Practical non-Euclidean geometry

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This note is concerned with programming routines to do practical drawing in non-Euclidean geometry.

There are several possible models of non-Euclidean geometry to choose from. The one I shall mostly work with is that in which the non-Euclidean plane is identified with the interior of the unit disk \( D \), and the geodesics are either (1) the arcs of circles inside the disk orthogonal to its boundary or (2) diameters. In this model, the non-Euclidean isometries are the Möbius transformations generated by the matrices

\[
\begin{bmatrix}
\alpha & \beta \\
\beta & \alpha
\end{bmatrix}
\]

with \( \det = |\alpha|^2 - |\beta|^2 = 1 \). This is the group \( SU(1,1) \), and it preserves the unit disk precisely because it preserves the Hermitian form \( |z_1|^2 - |z_2|^2 \). I recall that Möbius transformations take circles and lines to circles and lines, and preserve angles. So any Möbius transformation that takes the closed unit circle to itself takes the geodesic arcs defined here to other geodesic arcs.

Another model is that in which the plane is the Poincaré upper half plane \( \mathcal{H} \), and the geodesics are either vertical lines or semi-circles perpendicular to the real axis. Here the group of isometries is the projective image of \( SL_2(\mathbb{R}) \). Although the disc model is the one we shall be primarily interested in, it is sometimes convenient to do things on \( \mathcal{H} \) and then transform results into the unit disc. This is done by means of the Cayley transformation

\[
C: z \mapsto \frac{z - i}{z + i}
\]

which maps \( \mathcal{H} \) isometrically (in the sense of non-Euclidean geometry) onto the interior of \( D \). Thus the real matrix \( T \) in \( SL_2(\mathbb{R}) \) acts as \( CT C^{-1} \) on \( D \).

A third model is the Kleinian one where the plane is still the interior of the unit disk, but the geodesics are straight line segments. Here the isometries are the projective transformations generated by planar slices of the cone of \( 2 \times 2 \) matrices with trace 0, on which \( SL_2(\mathbb{R}) \) acts by conjugation. I shall discuss it in the last few sections. It is especially important, maybe indispensable, in dealing with tilings of the non-Euclidean plane associated to certain Coxeter groups.

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1. Geodesic segments in the disc: circular arcs

The basic problem in non-Euclidean geometry is to draw the non-Euclidean geodesic segment between two points $z_1$ and $z_2$ inside the unit disk.

This is generically an arc of a circle, but it also might be a diameter. The principal problem in non-Euclidean graphics is to deal well with these two cases, given the limitations of machine floating point computation. I start out by trying to find the circle on which the segment lies, but if this is not possible constructs the straight line segment between them. This involves only elementary geometry. This is a little more interesting than one might expect, because of the way in which computers store real numbers. This is to some extent unavoidable, because there is a basic mathematical question involved—points that are close in the Euclidean metric may be far apart in the non-Euclidean one, if they are near the boundary of $D$.

I start by assuming the two points to lie on the arc of a circle. Let $z_0$ be the midpoint of the Euclidean segment from $z_1$ to $z_2$, let $w$ be the vector halfway from $z_1$ to $z_2$. Thus

$$z_0 = \frac{z_1 + z_2}{2}$$

$$w = z_0 - z_1.$$
Let $O$ be the center of the non-Euclidean line—i.e. the circular arc through $z_1$ and $z_2$ intersecting the unit circle orthogonally. Since $O$ is at equal distance from $z_1$ and $z_2$, we have

$$O = z_0 + iwt$$

for some real number $t$.

Since the tangents from $O$ to the unit circle have the same length, and the length of a tangent segment is $|O|^2 - 1$, we therefore have also

$$|z_2 - O|^2 = |O|^2 - 1$$

$$z_2 - O = (z_0 + w) - (z_0 + itw) = w(1 - it)$$

$$|w(1 - it)|^2 = |z_0|^2 + t^2|w|^2 + 2(z_0 \cdot itw) - 1$$

$$|w|^2(1 + t^2) = |z_0|^2 + t^2|w|^2 + 2t(z_0 \cdot iw) - 1$$

$$t = \frac{|w|^2 + 1 - |z_0|^2}{2(z_0 \cdot iw)}.$$
This will be undefined when the denominator vanishes. This happens when $z_0$ and $w$ are parallel, which is precisely when the arc is part of a diameter. So the simple rule is this:

*If* $z_0 \cdot iw = 0$, *draw the straight line from* $z_1$ *to* $z_2$. Otherwise find $O$ and *draw the arc from* $z_1$ *to* $z_2$ with center $O$.

