F, L_h^{-1} f \rangle = \langle F, L_h^{-1} \tilde{f} \rangle$$

we must have

$$L_h D_F = \delta_H^{-1}(h) D_F$$

for all $h$ in $H$. Since $D_F$ is smooth, Proposition I.6.6 tells us that for some smooth function $\varphi$ on $G$

$$\langle D, f \rangle = \int_G \varphi(g) f(g) \, dg.$$
The condition (I.8.4) on $D$ translates to the condition $\delta_G(h)^{-1}L_k \varphi = \delta_H(h)^{-1} \varphi$ or, equivalently,

(I.8.5) \[ \varphi(hg) = \delta_H(h) \delta_G(h)^{-1} \varphi(g) \] for all $h$ in $H$, $g$ in $G$.

Now a converse.

Suppose $\varphi$ to be a smooth function on $G$ satisfying (I.8.5), $f$ in $C^\infty_c$. Suppose $K$ to be a compact open subgroup of $G$ such that $f$ is right-$K$-invariant and that $\varphi$ is right-$K$-invariant on the support of $f$. Then

\[ \int_G \varphi(x) f(x) \, d_x x = \sum_{H \backslash G / K} \int_{HgK} \varphi(x) f(x) \, d_x x \]

and since $h \mapsto hgK$ is a bijection between $H / H \cap gKg^{-1}$ and $HgK / K$

\[ \int_{HgK} \varphi(x) f(x) \, d_x x = \sum_{H / H \cap gKg^{-1}} \varphi(hg) f(hg) \text{meas}(hgK) \]

\[ = \sum_{H / H \cap gKg^{-1}} \delta_H(h) \delta_G(h)^{-1}(hg) \varphi(g) f(hg) \text{meas}(gK) \]

(I.8.6) \[ = \text{meas}(gK) \varphi(g) \sum_{H / H \cap gKg^{-1}} \delta_H(h) f(hg) \]

\[ = \varphi(g) \left[ \frac{\text{meas}_G(gK)}{\text{meas}_H(H \cap gKg^{-1})} \right] \int_H f(hg) \, d_x h \]

\[ = \varphi(g) \left[ \frac{\text{meas}_G(gK)}{\text{meas}_H(H \cap gKg^{-1})} \right] \tilde{f}(g) . \]

I.8.7. Theorem. Suppose given right-invariant measures on $H$ and $G$. The map $F \mapsto \varphi_F$ is an isomorphism of the space of smooth distributions on $H \backslash G$ with the space of smooth functions $\varphi$ on $G$ such that

\[ \varphi(hg) = \delta_H(h) \delta_G(h)^{-1} \varphi(g) . \]

Taking Lemma I.7.3 into account, this concludes the proof of Theorem I.8.1.

Proof. Because (I.8.6) shows that

\[ \int_G \varphi(x) f(x) \, d_x x \]

depends only on $\tilde{f}$.

I.8.8. Corollary. There exists a unique right $G$-invariant linear map $d_x \varphi$ from $\Omega^\infty_c(H \backslash G)$ to $\mathbb{D}$, which will be denoted simply by integration, such that

\[ \int_G f(g) \, d_x x = \int_{H \backslash G} \left( \int_H \delta_H^{-1}(h) \delta_G(h) f(hx) \, d_x h \right) d_x \varphi . \]

for any $f$ in $C^\infty_c(G)$.

This can be written

\[ \int_G f(g) \, d_x x = \int_{H \backslash G} \tilde{f}_\Omega(\varphi) \, d_x \varphi . \]
if
\[\mathcal{F}_\Omega(x) = \int_H \delta_H^{-1}(h)\delta_G(h)f(hx)\,d_r\,h.\]

The argument leading to the Theorem gives us an explicit formula for the integral of smooth, compactly supported one-densities:

**I.8.9. Proposition.** For \(\varphi\) in \(\Omega_\infty^c(H\backslash G)\) fixed on the right by elements of the compact open subgroup \(K\)

\[\int_{H\backslash G} \varphi(\mathfrak{x})\,d\mathfrak{x} = \sum_{H\backslash G/K} \varphi(g) \left[ \frac{\text{meas}_G(gK)}{\text{meas}_H(H \cap gKg^{-1})} \right].\]

The terms in this sum do not depend on the choice of representatives \(g\) in \(H\backslash G/K\).

Now to follow up on my remark on the analogy with differential forms. Given compact open subgroup \(K\) in \(G\), there exists a unique \(K\)-invariant one-density \(\mu_K\) on \(H\backslash H K\) whose integral over \(H\backslash H K\) is equal to 1. Given any one-density \(D\), there exists a compact open subgroup \(K\) such that \(D\) restricted to \(H\backslash H K\) is a scalar multiple of this one. In other words, locally \(D = c_K\mu_K\).

