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Essays on representations of p -adic groups

Analysis on profinite groups

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All p -adic groups have a topology in which every point has a countable basis of compact open neighbourhoods. Analysis on such spaces is essentially algebra, and in particular the theory of Haar measures is elementary. In order to emphasize this, I shall introduce such spaces in purely combinatorial terms.

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Part I. Topology

1. The p-adic numbers

For an integer $m \neq 0$, let $\text{ord}_p m$ be the exponent of p in its prime factorization and $|m|_p$ be the inverse of the p -factor itself. Thus $|p^m|_p = p^{-m}$ and has limit 0 as m goes to ∞ . Set $|0|_p = 0$. For rational numbers define the p -norm

$$\left| \frac{m}{n} \right|_p = |m|_p / |n|_p$$

This norm is multiplicative and satisfies the additive inequality

$$|x + y|_p \leq \sup |x|_p, |y|_p.$$

The field \mathbb{Q}_p of p -adic numbers is the completion of \mathbb{Q} with respect to this norm. That is to say, a p -adic number is defined to be Cauchy sequences (x_n) of rational numbers modulo a certain equivalence condition. A Cauchy sequence (x_n) is one satisfying the condition

for any $\varepsilon > 0$ there exists N such that $|x_n - x_m|_p < \varepsilon$ for all $m, n > N$.

and two sequences $(x_n), (y_n)$ are equivalent if

for any $\varepsilon > 0$ there exists N such that $|x_n - y_m|_p < \varepsilon$ for all $m, n > N$.

The ring of p -adic integers \mathbb{Z}_p is defined to be the closure of \mathbb{Z} . Every non-trivial ideal in it is (p^r) for some r , so \mathbb{Z}_p is a principal ideal domain.

[zp] Proposition 1.1. *The image in \mathbb{Q}_p of the rational number m/n (m, n relatively prime) lies in \mathbb{Z}_p if and only if $\text{gcd}(n, p) = 1$.*

Proof. If $\text{gcd}(n, p) = 1$, then we can find an integer n_* such that $nn_* \equiv 1$ modulo p . Then $m/n = mn_*/nn_*$ and it suffices to prove the lemma when $n \equiv 1$ modulo p . But if $n = 1 + \ell p$ then in the ring of p -adic integers

$$\frac{1}{1 + \ell p} = 1 - \ell p + \ell^2 p^2 - \dots$$

This proves one half the Lemma.

On the other hand, suppose m/n to lie in \mathbb{Z}_p . Suppose that p divides n . Since $\text{gcd}(m, n) = 1$, m is then relatively prime to p . The assumption implies that we can find some integer q such that $|m/n - q|_p > 0$. But

$$\frac{m}{n} - q = \frac{m - nq}{n}.$$

But $m - nq$ must be relatively prime to p , a contradiction. □

The map from \mathbb{Z} to \mathbb{Z}/p^r induces a homomorphism from \mathbb{Z}_p to \mathbb{Z}/p^r as well. This leads to:

[zp-projlim] Proposition 1.2. *The ring \mathbb{Z}_p may be identified with the projective limit of the finite rings \mathbb{Z}/p^r .*

Proof. Any sequence of integers (x_n) such that $x_{n+1} \equiv x_n \pmod{p^n}$ is a Cauchy sequence in the p -adic norm, and the equivalence class of the sequence depends only on x_n modulo p^n . □

There is a very concrete way to represent p -adic numbers—every p -adic rational can be expressed uniquely as an infinite reduced sum

$$\sum c_i p^i$$

where only a finite number of the c_i with $i < 0$ are non-zero, and $0 \leq c_i < p$ for all i . Indeed, this is perhaps the most straightforward way to define p -adic numbers. A key step in this definition is the reduction of any series $\sum c_i p^i$ with c_i in \mathbb{Z} to one with $0 \leq c_i < p$ for all i .

And, finally, there is yet one more way to define the p -adic numbers—in terms of **Witt vectors**. This is a very interesting matter, but it won't be relevant to what I say here.

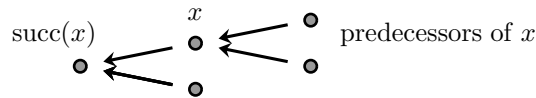
There is another example of a topological group with a similar structure. The group \mathbb{A}_f of finite adèles is the **restricted product** of the fields \mathbb{Q}_p —i.e. the subset of all (x_p) in $\prod_p \mathbb{Q}_p$ for which all but a finite number of the x_p lie in \mathbb{Z}_p . This has as basis of neighbourhoods of 0 the products $\prod K_p$ where each K_p is open in \mathbb{Q}_p and $K_p = \mathbb{Z}_p$ for all but a finite number of p . The group $\prod \mathbb{Z}_p$ is the projective limit of finite quotients by open subgroups.

2. Trees and topologies

A p -adic integer can be identified with a sequence (x_n) of compatible integers in the finite rings \mathbb{Z}/p^n . We can make a graph with directed edges out of these data: there is one node in the graph for each pair (n, x) with $n > 0$ and x in \mathbb{Z}/p^n , and an edge from (n, x) to $(n - 1, y)$ if $n \geq 2$ and $y \equiv x \pmod{p^{n-1}}$. A p -adic integer then amounts to a one-way path of nodes in this graph, coming from infinity and terminating at one of the initial nodes in \mathbb{Z}/p . These observations should motivate the following discussion.