For this, we need to know how to draw the circular arc from $z_1$ to $z_2$ centred at $O$. This is simple—if $\alpha_i$ is the argument of $z_i - O$ then we draw the arc from $\alpha_1$ to $\alpha_2$, centred at $O$, in the positive direction if $T > 0$ and in the negative direction if $T < 0$.

The problem with this is that computer calculations with real numbers are not all that accurate, and it is not possible to tell if $z_0 \cdot iw$ is exactly 0. We want a test that is a bit more stable under perturbations. Also, we don’t want to find ourselves drawing arcs of circles of very large radii.

Set

$$T = \frac{1}{t} = \frac{2(z_0 \cdot iw)}{|w|^2 + 1 - |z_0|^2}.$$

This will be 0 if and only if the geodesic is a straight line. If the denominator is 0, then $|w|^2 + 1 = |z_0|^2$. Since the disc is convex, $z_0$ must lie inside it and $|z_0| < 1$, unless $z_0$ lies in the unit circle and $z_1 = z_2$. That is to say, as long as $z_1$ and $z_2$ are distinct points in the closed unit disk, $t$ will not be 0 and $T$ well defined.

We shall see in the next section that the number $T$ is in fact an accurate measure of the straightness of the geodesic from $z_1$ to $z_2$. The new rule is therefore:

*Calculate $T$. If it is suitably small, draw the arc from* $z_1$ *to* $z_2$ *as Bézier curve. Otherwise find* $O$ *and draw the arc from* $z_1$ *to* $z_2$ *with center* $O$.

It remains to explain how to find that Bézier arc in a stable way. Also, because of machine treatment of floating point numbers, evaluating the denominator in the formula for $T$ is not straightforward.

### 2. Geodesic segments in the disc: using Bézier curves

I’ll begin by justifying the use of $T$ in a test for straightness of the non-Euclidean path from $z_1$ to $z_2$. The obvious measure of straightness is the angle $\theta$ spanned by the arc. But $\theta$ and $T$ are closely related.

We know that

$$z_2 - O = z_2 - (z_0 + itw) = (z_2 - z_0) - itw = w - itw = w(1 - it)$$

$$z_1 - O = z_1 - (z_0 + itw) = -w - itw = -w(1 + it).$$

So we get

$$\frac{z_2 - O}{z_1 - O} = \frac{1 - it}{1 + it} = \frac{it - 1}{it + 1} = \frac{1 - 1/it}{1 + 1/it} = \frac{1 + iT}{1 - iT}.$$
This tells us that

If $\theta$ is the angular span of the arc then $\theta/2$ is the argument of $1 + iT$.

In other words

$$\frac{\theta}{2} = \arctan(T) = \int_0^T \frac{ds}{1 + s^2} = T - \frac{T^3}{3} + \frac{T^5}{5} - \cdots$$

if $T$ is small. But then

$$\frac{\theta}{2T} = 1 - \frac{T^2}{3} + \frac{T^4}{5} - \cdots$$

Thus $T$ is nearly the same as $\theta/2$ if it is small. So our procedure for drawing the segment is justified.

If $T$ is small, we are going to draw a Bézier curve from $z_1$ to $z_2$. For this purpose, we must calculate the velocity vectors for the normalized path from one to the other. This is just the same as the very general problem of finding velocity vectors at the start and end of a trajectory with uniform speed along a circular arc of angular span $\theta$ whose radius vector $u$ at the start of the arc is given.

Let $r = ||u||$. The unit tangent vector (in the positive direction of travel) at the end of $u$ is $iu/r$. If we traveled along the circle for time 1 at constant speed, starting out at velocity $iu/r$, we would travel a unit length of arc. We want to travel an angle $\theta$ along the arc. Since $\theta$ is measured in radians, and a radian of angle $\theta$ means arc distance $\theta r$. So we start out at velocity $i(\theta r)(u/r) = i\theta u$. 
In our case, the radius vector is \(-w(1 + it)\), so we get

\[
v_1 = i \cdot -w(1 + it) \cdot \theta
\]

\[
= -2T \cdot i \cdot w(1 + it) \left( \frac{\theta}{2T} \right)
\]

\[
= -2T w(i - t) \left( \frac{\theta}{2T} \right)
\]

\[
= -2w(i/t - 1) \left( \frac{\theta}{2T} \right)
\]

\[
= 2w(1 - iT) \left( \frac{\theta}{2T} \right).
\]

and similarly it is

\[
v_2 = 2w(1 + iT) \left( \frac{\theta}{2T} \right)
\]

at the end. Because \(\theta/2T \sim 1\) as \(T \to 0\), these formulas make sense even in the case where the geodesic line is a diameter.