I add a supplement. Suppose
\[\int_{H\backslash G} F(\mathfrak{x})\,d\mathfrak{x} = 0.\]

Then we can find \(f\) in \(C_\infty^c(G)\) such that \(\mathcal{F}_\Omega = F\), and then
\[\int_G f(g)\,d_r\,g = 0\]
as well. According to Proposition I.6.5, we may express \(f\) as a linear combination of \(R_g\varphi - \varphi\). We deduce the following generalization of Proposition I.6.5:

**I.8.10. Corollary.** If \(F\) lies in \(\Omega_\infty^c(H\backslash G)\) then
\[\int_{H\backslash G} F(\mathfrak{x})\,d\mathfrak{x} = 0\]
if and only if \(F\) is a linear combination of smooth densities of the form \(\Phi - R_g\Phi\).

9. Measures on \(p\)-adic manifolds

In the special case that \(G\) is an algebraic group defined over the \(p\)-adic field \(\mathfrak{f}\), we can derive measures on \(G\) in a way that one must often use when one wants to treat coherently all local fields associated to a global one.

First let \(V\) be the additive group \(\mathfrak{f}^n\). A translation-invariant measure \(dx\) on \(V\) is a constant multiple of the measure \(dx = dx_1 \cdots dx_n\) that assigns measure 1 to \(o^n\). This assigns measure \(q^{-nk}\) to \((p^n)^k\). This determines the measure on any compact open subgroup, since it will be the disjoint union of translates of one of these neighbourhoods of 0.

If \(T\) is any matrix in \(\text{GL}_n(\mathfrak{f})\) and \(dx\) is translation-invariant then the transform by \(T\) is also translation-invariant, hence a multiple of \(dx\). The constant is determined by what \(T\) does to \(o^n\). If \(D\) is a diagonal matrix with diagonal entries \(d_i\) the volume of \(T o^n\) is \(\mu \prod |d_i|\). If \(\gamma\) is in \(\text{GL}_n(\mathfrak{o})\) then \(\gamma o^n = o^n\), hence it preserves volumes. The principal divisor theorem tells us that any matrix \(T\) in \(\text{GL}_n(\mathfrak{f})\) is a product \(\gamma_1 \gamma_2 \cdots \gamma_k\) with the \(\gamma_i\) in \(\text{GL}_n(\mathfrak{o})\) and \(D\) diagonal, so \(T\) multiplies volumes by \(\prod |\det(T)|\). Consequently:
I.9.1. Proposition. If \( \varphi \) is an analytic isomorphism of subsets \( X \) and \( Y \) of \( \mathbb{k}^n \), then setting \( y = \varphi(x) \), \( g(x) = f(\varphi(x)) \) we have the change of variables formula

\[
\int_Y f(y) \, dy = \int_X g(x) \left| \det(\partial y/\partial x) \right| \, dx.
\]

Now suppose \( X \) to be an arbitrary \( \mathbb{k} \)-analytic manifold, given with a countable, locally finite atlas \( \{X_i\} \). For each \( X_i \) we are given an open embedding into \( \mathbb{k}^n \), and on overlaps two coordinate systems differ by an invertible analytic function. Details about what this means can be found in [Serre:1965]. Because of the implicit function theorem for \( \mathbb{k} \)-analytic maps, every non-singular algebraic variety over \( \mathbb{k} \) has such a structure.

Following what happens for real manifolds, I define a smooth one-density on \( X \) to be a family of compatible smooth measures on the sets \( X_i \). Compatibility means only that the measures agree on overlaps. In particular, if one is given a non-vanishing differential form \( \omega \) on an algebraic variety \( M \), then \( |\omega| \) defines a smooth one-density. If one applies this to an algebraic group, one obtains:

I.9.2. Proposition. If \( G \) is an algebraic group defined over \( \mathbb{k} \), then the modulus character is \( |\det(\text{Ad}_g)| \).

I.9.3. Theorem. If \( G \) is an algebraic group defined over \( \mathbb{k} \) and \( H \) a closed subgroup, then the space of smooth one-densities on \( H \setminus G \) may be identified with the space of smooth \( \mathbb{C} \)-valued functions on \( G \) such that

\[
f(hg) = |\det^{-1}\text{Ad}_{h\setminus g}(h)| f(g)
\]

for all \( h \) in \( H \), \( g \) in \( G \).

If \( G \) is unimodular

\[
|\det^{-1}\text{Ad}_{h\setminus g}(h)| = |\det\text{Ad}_g(h)|.
\]

10. References


5. ——, *Adèles and algebraic groups*, Birkhäuser, 1982.