A **rooted tree** consists of (1) a set of **nodes**, (2) a designated **root node**, and (3) for every node other than the root an assignment of **immediate successor** node $\text{succ}(x)$, satisfying the condition that from any node x there exists a unique sequence $x_n = x, x_{n-1}, \dots, x_0$ with each x_{i-1} the successor of x_i and x_0 the root node. A **successor** is defined inductively by the condition that it be either an immediate successor or a successor of an immediate successor.

A node x is an **immediate predecessor** of another node y if $y = \text{succ}(x)$, and a predecessor if linked by a chain of immediate predecessors.



A rooted tree is said to be **locally finite** if the number of immediate predecessors of every node is finite. A **chain** in the tree rooted at the node x is a finite sequence of nodes $x_0 = x, x_1, \dots, x_n$ where each x_{i+1} is an immediate predecessor of x_i . It is said to have length n . A **branch** of a rooted tree is a sequence (finite or infinite) of nodes (x_i) where x_0 is the root and each x_{i+1} is an immediate predecessor of x_i , satisfying the condition that the sequence stops only at a node with no predecessors.

Locally finite rooted trees possess a recursive structure, since if T is any locally finite rooted tree and x is a node of T then the set of all predecessors of x in T , together with x itself, make up a locally finite tree with root x . Any subset of T that contains along with a node its successor will also be a locally finite tree with the same root as T .

[finite-depth] **Lemma 2.1.** *In any locally finite tree, the number of chains of length n rooted at a given node is finite.*

Proof. That this is true for chains of length 1 is the definition of locally finite. The proof proceeds by induction on chain length. □

[konigs-lemma] **Lemma 2.2.** (König's Lemma) *In every locally finite rooted tree with an infinite number of nodes there exists an infinite branch.*

Proof. Let ρ be the root of the given tree, assumed to possess an infinite number of nodes. It follows from the previous result that there exist chains rooted at ρ of arbitrary length, and in particular that the set C_0 of all chains rooted at $x_0 = \rho$ is infinite. Since the number of immediate predecessors of x_0 is finite, the subset C_1 of chains in C_0 passing through some one of them, say x_1 , is infinite. Similarly there must exist an infinite number among the chains in C_1 whose third nodes agree. By induction, we obtain for each n a sequence of sets of chains

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_n \supseteq C_{n+1} \supseteq \dots$$

where all the chains in C_n agree with each other in their first $n + 1$ nodes, and agree in their first n nodes with the chains in C_{n-1} . By choosing x_n to be the common n -th node of the chains in C_n we assemble an infinite branch. □

The reasoning here, although plausible, is specious or at least highly subtle, since disguised in it is a weak form of the Axiom of Choice. König's Lemma is one of those results that hovers on the edge of obviousness, like a dim star one cannot quite focus on directly. The degree to which it is not obvious becomes more apparent when one sees (in, for example, the enlightening discussion to be found in §2.3.4.3 of [Knuth:1973]) some of its immediate consequences.

Suppose T to be a locally finite rooted tree with root ρ . For every node x in T define Ω_x to be the set of all branches passing through x . König's Lemma guarantees that this is never empty. Let Ω_T be the set of all branches of T , which is the same as Ω_ρ . We can make a topological space out of Ω_T by defining as basis of open sets the sets Ω_x —if ω is any branch in the tree then the sets Ω_x for each of its nodes x define a basis of neighbourhoods of ω . Two distinct branches must eventually diverge, and therefore it is easy to see that this topology is Hausdorff. The next result shows that this topology is otherwise somewhat special.

[omega-closed] Proposition 2.3. *If T is a locally finite rooted tree and x a node in T then Ω_x is closed as well as open in Ω_T .*

In other words, any point of Ω_T has a basis of neighbourhoods that are both open and closed.

Proof. Let $x_0 = \rho, x_1, \dots, x_n = x$ be the chain from ρ to x , and let $X = \{x_i\}$. Let Y be the union of all of the immediate predecessors of the x_i for $i < n$, except for x_{i+1} . In other words, y is in Y if it is an immediate predecessor of some x_i and is not in X . Since T is locally finite, Y is finite. Every branch in T that does not pass through x has to branch off at one of the x_i with $i < n$, and therefore the complement of Ω_x is the union of the Ω_y for y in Y . □

[compactness] Proposition 2.4. *If T is a locally finite rooted tree then the topological space Ω_T is compact.*

Proof. This amounts to the following assertion:

Suppose X to be a set of nodes of T such that the sets Ω_x (x in X) cover Ω_T . Then there exists a finite set of Ω_x (x in X) covering Ω_T .

The assumption means that every branch in Ω_T lies in some Ω_x with x in X , or equivalently that every branch in Ω_T has a node in X . If X contains the root node, we are immediately through.

Otherwise, let X_{\min} be the set of nodes in X that are minimal—that is to say, x lies in X_{\min} if it lies in X and in the path from the root to x there are no elements of X other than x . It is clear that that the Ω_x for x in X_{\min} cover Ω_T , since every branch has to have a first element in X .