Given the normalized tangent vectors \(v_i\), we can find the intermediate control points of the approximating Bézier curve (as explained in §6.5 of *Mathematical Illustrations*):

\[
P_1 = z_1 + (1/3)v_1
\]

\[
P_2 = z_2 - (1/3)v_2.
\]

### 3. Technical problems

Because of the way computers deal with real numbers (as opposed to integers), the input data, which is in this case the two points \(z_1\) and \(z_2\), will usually only approximate in the sense that they will be the result of computations which are only approximate. I ask a question one should always ask in dealing with real numbers in computers: do small variations in the input data affect the output? Do small changes in \(z_1\) and \(z_2\) affect the arc that’s drawn? Is this an artefact of the particular technique used to do the drawing, or can we do better with a better one?

First of all, I recall that computers handle real numbers in floating point format, which means that it stores significant digits. Thus a machine with 10 figure accuracy would store \(\pi/1000\) as \(3.141592654 \cdot 10^{-3}\) instead of 0.003141593. On the whole this scheme is very efficient, but it also causes misunderstanding if one is not careful.

Let me illustrate what I mean by a simple example from high school algebra. The solutions to the quadratic equation

\[x^2 + ax + b = 0\]

are given by the formula

\[x = -a/2 \pm \sqrt{(a/2)^2 - b}.\]

Most of the time this formula can be used without trouble, but occasionally a difficulty arises. Consider

\[x^2 - 1634x + 2 = 0.\]
On my calculator, which handles real numbers with 10 significant figures, I get solutions
\[ x = 817 \pm \sqrt{667.487} \]
\[ x_1 = 817 + 816.9987760 = 1633.998776 \]
\[ x_2 = 817 - 816.9987760 = 0.0012240 . \]

What’s going on here is that the original data is specified exactly, and one of the roots is given correctly to 10 significant decimals, but the accuracy of the second root is only 4 significant figures. The problem arises in the subtraction 817 − 816.9987760. Both figures are accurate to 10 significant figures, and this cannot be improved. But the subtraction involves a loss of accuracy. This is not necessary. We know that the product \( x_1x_2 = 2 \), so we can set
\[ x_2 = 2/x_1 = 0.001223991125 . \]
which is correct to 10 significant figures. This leads to a modification of the usual rule for solving quadratic equations:
\[ x^2 + ax + b = 0 . \]

One root is calculated by the formula
\[ x_1 = -(a/2) - \varepsilon\sqrt{(a/2) - b} \]
where (a) the square root here always means the positive root, and (b) \( \varepsilon \) is chosen to be ±1, so that the second term has the same sign as \( a \). The other root is given by
\[ x_2 = b/x_1 . \]

In other words, it is the careless use of the quadratic formula that causes difficulty here, not something intrinsic to the problem. This phenomenon is called cancellation error, and is discussed elegantly in Chapter 1 of [Henrici:1982], from which this example was taken.

In our case, if \(|z_0|\) is near 1 we get cancellation problems in calculating \( 1 - |z_0|^2 \), which occurs in the denominator of the formula for \( T \). In some sense, this phenomenon is unavoidable, if we are given only the points \( z_1 \) and \( z_2 \) as data. If are are placed at a point at radius \( r \) from the origin in the unit disk and move a small Euclidean distance \( dr \), then the non-Euclidean distance moved is \( 1 dr/(1 - r^2) \). (This can be justified by using the fact that along the imaginary axis in the upper half plane \( \mathcal{H} \) the metric is \( dy/y \).) Hence, as I have already said, perturbations in the data \( z_1, z_2 \) are magnified.

But in practice the points \( z_1 \) and \( z_2 \) are found as the non-Euclidean transformations of other points, and this gives us a little more information. There is a very useful formula in these circumstances.

If \( z \) is any point of \( \mathbb{C} \) and
\[ \gamma = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \]
then
\[ 1 - |\gamma(z)|^2 = \frac{1 - |z|^2}{\beta z + \overline{\alpha} . \]

This is easy to verify, and explains among other things why the matrices in \( \text{SU}(1, 1) \) take \( \mathbb{D} \) into itself.
4. Endpoints of geodesics in the disc

In the rest of this essay, I’ll discuss how to solve other problems involving non-Euclidean computation, particularly involving graphics.

The first is, given points $z_1$ and $z_2$, how does one find the endpoints on the unit circle of the arc through them?

Let $O$ be the centre of the arc, with argument $\alpha$. Then the arguments of the endpoints are $\pm \theta + \alpha$ where $\cos \theta = 1/|O|$.

This makes sense even when $O$ is very large, since $O$ is never $0$, and

$$\frac{1}{O} = \frac{1}{z_0 + iw} = \frac{1}{i} \frac{1}{z_0/t + iw} = \frac{T}{iw + z_0T}.$$  

We can also see this as a more algebraic problem. Given $O$, we look for $\zeta$ such that

$$|\zeta|^2 = 1, \quad |\zeta - O|^2 = |O|^2 - 1$$

which implies that $\zeta \cdot O = 1$, or $\overline{O}\zeta + \zeta O = 2$, which means that $\overline{O}\zeta$ is a root of the equation

$$(Z - \overline{O}\zeta)(Z - O\zeta) = Z^2 - 2Z + |O|^2 = 0$$

with roots

$$Z = 1 \pm \sqrt{1 - |O|^2}.$$  

Because of the formula for $1/O$, we can put the formula we get for $\zeta$ in terms of $w, z_0,$ and $T$.

No matter which of these methods we use, we have to decide which endpoint goes with which $z_i$. This I do by calculating $(e_2 - e_1) \cdot (z_2 - z_1)$, where the $e_i$ are the two endpoints computed.
5. Triangles in the disc

The next problem is one of drawing triangles. In non-Euclidean geometry, unlike Euclidean geometry, the congruence class of a triangle is completely determined by its interior angles, say \( \alpha, \beta, \gamma \) with \( \alpha + \beta + \gamma < \pi \). For example, the non-Euclidean area of the triangle is

\[
\pi - (\alpha + \beta + \gamma).
\]

That is to say, there is no notion of similarity in non-Euclidean geometry.

The problem then arises, if we are given the interior angles, one of the vertices, say \( C \), and the ray from \( A \) along one side, how do we find the other vertices \( A \) and \( B \)?

We may assume that \( C \) is the origin, that \( A \) is on the positive x-axis, and that \( B \) is above \( A \). The triangle is now uniquely determined, as in this figure:

\[
\alpha \quad \beta \quad \gamma \quad \delta
\]

We must find the numbers \( r_\alpha = |C - A|, r_\beta = |B - C| \) (Euclidean lengths). This can be done by following a recipe explained to me by Robert Bédard. First, extend the diagram to include the center of the non-Euclidean geodesic through \( A \) and \( B \). This is possible, because in these circumstances the side through \( A, B \) is certainly not a diameter.

The angle \( \delta \) satisfies

\[
(\alpha + \pi/2) + (\beta + \pi/2) + \gamma + \delta = 2\pi, \quad \delta = \pi - (\alpha + \beta + \gamma).
\]
Then, the figure above is rotated and shifted, so that \( \delta \) becomes the origin, and the line through the origin and \( B \) becomes the positive \( x \)-axis. We are going to calculate a ratio, so we may scale the diagram, too, making \( B = (1, 0) \).

We must now find the coordinates \((x, y)\) of \( C \) in the new coordinate system. A simple calculation of angles shows that the top line \( AC \) now has slope \( \tau \) and the bottom one \( BC \) now has slope \( \sigma \), where

\[
\sigma = \tan\left(\frac{\pi}{2} - \beta\right), \\
\tau = \tan\left(\frac{\pi}{2} - \beta - \gamma\right).
\]

The point \( C \) is the intersection of the two lines. Hence to find \((x, y)\) we have to solve for \( x, y \) in the pair of equations

\[
-\tau x + y = -\tau x_A + y_A \\
-\sigma x + y = -\sigma x_B + y_B.
\]

But \( B = (1, 0) \) and \( A = (\cos \delta, \sin \delta) \), so we get

\[
x = \frac{\sigma + \sin \delta - \tau \cos \delta}{\sigma - \tau}, \\
y = \sigma(x - 1), \\
C = x + iy.
\]

Finally, we want to know in these units the radius \( R \) of the circle centred at \( C \).
\[ \rho = C - D \]
\[ R = \sqrt{\rho^2 - 1} \]

and finally get
\[ r_\alpha = |A - C|/R \]
\[ r_\beta = |B - C|/R \] .