It suffices to show that X_{\min} is finite, since a path from x in X to the root must pass through a node in X_{\min} . Let Y be the set of nodes in T that are not in X and none of whose successors are in X . In particular, Y contains the root node of T . Any successor of a node in Y will also be in Y , so that Y itself is a rooted tree and we can apply König's Lemma to it. It is not possible for Y to contain an infinite branch, since any infinite branch in Y would also be a branch in T , and by assumption every branch in T must contain a node in X . König's Lemma tells us that Y must be finite. But since X does not contain the root node, every element of X_{\min} is the predecessor of some node in Y , and since the tree is locally finite X_{\min} must be finite. □

Applying this to each of the rooted trees Ω_x :

[totally-disconnected] Proposition 2.5. *Every point in the topological space Ω_T possesses a countable basis of compact open neighbourhoods.*

3. Locally profinite spaces

Suppose we are given a sequence of finite sets X_0, X_1, \dots and for each $n > 0$ a surjection $\pi_n: X_n \rightarrow X_{n-1}$. We can make a finite union of trees from these data by taking the nodes of our graph to be the points of the X_n and defining the successor of x in X_n to be $\pi_n(x)$. The branches of this tree are the infinite sequences (x_n) where $\pi_n(x_n) = x_{n-1}$, and for a given point x in X_n the set Ω_x consists of all sequences with $x_n = x$. Let X be the set of such sequences, and let Π_n be the canonical surjection $(x_n) \mapsto x_n$ from X to X_n . The topological space X is the **projective limit** of the given sequence of finite sets. According

♣ [compactness] to Proposition 2.4 it possesses a natural Hausdorff topology with respect to which it is compact. The point $x = (x_n)$ has as a basis of neighbourhoods the sets $\Omega_{x_n} = \Pi_n^{-1}(\Pi_n(x))$.

More generally, suppose Σ to be a directed set: Σ is ordered, and given any two α, β in Σ there exists $\gamma \geq \alpha, \beta$. Suppose given (a) for each α in Σ a finite set X_α and (b) for each $\alpha > \beta$ a map

$$p_{\beta,\alpha}: X_\alpha \rightarrow X_\beta.$$

The maps are required to be (a) surjective and (b) consistent, in the sense that

$$p_{\gamma,\beta} p_{\beta,\alpha} = p_{\gamma,\alpha}$$

whenever $\alpha > \beta > \gamma$. The projective limit of this system is defined to be the subset of all (x_α) in $\prod X_\alpha$ such that $p_{\beta,\alpha} x_\alpha = x_\beta$. The product has a natural topology with respect to which it is compact, and the projective limit is closed in it, hence also compact. Proving compactness of the product (Tychonov's Theorem) requires the full Axiom of Choice, so that the sequentially profinite systems I have examined are definitely simpler.

A topological space X is said to be profinite if it is a projective limit of finite sets. It is said to be totally disconnected if every point has a basis of neighbourhoods that are both open and closed.

[lctd-lp] **Theorem 3.1.** *Suppose X to be a topological space. The following are equivalent:*

- (a) *it is profinite;*
- (b) *it is Hausdorff, compact, and totally disconnected.*

I'll not prove this result, because it will not be used subsequently, except to justify terminology.

[lctd] **Corollary 3.2.** *If X is a Hausdorff topological space, the following are equivalent:*

- (a) *it is locally profinite;*
- (b) *every point has a basis of compact, open sets.*

For our purposes, it is the second condition that is important. The equivalence only justifies applying the convenient term 'locally profinite' to spaces satisfying it. A profinite space is the same as a compact, Hausdorff, locally profinite space, and is also just a compact Hausdorff space in which every point has a basis of compact open sets. The spaces dealt with in earlier sections are **sequentially profinite**, and this gives rise to the notion **locally sequentially profinite**.

[good-refinement] **Lemma 3.3.** *Every covering of a profinite space by compact open subsets possesses a disjoint refinement by compact open sets.*

Proof. Suppose X to be a profinite space and suppose given a covering by compact open subsets U_i , which we may assume to be a finite covering. The union of any of the U_i is compact, hence closed in X , so that the sets

$$U_{*,i} = U_i - \bigcup_{j < i} U_j$$

are also open, and also cover X . □

[product-locprof] **Proposition 3.4.** *If (X_n) is a sequence of sequentially profinite spaces then the product $X = \prod X_n$ is sequentially locally profinite.*

Proof. I recall that open sets in the product topology are of the form $\prod U_n$ with each U_n open in X_n and all but a finite number of $U_n = X_n$.

Let $\Pi_{n,m}$ be the canonical surjection from X_n to the finite set $X_{n,m}$. As a basis of open sets in the topology of X we therefore have the finite products $\prod_{n \leq N} \Pi_{n,m}^{-1}(x_{n,m})$ with $x_{n,m}$ in $X_{n,m}$. In order to prove the Proposition we need to find a cofinal sequence of finite sets onto which X surjects. This can be the sequence

$$X_{1,1}, X_{1,2} \times X_{2,1}, X_{1,3} \times X_{2,2} \times X_{3,1}, \dots \quad \square$$

4. p -adic fields

The ring of p -adic integers \mathbb{Z}_p is the projective limit of the finite rings \mathbb{Z}/p^n . The ordinary integers \mathbb{Z} may be embedded in \mathbb{Z}_p , since m may be identified with the sequence $(m \bmod p^n)$. If q is an integer relatively prime to p then for every $n > 0$ there exists a multiplicative inverse of q modulo p^n , so that all rational numbers m/q with q prime to p may also be identified with elements of \mathbb{Z}_p . More generally, the p -adic integers with multiplicative inverses in \mathbb{Z}_p are precisely those whose image in \mathbb{Z}/p does not vanish. A p -adic rational number other than 0 can be identified with a unique expression m/p^k where m is a unit in \mathbb{Z}_p . These make up the field \mathbb{Q}_p .