6. Shifts

In this and the next section I’ll explain how to implement the effects of certain non-Euclidean isometries. In this one, I’ll deal with shifts.

Among the isometries of the upper half plane \( \mathcal{H} \) are those associated to matrices like
\[
\begin{bmatrix}
  t \\
  1/t
\end{bmatrix}
\]

which takes \( z \) to \( t^2 z \). All these fix both 0 and \( \infty \). Conjugated by the Cayley transform, this becomes
\[
\begin{bmatrix}
  \frac{t + 1/t}{2} & \frac{t - 1/t}{2} \\
  \frac{t - 1/t}{2} & \frac{t + 1/t}{2}
\end{bmatrix}
\]

which is
\[
\begin{bmatrix}
  \cosh x & \sinh x \\
  \sinh x & \cosh x
\end{bmatrix}
\]

if \( t = e^x \). All these fix the endpoints \( \pm 1 \). A matrix in this form is equivalent to (induces the same Möbius transformation as) one of the form
\[
\begin{bmatrix}
  1 & \tau \\
  \tau & 1
\end{bmatrix}
\]

where \( \tau = \sinh(x)/\cosh(x) \). This is very convenient to work with. I call the associated transformation a shift. The explicit formula for a shift is
\[
\begin{align*}
  z \mapsto \frac{z + \tau}{\tau z + 1} &= \frac{(z + \tau)(\tau^2 + 1)}{\tau^2 + 1} = \frac{\tau|z|^2 + \tau^2 + z + \tau}{|\tau z + 1|^2} .
\end{align*}
\]

The important thing about shifts is that the real axis is taken into itself. The shift parametrized by \( \tau \) takes 0 to \( \tau \), and \( \sigma(0) \) approaches 1 as \( t \) approaches \( \infty \), or \( \tau \to 1 \).

The orbits of the shifts are arcs of circles passing through \(-1\) and \(1\). 

![Diagram of a circle passing through -1 and 1 with a marked point inside the circle]
7. Reflections

The data determining a non-Euclidean reflection are two points \( z_1, z_2 \) and a third \( z \). What is the non-Euclidean reflection of \( z \) in the non-Euclidean geodesic through \( z_1 \) and \( z_2 \)? If \( z_1 \) and \( z_2 \) are on the \( x \)-axis, this is just the Euclidean reflection in that axis, and the steps below reduce to that case.

(1) Rotate so as to make \( z_1 \) on the positive \( x \)-axis.
(2) Find the shift \( \sigma \) taking 0 to \( z_1 \), and then apply its inverse to \( z_1 \) (getting 0) and \( z_2 \).
(3) Rotate to get \( z_2 \) on the \( x \)-axis.
(4) Reflect in the \( x \)-axis.
(5) Unrotate, unshift, unrotate.

8. The Klein model

The group \( \text{SL}_2(\mathbb{R}) \) acts on the three-dimensional space of symmetric real \( 2 \times 2 \) matrices with trace 0 by conjugation:

\[ x \mapsto g x g^{-1}. \]

This action preserves the negative determinant, the quadratic form

\[-\det \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a^2 + bc.\]

and of course each ‘sphere’ is taken into itself. This signed determinant can also be written

\[ a^2 + \left( \frac{b + c}{2} \right)^2 - \left( \frac{b - c}{2} \right)^2 = x^2 + y^2 - z^2 \]

where

\[ x = a, \quad y = \frac{b + c}{2}, \quad z = \frac{b - c}{2}. \]

The point

\[ \kappa = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

is fixed by the subgroup \( \text{SO}_2 \) of rotations

\[ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

and its orbit \( \mathcal{X} \), which is one connected component of the sphere \( -\det = -1 \), may therefore be identified with \( \text{SL}_2(\mathbb{R})/\text{SO}_2 \). The metric induced on \( \mathcal{X} \) by the signed determinant is also \( \text{SL}_2(\mathbb{R}) \)-invariant, so in other words we have another model of the non-Euclidean plane. If \( H \) is the slice \( z = (b - c)/2 = 1 \) through the cone \( x^2 + y^2 < z^2 \), then there is a canonical projection of \( \mathcal{X} \) onto \( H \). This defines the **Klein model** of non-Euclidean geometry.
The advantage of this model over the others is that the action of $\text{SL}_2(\mathbb{R})$ is essentially linear. In this model, the geodesics are the intersections of $H$ with planes through the origin.

Reflections in this model are by reflections that are orthogonal with respect to the quadratic form $x^2 + y^2 - z^2$. For the moment, let the dot product be with respect to it. Thus reflection in the linear plane

$$\alpha \cdot v = (a, b, c) \cdot (x, y, z) = ax + by - cz = 0$$

is

$$v \mapsto v - 2 \left( \frac{v \cdot \alpha}{\alpha \cdot \alpha} \right) \alpha.$$

The correspondence between the Klein model and the Poincaré model is not too complicated. One erects a hemisphere over the slice $H$, projects points vertically on $H$ to those on the hemisphere, and then maps this hemisphere in turn onto the unit disk by stereographic projection.

**9. Coxeter tilings**

A **Coxeter matrix** is an integral matrix $(m_{i,j})$ with $m_{i,i} = 1$, $m_{i,j} = m_{j,i} \geq 2$. To every Coxeter matrix is associated a **Coxeter group** $W$ defined by generators $s_i$ in a set $S$ and relations $(s_is_j)^{m_{i,j}} = 1$.

Every Coxeter group possesses a **standard representation** on a real vector space $V$ of dimension $|S|$. If $e_i$ is a basis of the dual $V^\vee$, we first define an inner product:

$$e_i \cdot e_j = -\cos(\pi/m_{i,j})$$

so that in particular $e_i \cdot e_i = 1$. Then, in $V$ itself we define vectors $e_i^\vee$ by the formula

$$\langle e_i, e_j^\vee \rangle = 2 (e_i \cdot e_j).$$

These will be linearly dependent if the inner product has non-trivial radical. This happens, for example, if the matrix $e_i \cdot e_j$ is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

when $e_i^\vee = -e_j^\vee$. If the inner product is non-degenerate, we may use it to identify $V$ with $V^\vee$.

We then associate to $s_i$ the reflection

$$r_i: v \mapsto v - \langle e_i, v \rangle e_i^\vee.$$

Let $C$ be the region

$$\langle e_i, v \rangle > 0 \text{ for all } i$$

in $V$. Its walls are the hyperplanes of reflection for the generators in $S$. Let $\mathcal{C}$ the interior of the union of the transforms of $C$ by products of the reflections $r_i$, which is of course stable under $W$. 
This means that we can certainly calculate all elements up to any given length. But how do we assure listing the calculation of the minimal.
The cone $C$ is all of $V$ if and only if the group is finite and the inner product is positive definite. It will be a half-space if and only if the inner product is semi-definite, and in this case it is called an affine Coxeter group, because it may also be realized as a group of affine transformations in one dimension less. In all other cases, it will be properly contained in a half-space. It is this last case that we shall be eventually interested in.

The real point of looking at the geometry of the standard representation of a Coxeter group is that there is a strong relationship between this geometry and the combinatorics of the group. An expression for $w$ in terms of the generators $s_i$ is reduced if there is no shorter expression for it. The length $\ell(w)$ is the length of a reduced word for $w$.

We have $\ell(s_i w) > \ell(w)$ if and only if $wC$ lies on the same side of $(e_i, v)$ as $C$, or equivalently if and only if $(e_i, wC) > 0$.

This leads to a simple way to do computations in Coxeter groups. It will not in practice be infallible, but it will work well most of the time. Suppose $\rho$ to be the unique vector in $V$ such that $(e_i, \rho) = 1$ for all $i$. It lies in the interior of $C$, and since $C$ is a fundamental domain for $W$ an element $w$ in $W$ is determined uniquely by the vector $w\rho$. Thus $(e_i, wC) > 0$ if and only if $(e_i, w\rho) > 0$.