Generalizing this construction, I define **p -adic field** to be a field \mathfrak{k} containing a ring \mathfrak{o} and an ideal \mathfrak{p} of \mathfrak{o} satisfying these conditions:

- (a) $\mathfrak{o}/\mathfrak{p}$ is a finite field, say of q elements;
- (b) the canonical projections from \mathfrak{o} to $\mathfrak{o}/\mathfrak{p}^n$ and from $\mathfrak{o}/\mathfrak{p}^{n+1}$ to $\mathfrak{o}/\mathfrak{p}^n$ identify the ring \mathfrak{o} with the projective limit of the quotients $\mathfrak{o}/\mathfrak{p}^n$;
- (c) if ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$, then every non-zero element of \mathfrak{k} may be expressed uniquely as $u\varpi^n$ where u lies in $\mathfrak{o} - \mathfrak{p}$.

If x is an element of $\mathfrak{o} - \mathfrak{p}$ there exists an element y of $\mathfrak{o} - \mathfrak{p}$ such that $xy \equiv 1$ modulo \mathfrak{p} . If then $m = xy - 1$ the series

$$u = 1 - m + m^2 - m^3 + \dots$$

converges to an element of \mathfrak{o} because of condition (b), since modulo any power of \mathfrak{p} the series terminates. The limit will be a multiplicative inverse of xy , so that yu will be a multiplicative inverse of x . Hence every element of $\mathfrak{o} - \mathfrak{p}$ is a unit of \mathfrak{o} . The ideal \mathfrak{p} is the only prime ideal of \mathfrak{o} other than (0) , and every non-zero ideal of \mathfrak{o} is a power of \mathfrak{p} . If ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$ then multiplication by ϖ^n induces a bijection of $\mathfrak{o}/\mathfrak{p}$ with $\mathfrak{p}^n/\mathfrak{p}^{n+1}$, so that $\mathfrak{o}/\mathfrak{p}^n$ is a finite ring of cardinality q^n .

The field \mathbb{Q}_p of p -adic integers is a p -adic field with $\mathfrak{o} = \mathbb{Z}_p$ and $\mathfrak{p} = (p)$. Every finite algebraic extension of a p -adic field is a p -adic field. The completion of any algebraic number field of finite degree with respect to any non-zero prime ideal of its ring of algebraic integers is a p -adic field, and is a finite extension of \mathbb{Q}_p . The field of power series in x with coefficients in \mathbb{F}_q and a finite number of negative powers of x

$$c_{-n}x^{-n} + c_{-(n-1)}x^{-(n-1)} + \dots$$

make up the quotient field $\mathbb{F}_q((x))$ of the ring of formal power series $\mathbb{F}_q[[x]]$. It is a p -adic field with $\mathfrak{p} = (x)$. The completion of any \mathbb{F}_q -rational local ring on a non-singular algebraic curve over \mathbb{F}_q is isomorphic to it. Conversely, any p -adic field is either some $\mathbb{F}_q((x))$ or a finite algebraic extension of \mathbb{Q}_p .

Throughout this book we will work with a fixed p -adic field $(\mathfrak{k}, \mathfrak{o}, \mathfrak{p})$, where $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$. If $x = u\varpi^n$ with u a unit in \mathfrak{o} then its norm $|x| = |x|_{\mathfrak{p}}$ is defined to be

$$|x| = q^{-n}.$$

Thus when $n \geq 0$ the index of the ideal (x) in \mathfrak{o} is $|x|^{-1}$.

The ring \mathfrak{o} is a profinite space. The ideals \mathfrak{p}^n form a basis of neighbourhoods of 0. Any closed or open subspace of a finite dimensional vector space over \mathfrak{k} will also be a locally profinite space.

Now let F be a global field. The ring $\mathbb{A} = \mathbb{A}_{F,f}$ of finite adèles is the **restricted product** of the non-Archimedean completions F_v —the subset of (x_v) in $\prod F_v$ for which all but a finite number of the x_v lie in the integer ring \mathfrak{o}_v . The product $\prod \mathfrak{o}_v$ is an open neighbourhood of 0 in \mathbb{A} , and according to Proposition 3.4:

♣ [product-locprof]

[adeles-locprof] **Proposition 4.1.** *The ring of finite adèles of F is locally sequentially profinite.*

5. References

1. D. E. Knuth, **Fundamental algorithms**, volume 1 of **The art of computer programming**, Addison-Wesley, second edition, 1973. König's Lemma is discussed in §2.3.4.3, on the "Infinity Lemma".
2. J-P. Serre, **Cohomologie Galoisienne**, *Lecture Notes in Mathematics* **5**, 1964.

Part II. Analysis

6. Smooth functions on a locally profinite space

Fix a coefficient field \mathbb{D} of characteristic 0. Much later it will be assumed to be algebraically closed, and even more specifically \mathbb{C} , but for the moment it might even be just \mathbb{Q} .

Suppose X to be a locally profinite space and V a vector space over \mathbb{D} . Define

$$\begin{aligned} C(X, V) &= \text{space of continuous functions on } X \text{ with values in } V \\ C_c(X, V) &= \text{subspace of those with compact support} \\ C^\infty(X, V) &= \text{space of locally constant functions with values in } V \\ C_c^\infty(X, V) &= \text{subspace of those with compact support} \\ \mathcal{D}(X) &= \text{linear dual of } C_c^\infty(X) = C_c^\infty(X, \mathbb{C}). \end{aligned}$$

The functions in C^∞ are called **smooth**, and they are continuous. The elements of \mathcal{D} are **distributions**.

For any open set U of X let char_U be its characteristic function.

[excision-lemma] Proposition 6.1. *If Y is a closed subset of the locally profinite space X and f in $C_c^\infty(Y, V)$, then there exists a finite sum $\sum \text{char}_U \cdot v_U$ whose restriction to Y is f .*

♣ [totally-disconnected] Proof. Suppose f to be in $C_c^\infty(Y)$. Because of Proposition 2.5, we can find a covering $\{U_i\}$ of the support of f by compact open subsets U_i of X with the property that f is constant on each $U_i \cap Y$. Apply Lemma 3.3 to get a refinement by disjoint sets $U_{*,i}$. 0

On each $U_{*,i} \cap Y$ the function f takes a constant value v_i . The linear combination $\sum \text{char}_{U_{*,i}} \cdot v_i$ then lies in $C_c^\infty(X, V)$ and has image f in $C_c^\infty(Y, V)$. 0

If $Y = X$:

[characteristic] Corollary 6.2. *Every function in $C_c^\infty(X, V)$ is a finite sum of functions $\text{char}_U \cdot v$ with v in V .*

[excision] Proposition 6.3. *(Excision Lemma) If Y is a closed subset of the locally profinite space X , then the natural maps make an exact sequence*

$$0 \rightarrow C_c^\infty(X - Y, V) \rightarrow C_c^\infty(X, V) \rightarrow C_c^\infty(Y, V) \rightarrow 0.$$

♣ [excision-lemma] Proof. The only interesting point is the final surjectivity, which follows from Proposition 6.1. 0

7. Locally profinite groups

I define a **locally profinite group** to be a Hausdorff topological group possessing a basis of neighbourhoods of 1 that are compact open subgroups.

In $\text{GL}_n(\mathfrak{k})$ the subgroup $\text{GL}_n(\mathfrak{o})$ of invertible matrices with coefficients in \mathfrak{o} is a compact open subgroup, with the congruence subgroups

$$\text{GL}_n(\mathfrak{p}^m) = \{g \in \text{GL}_n(\mathfrak{o}) \mid g \equiv I \pmod{\mathfrak{p}^m}\}$$

forming a basis of neighbourhoods of the identity. The group $\text{GL}_n(\mathfrak{o})$ may be identified with the projective limit of the groups $\text{GL}_n(\mathfrak{o}/\mathfrak{p}^n)$. Hence the group of \mathfrak{k} -rational points on any closed subgroup of $\text{GL}_n(\mathfrak{k})$, and in particular the group of \mathfrak{k} -rational points on any affine algebraic group defined over \mathfrak{k} , is a locally profinite group.

LOCAL STRUCTURE.

[k-stable] **Proposition 7.1.** *Suppose X to be any locally profinite space on which the locally profinite group G acts continuously. Then for every compact open set Ω in X there exists a compact open subgroup K of G such that $K\Omega = \Omega$.*

Proof. Because the action of G is continuous, left multiplication in G is continuous. For any point x of Ω there exists a compact subgroup K and a neighbourhood U of x such that $KU \subseteq \Omega$. Since Ω is compact, Ω will be covered by a finite number of these, say by the $K_i U_i$. Then Ω will be stable with respect to the intersection of the K_i . □

If G is a locally profinite group, then it acts by the right- and left-regular representations on $C^\infty(G)$, $C_c^\infty(G)$, and $\mathcal{D}(G)$ according to the recipes

$$\begin{aligned} L_g f(x) &= f(g^{-1}x) \\ R_g f(x) &= f(xg) \\ \langle L_g \Phi, f \rangle &= \langle \Phi, L_{g^{-1}} f \rangle \\ \langle R_g \Phi, f \rangle &= \langle \Phi, R_{g^{-1}} f \rangle . \end{aligned}$$

We have

$$L_{g_1 g_2} = L_{g_1} L_{g_2}, \quad R_{g_1 g_2} = R_{g_1} R_{g_2} .$$

SMOOTHNESS. I have already defined a **smooth** function on G as one that is locally constant. A **smooth distribution** is one that is locally right-invariant under some open subgroup of G . **Uniform** smoothness means global right-invariance under some fixed open subgroup.

[right-left] **Lemma 7.2.** *If G is a locally profinite group and f a function on G with values in V with compact support, the following are equivalent:*

- (a) *the function f lies in $C_c^\infty(G, V)$;*
- (b) *there exists a compact open subgroup K of G such that $L_k f = f$ for all k in K ;*
- (c) *there exists a compact open subgroup K of G such that $R_k f = f$ for all k in K .*

This is straightforward. As a consequence:

[locally-smooth] **Lemma 7.3.** *A smooth function is one that is locally left- or right-invariant with respect to multiplication by elements of some compact open subgroup.*

HAAR MEASURES. The theory of Haar measures on locally profinite groups is very simple. In practice, as will be explained in a moment, integrals are always sums—even though occasionally infinite.

[haar] **Proposition 7.4.** *Let G be a locally profinite group. Given a compact open subgroup K and a constant c_K in \mathbb{Q}^\times , there exists a unique right G -invariant measure μ on G such that*

$$\langle \mu, \text{char}_K \rangle = c_K .$$

Proof. First I assign measures to compact open subsets of G .

Suppose that the measure μ is known to exist. If K_* is a compact open subgroup contained in K then

$$\langle \mu, \text{char}_{K_*} \rangle = [K: K_*]^{-1} c_K$$

since K is the disjoint union of the $K_* x$ as x runs over representatives of $K_* \backslash K$, and $\langle \mu, \text{char}_{K_* x} \rangle = \langle \mu, \text{char}_{K_*} \rangle$. If K_* is an arbitrary compact open subgroup, then

$$\langle \mu, \text{char}_{K_*} \rangle = \frac{[K_*: K \cap K_*]}{[K: K \cap K_*]} c_K .$$

But knowing $\langle \mu, \text{char}_{K_*} \rangle$ for all compact open subgroups K_* , together with right G -invariance, determines $\langle \mu, f \rangle$ for any smooth function f of compact support, since f is a linear combination of $\text{char}_{K_* x}$ for some one K_* and a finite set of x in G . This argument when run backwards gives the recipe for constructing μ as a distribution.