We shall use this observation in order to figure out how to make a list of all elements of $W$ up to some fixed length. We represent each $w$ by the array $(e_i, w\rho)$, so that the identity, for example, is the array $(1, 1, \ldots, 1)$. Reflection in these terms is simple:

$$(e_i, s_j w\rho) = (e_i, w\rho - (e_j, w\rho)e_j') = (e_i, w\rho) - (e_j, w\rho)(e_i, e_j').$$

This means that we can certainly calculate all elements up to any given length. But how do we assure listing each $w$ only once? We shall count a possibly new element $v = s_i w$ only if $i$ is least such that $(e_i, v) < 0$. In this way, every $w$ in $W$ is effectively assigned the unique expression in terms of the $s_i$ which is lexicographically minimal.

The problem with this technique is that on a computer real numbers possess only limited accuracy, so that the calculation of the $w\rho$ is not exact. There are ways to get around this, very elegant and sophisticated ones, but for our purpose we shall not need them.

There is just one case we are going to deal with, where $|S| = 3$. The nature of the group depends on the sum

$$\frac{1}{m_{1.2}} + \frac{1}{m_{1.3}} + \frac{1}{m_{2.3}}.$$

If it is $> 1$, the group $W$ is finite; if it is exactly equal to 1 the region $C$ is half of $\mathbb{R}^3$ and $W$ is the symmetry group of a planar tiling; and if it is $< 1$ then $W$ may be realized as a group of non-Euclidean tiling symmetries. These assertions can be verified by figuring out the signature of the matrix of the inner product

$$\begin{bmatrix}
1 & -\cos \pi/m_{1.2} & -\cos \pi/m_{1.3} \\
-\cos \pi/m_{1.2} & 1 & -\cos \pi/m_{2.3} \\
-\cos \pi/m_{1.3} & -\cos \pi/m_{2.3} & 1
\end{bmatrix}.$$

We are interested only in the third case. For the moment, let $u \cdot v$ be the inner product with respect to the metric $x^2 + y^2 - z^2$. Under the assumption that $\sum 1/m_{i,j} < 1$ we deduce from the assertion above that we can find vectors $e_i$ in $\mathbb{R}^3$ with

$$e_i \cdot e_j = -\cos \pi/m_{i,j}.$$
Normalizing suitably, we may assume that

\[ e_1 = (x_1, 0, 0) \]
\[ e_2 = (x_2, y_2, 0) \]
\[ e_3 = (x_3, y_3, z_3) \]

with all last coordinates positive. This gives us formulas

\[ e_1 \cdot e_1 = x_1^2 = 1 \]
\[ x_1 = 1 \]
\[ e_2 \cdot e_1 = x_2 x_1 = -\cos \pi/m_{1,2} \]
\[ x_2 = -\cos \pi/m_{1,2} \]
\[ e_2 \cdot e_2 = x_2^2 + y_2^2 \]
\[ y_2 = \sqrt{1 - x_2^2} \]
\[ e_3 \cdot e_1 = x_3 x_1 = -\cos \pi/m_{1,3} \]
\[ x_3 = -\cos \pi/m_{1,3} \]
\[ e_3 \cdot e_2 = x_3 x_2 + y_3 y_2 = -\cos \pi/m_{2,3} \]
\[ y_3 = \frac{-\cos \pi/m_{2,3} - x_2 x_3}{y_2} \]
\[ e_3 \cdot e_3 = x_3^2 + y_3^2 - z_3^2 = 1 \]
\[ z_3 = \sqrt{x_3^2 + y_3^2} - 1 \]

Now we are ready to describe the process of constructing a non-Euclidean tiling of the plane by triangles with angle \( \pi/m_{i,j} \) where \( \sum 1/m_{i,j} < 1 \). We proceed by recursion. We start with the normalized triangle \( T \) found by Bédard’s technique. The state at any moment (the collection of arguments to the recursive procedure) after that is characterized by (1) the triangle \( wT \) to draw, (2) the vector \( wp \), (3) the length left to go. When the recursive procedure is called, the triangle is drawn, and if \( n > 0 \) then for each \( s_i w > w \) the procedure is called for \( s_i w T \), \( s_i wp \), and \( n - 1 \).

The following picture took less than a second to draw.

Another, very interesting, approach to drawing Coxeter tilings can be found in [Gagern & Richter-Gebert:2009]. Ruler and compass constructions in non-Euclidean geometry can be found in [Goodman-Strauss:2001].
10. References