To define it as a measure, one must evaluate $\langle \mu, f \rangle$ for any continuous function of compact support. But if f is a continuous function of compact support, there exist arbitrarily close functions in $C_c^\infty(G)$, with the same support, and one can define $\langle \mu, f \rangle$ as a limit, exactly as one defines Riemann sums. ◻

I shall call such any \mathbb{Q} -distribution on G with positive c_K that is right G -invariant a (rational) **right Haar measure** on G , and if $d_r x$ is one write

$$\begin{aligned} \text{meas}(U, d_r x) &= \langle d_r x, \text{char}_U \rangle \\ \int_G f(x) d_r x &= \langle d_r x, f \rangle . \end{aligned}$$

I have said that, in practice, integration on G amounts to summation. Let's make this explicit.

[integration-is-sum] Lemma 7.5. *Suppose (a) V is a vector space over \mathbb{Q} , (b) f lies in $C_c^\infty(G, V)$, and (c) K is a compact open subgroup of G . If f is left-invariant with respect to K , then*

$$\int_G f(x) d_r x = \text{meas}(K) \sum_{K \backslash G} f(Kg) ,$$

and if it is right-invariant then

$$\int_G f(x) d_r x = \text{meas}(K) \sum_{G/K} \delta_G(g) f(gK) ,$$

Proof. Straightforward. For example:

$$\begin{aligned} \int_G f(x) d_r x &= \sum_{K \backslash G} \int_{Kg} f(x) d_r x \\ &= \text{meas}(K) \sum_{K \backslash G} f(Kg) . \end{aligned} \quad \bullet \quad \square$$

Here is a frequently useful characterization:

[h0] Lemma 7.6. *Suppose G to be a locally profinite group, assigned a right-invariant Haar measure $d_r g$, and f to lie in $C_c^\infty(G)$. The following are equivalent:*

(a) *We have*

$$\int_G f(g) d_r g = 0 ;$$

(b) *the function f is a linear combination of functions of the form $R_g \varphi - \varphi$.*

Proof. One way is trivial. As for the other, suppose that

$$\int_G f(g) d_r g = 0 .$$

If f is left-invariant under K , this means that

$$\sum_{K \backslash G} f(g_i) = 0 ,$$

if the support of f is the disjoint union of the Kg_i . Let char_g be the characteristic function of Kg . Then $\text{char}_g = R_{g^{-1}} \text{char}_1$ and so

$$\begin{aligned} f &= \sum_i f(g_i) \text{char}_{g_i} \\ &= \left(\sum_i f(g_i) \text{char}_{g_i} \right) - \left(\sum_i f(g_i) \right) \text{char}_1 \\ &= \sum_i f(g_i) (R_{g_i^{-1}} \text{char}_1 - \text{char}_1) . \quad \mathbf{0} \end{aligned}$$

THE MODULUS CHARACTER. If $d_r x$ is any right Haar measure on G then any left translation $L_x d_r x$ is also a right Haar measure, and must be therefore a scalar multiple of $d_r x$. In other words for each g in G there exists a scalar $\delta_G(g)$ such that

$$\int_{gU} d_r x = \delta_G(g) \int_U d_r x$$

for all compact open subsets U of G , or in brief

$$d_r g x = \delta_G(g) d_r x .$$

The constant $\delta_G(g)$ is independent of the right Haar measure chosen, since all others are just scalar multiples of it. It depends multiplicatively on g :

$$\delta_G(g_1 g_2) = \delta_G(g_1) \delta_G(g_2) .$$

This multiplicative character of G with values in the positive rational numbers is called the **modulus character** of G . If $d_r x$ is a right Haar measure on G then $\delta_G(x)^{-1} d_r x$ is a left Haar measure.

The modulus character also clearly characterizes how conjugation affects measures, since

$$\text{meas}(g U g^{-1}) = \delta_G(g) \text{meas}(U) .$$

The group is called **unimodular** if δ_G is trivial. If K is a compact subgroup of G then the image of any character of K with values in the positive rational numbers has to be a torsion group, hence trivial. Therefore every compact group is unimodular.

SMOOTH DISTRIBUTIONS. Assume for the moment a right-invariant Haar measure $d_r g$ chosen on G . If φ is a smooth function on G with values in V then the formula

$$\langle D_\varphi, f \rangle = \int_G \varphi(x) f(x) d_r x \quad (f \in C_c^\infty(G, \mathbb{Q}))$$

defines a distribution D_φ on G with values in V , and

$$R_g D_\varphi = D_{R_g \varphi}, \quad L_g D_\varphi = \delta_G(g)^{-1} D_{L_g \varphi} .$$

If φ is right-invariant under an open group K , then D_φ will also be right-invariant under K .

Conversely, suppose D to be a smooth distribution, which will be locally right-invariant by some compact open subgroup K . We can associate to it a function value at g in G by the formula

$$\varphi(g) = \frac{\langle D, \text{char}_{gK} \rangle}{\text{meas}(gK)} .$$

Local right-invariance of D implies immediately that

$$\langle D, \text{char}_{gK} \rangle = \langle D, \text{char}_{gK_*} \rangle \left(\frac{\text{meas}(gK)}{\text{meas}(gK_*)} \right).$$

for any compact open subgroup K_* of K , which means that the definition of $\varphi(g)$ is independent of the choice of K with respect to which D is right-invariant. It is also straightforward to see that D is then the same as D_φ . We have proven:

[smooth-distribution] Proposition 7.7. *Suppose V to be a vector space over \mathbb{Q} . Given a right-invariant Haar measure on G , the correspondance $\varphi \mapsto D_\varphi$ is a right- G -covariant isomorphism between the space of smooth functions with values in V and that of V -valued smooth distributions on G .*

There is one large class of distributions we shall often use. Suppose H to be any compact subgroup of G (not necessarily open). Then associated to H is the distribution μ_H (not necessarily smooth) defined by the formula

$$\langle \mu_H, f \rangle = \frac{1}{[H:H \cap K]} \sum_{H/H \cap K} f(h)$$

if f in $C_c^\infty(G)$ is right-invariant under K . In effect it evaluates the average value of f on H .

8. Quotients

If G is a locally profinite group and H a closed subgroup, the quotient $H \backslash G$ is also a locally profinite space. It can be covered by translates of the compact quotients $(H \cap K) \backslash K$, where K is a compact open subgroup of G . If G is locally sequentially profinite, this quotient may be identified with the projective limit of finite quotients $(H \cap K)K_n \backslash K$ where K_n is a sequence of open normal subgroups of K .

[sections] Lemma 8.1. *There exists a continuous section of the canonical projection $G \rightarrow H \backslash G$ over all of $H \backslash G$.*

Proof. It suffices to prove this when G is compact. I'll give the proof, in fact, only when G is sequentially profinite—for the general case, I refer to [Serre:1964].

Let H_n be a shrinking sequence of compact open subgroups of H . The first claim is that for each n there exist continuous sections of the canonical projections

$$H_n \backslash G \rightarrow H \backslash G.$$

This is because if K is a compact open subgroup of G small enough so that $K \cap H \subseteq H_n$ then we have the following commutative diagram:

$$\begin{array}{ccc} H_n \cap K \backslash K & \hookrightarrow & H_n \backslash G \\ \cong & & \downarrow \\ H \cap K \backslash K & \hookrightarrow & H \backslash G \end{array}.$$

♣ [good-refinement] So this projection is a bijection on $H \cap K \backslash K$ and any of its G -translates. Apply Lemma 3.3.

We now look at a tree whose nodes are continuous sections

$$s_n: H \backslash G \rightarrow H_n \backslash G$$

and where the successor of a section is its composition with the canonical projection from $H_{n+1} \backslash G$ to $H_n \backslash G$. Since H_{n+1} has finite index in H_n , such systems form a locally finite tree. By König's Lemma

♣ [konigs-lemma] (Lemma 2.2) there exists an infinite branch in this tree, hence a sequence of compatible continuous sections $s_n: H \backslash G \rightarrow H_n \backslash G$, hence a map from $H \backslash G$ to the projective limit of the $H_n \backslash G$, which can be canonically identified with G . □

[h-projection] **Lemma 8.2.** Suppose H to be a closed subgroup of G and $d_r h$ to be a right Haar measure on H . The map taking f to \bar{f} where

$$\bar{f}(g) = \int_H f(hg) d_r h$$

is a surjection from $C_c(G)$ to $C_c(H \backslash G)$, and commutes with the right action of G . It also induces a surjection from $C_c^\infty(G)$ to $C_c^\infty(H \backslash G)$.

♣ [sections] *Proof.* The first assertion follows from Lemma 8.1.

Every function in $C_c^\infty(H \backslash G)$ is a linear combination of the functions char_{HxK} for some x in G and compact open subgroup K . This is the image under the map given of $\text{meas}(H \cap xKx^{-1}, d_r h)^{-1} \text{char}_{xK}$.

If H and G are both unimodular then invariant Haar measures on G and H will determine a G -invariant measure on the quotient $H \backslash G$ (as we'll see in a moment). But in the situation we'll be mostly concerned with, G will be unimodular and H not, and one has to be more careful, since there is then no G -invariant integral on $H \backslash G$. What one integrates instead are **one-densities**, which are for arbitrary locally compact spaces the analogue of differential forms on oriented manifolds.

I find it hard to motivate what is to come, and have experimented with many different approaches. What I shall do now is simply start out with the integral formula for G , and then try to take into account that we are interested in $H \backslash G$. Suppose f to be in $C_c^\infty(G)$, fixed on the right by the compact open subgroup K . Then

$$\int_G f(g) d_r g = \sum_{H \backslash G / K} \int_{HgK} f(x) d_r x.$$

Since

$$HGK/K = \bigsqcup_{h \in H/H \cap gKg^{-1}} hgK$$

the integral becomes

$$\begin{aligned} \sum_{H \backslash G / K} \int_{HgK} f(x) d_r x &= \sum_{H \backslash G / K} \sum_{HgK/K} \text{meas}(xK) f(x) \\ &= \sum_{H \backslash G / K} \sum_{H/H \cap gKg^{-1}} \text{meas}(hgK) f(hg) \\ &= \sum_{H \backslash G / K} \sum_{H/H \cap gKg^{-1}} \text{meas}(gKg^{-1}) \delta_G(h) f(hg) \\ &= \sum_{H \backslash G / K} \frac{\text{meas}_G(gKg^{-1})}{\text{meas}_H(H \cap gKg^{-1})} \int_H \delta_G(h) f(hg) d_\ell h \\ &= \sum_{H \backslash G / K} \frac{\text{meas}_G(gKg^{-1})}{\text{meas}_H(H \cap gKg^{-1})} \cdot f_\Omega(g) \end{aligned}$$

where

$$f_\Omega(g) = \int_H \delta_G(h) f(hg) dh.$$

The function $F(g) = f_\Omega(g)$ satisfies these conditions:

(a)

$$F(hg) = \frac{\delta_H(h)}{\delta_G(h)} \cdot F(g)$$

for all h in H and g in G ;

(b) it has compact support modulo H on the left;

(c) it is right invariant by a compact open subgroup of G .

Define $\Omega^\infty(H \backslash G)$ to be the space of all such functions on G .

[int-quotients] Proposition 8.3. *There exists a unique linear map from $\Omega^\infty(H \backslash G)$ to \mathbb{D} evaluated by the formula*

$$\sum_{H \backslash G / K} \frac{\text{meas}_G(gKg^{-1})}{\text{meas}_H(H \cap gKg^{-1})} \cdot F(g)$$

whenever F is right K -invariant. It is right G -invariant, and extends to a continuous linear functional on the space of all continuous functions F on G satisfying condition (a) above.

♣ **[h-projection]** I leave this as an exercise. In doing it, you will want to know the analogue of Lemma 8.2 for the map $f \mapsto f_\Omega$.

♣ **[int-quotients]** I shall write the functional of Proposition 8.3 as

$$\int_{H \backslash G} F(x) dx .$$

This functional is characterized by the equation

$$\int_G f(g) d_r g = \int_{H \backslash G} dx \int_H \delta_G(h) d_\ell h = \int_{H \backslash G} dx \int_H \delta_G(h) \delta_H(h)^{-1} d_r h .$$

When G and H are unimodular the formula becomes

$$\int_{H \backslash G} F(g) dx = \sum_{H \backslash G / K} \frac{\text{meas}_G(K)}{\text{meas}_H(H \cap gKg^{-1})} \cdot F(g)$$

for smooth functions F on $H \backslash G$.

9. Measures on \mathfrak{p} -adic manifolds

In the special case that G is an algebraic group defined over the \mathfrak{p} -adic field \mathfrak{k} , we can derive measures on G in a way that one must often use when one wants to treat coherently all local fields associated to a global one.

First let V be the additive group \mathfrak{k}^n . A translation-invariant measure dx on V is a constant multiple of the measure $dx = dx_1 \dots dx_n$ that assigns measure 1 to \mathfrak{o}^n . This assigns measure q^{-nk} to $(\mathfrak{p}^n)^k$. This determines the measure on any compact open subgroup, since it will be the disjoint union of translates of one of these neighbourhoods of 0.

If T is any matrix in $\text{GL}_n(\mathfrak{k})$ and dx is translation-invariant then the transform by T is also translation-invariant, hence a multiple of dx . The constant is determined by what T does to \mathfrak{o}^n . If D is a diagonal matrix with diagonal entries d_i the volume of $T\mathfrak{o}^n$ is $\mu \prod |d_i|$. If γ is in $\text{GL}_n(\mathfrak{o})$ then $\gamma\mathfrak{o}^n = \mathfrak{o}^n$, hence it preserves volumes. The principal divisor theorem tells us that any matrix T in $\text{GL}_n(\mathfrak{k})$ is a product $\gamma_1 D \gamma_2$ with the γ_i in $\text{GL}_n(\mathfrak{o})$ and D diagonal, so T multiplies volumes by $|\det(T)|$.

If φ is an analytic isomorphism of subsets X and Y of \mathfrak{k}^n , then setting $y = \varphi(x)$, $g(x) = f(\varphi(x))$ we have the change of variables formula

$$\int_Y f(y) dy = \int_X g(x) |\det(\partial y / \partial x)| dx .$$

Now suppose X to be an arbitrary \mathfrak{k} -analytic manifold, given with a countable, locally finite atlas $\{X_i\}$. For each X_i we are given an open embedding into \mathfrak{k}^n , and on overlaps two coordinate systems differ by an invertible analytic function. Details about what this means can be found in [Serre:1965]. Because of

the implicit function theorem for \mathfrak{k} -analytic maps, every non-singular algebraic variety over \mathfrak{k} has such a structure.

Following what happens for real manifolds, I define a **smooth one-density** on X to be a family of compatible smooth measures on the sets X_i . Compatibility means only that the measures agree on overlaps. In particular, if one is given a non-vanishing differential form ω on an algebraic variety M , then $|\omega|$ defines a smooth one-density. If one applies this an algebraic group, one obtains:

[p-adic-modulus] **Proposition 9.1.** *If G is an algebraic group defined over \mathfrak{k} , then the modulus character is $|\det(\text{Ad}_{\mathfrak{g}})|$.*

[density-on-pg] **Corollary 9.2.** *If G is an algebraic group defined over \mathfrak{k} and H a closed subgroup, then the space of smooth one-densities on $H \backslash G$ may be identified with the space of smooth \mathbb{C} -valued functions on G such that*

$$f(hg) = |\det^{-1} \text{Ad}_{\mathfrak{h} \backslash \mathfrak{g}}(h)| f(g)$$

for all h in H , g in G .

10. References

1. J-P. Serre, Lie algebras and Lie groups, Benjamin, 1965. Chapters I-III of LG are about p-adic analytic manifolds.
2. A. Weil, **Adèles and algebraic groups**, Birkhäuser, 1982.