Computing weight multiplicities

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Let $\pi = \pi^\lambda$, $V = V^\lambda$ be the representation of a reductive group G with highest weight λ. I shall discuss in this essay formulas of [Freudenthal:1954] (see also [Freudenthal-de Vries:1969]) and [Moody-Patera:1982] that provide a reasonably efficient way to compute the dimensions of all its weight subspaces V_μ.

The version of Moody and Patera replaces the set of positive roots in Freudenthal's formula by orbits among the roots of 'parabolic' subgroups of the Weyl group, and is more efficient only when the number of singular weights is relatively large. Sadly, I am not aware of any dramatic improvement of Freudenthal's formula for groups of low rank.

Contents

1. SL(2)

Freudenthal's formula depends on a clever observation about representations of SL_2.

Let

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Here $[x, y] = h$, $[h, x] = 2x$, and $[h, y] = -2y$.

Suppose V to be a space on which $\mathfrak{sl}_2$ acts, with highest weight λ and highest weight vector v_0. For each $n > 0$, let $v_n = y^n \cdot v_0$. Thus

$$\begin{aligned} x \cdot v_0 &= 0 \\ h \cdot v_n &= (\lambda - 2n)v_n \\ y \cdot v_n &= v_{n+1} . \end{aligned}$$

Since all weights have multiplicity one in V, we must have $x \cdot v_n = c_n v_{n-1}$ for all $n \geq 1$. *We may take $c_0 = 0$, but what are the other c_n?* These can be found by induction. For $n > 0$

$$x \cdot v_n = x \cdot y \cdot v_{n-1} = [x, y] \cdot v_{n-1} + y \cdot x \cdot v_{n-1} = [x, y] \cdot v_{n-1} + c_{n-1}v_{n-1} = h \cdot v_{n-1} + c_{n-1}v_{n-1} ,$$

so that

$$c_n = (\lambda - 2(n - 1)) + c_{n-1} .$$

This gives us

$$\begin{aligned} c_0 &= 0 \\ c_1 &= \lambda \\ c_2 &= (\lambda - 2) + c_1 \\ &= 2\lambda - 2 \\ c_3 &= (\lambda - 4) + c_2 \\ &= 3\lambda - 6 \\ &\ldots \end{aligned}$$

This leads to an explicit formula $c_n = n\lambda - n(n-1)$, but that's not what we want. We can also express what we have found as

$$\pi(y)\pi(x)v_n = c_n v_n \,.$$

with

$$\begin{aligned} c_n &= \sum\nolimits_{k=0}^{n-1} (\lambda - 2k) \\ &= \sum\nolimits_{k=1}^{n} (\mu + 2k) \quad (\mu = \lambda - 2n)) \,. \end{aligned}$$

Now $\mu = \lambda - 2n$ is the eigenvalue of $\pi(h)$ for the eigenvector v_n. So I can also put this in a slightly more general form: *if v is an eigenvector of $\pi(h)$ in the weight space V_μ of an irreducible representation of $\mathfrak{sl}_2$, then*

$$\text{trace } \pi(y)\pi(x) \,\big|\, V_\mu = \sum_{k \ge 1} \langle \mu + k\gamma, h \rangle \dim V_{\mu + k\gamma} \,.$$

Freudenthal's clever observation is that this makes sense, and remains valid, for an arbitrary representation (π, V) of $\mathfrak{sl}_2$, since it will be a direct sum of irreducible ones and the terms are additive.

There is one last transformation to be made. First of all, introduce the Killing form on $\mathfrak{sl}_2$, and replace y by the element $x^\bullet$ such that $x \bullet x^\bullet = 1$. It will be a scalar multiple of y, specified in Proposition 5.6. If we multiply both sides of this last equation by $2/\|h\|^2$ then according to Proposition 5.5 we therefore get:

1.1. Proposition. *If V_μ is the μ-weight subspace of any finite-dimensional representation π of $\mathfrak{sl}_2$, then*

$$\text{trace } \pi(x)\pi(x^\bullet) \,\big|\, V_\mu = \sum_{k \ge 1} \big((\mu + k\gamma) \bullet \gamma\big) \dim V_{\mu + k\gamma} \quad (x \in \mathfrak{g}_\gamma) \,.$$

2. Freudenthal's formula

Now suppose $\mathfrak{g}$ arbitrary, (π, V) the representation of *highest weight* λ. Restricted to the copy of $\mathfrak{sl}_2$ associated to the root γ the last formula becomes

$$\text{trace } \pi(x_\gamma)\pi(x_\gamma^\bullet) \,|\, V_\mu = \sum_{k \ge 1} ((\mu + k\gamma) \bullet \gamma) \dim V_{\mu + k\gamma} \,.$$

Recall from Proposition 5.7 that the Casimir operator is

$$\sum_\alpha h_\alpha h_\alpha^\bullet + \sum_{\gamma \in \Sigma} x_\gamma x_\gamma^\bullet \,.$$

Recall from Theorem 5.8 that it acts on V^λ as the scalar

$$\|\lambda + \rho\|^2 - \|\rho\|^2 \,.$$

We now have for each weight μ two different ways to evaluate the trace of the Casimir operator on V^λ_μ. On the one hand

$$\text{trace } \pi(\mathfrak{C}) \,|\, V_\mu = \big(\|\lambda+\rho\|^2 - \|\rho\|^2\big) \dim V_\mu \,.$$

On the other, we can evaluate terms explicitly. By Proposition 5.1 the first part is

$$\sum_\alpha h_\alpha h^\bullet_\alpha \,|\, V_\mu = \|\mu\|^2 \dim V_\mu \,.$$

The second part is

$$\sum_{\gamma\in\Sigma} \text{trace } \pi(x_\gamma x^\bullet_\gamma) \,|\, V_\mu = \sum_{\gamma\in\Sigma}\sum_{k\ge1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} \,.$$

Combining these, we deduce what I call the *raw* formula of Freudenthal:

2.1. Proposition. *For any weight μ of the representation on $V = V^\lambda$ with highest weight λ*

$$(\|\lambda+\rho\|^2 - \|\rho\|^2) \dim V_\mu = \|\mu\|^2 \dim V_\mu + \sum_{\gamma\in\Sigma} \sum\nolimits_{k\ge1} ((\mu+k\gamma)\bullet\gamma) \dim V_{\mu+k\gamma} \,.$$

More practical formulas rely on two simple facts. For every root γ let

$$S_\gamma = \sum\nolimits_{k\ge1} ((\mu+k\gamma)\bullet\gamma) \dim V_{\mu+k\gamma} \,.$$

Then

(a) if $w(\mu) = \mu$ then $S_{w(\gamma)} = S_\gamma$;
(b) $S_{-\gamma} = (\mu\bullet\gamma) + S_\gamma$.

For formula (a): if $w(\mu) = \mu$ then $\mu + w(\gamma) = w(\mu+\gamma)$ and

$$S_{w(\gamma)} = \sum\nolimits_{k\ge1} \big((w(\mu+k\,\gamma)\bullet w(\gamma)\big) \dim V_{w(\mu+k\,\gamma)} = S_\gamma \,.$$

Now for formula (b). Since the root reflection s_γ preserves the λ-string through μ, and the middle of the string is orthogonal to γ,

$$\sum\nolimits_{k\in\mathbb{Z}} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} = 0 \,.$$

Therefore

$$\begin{aligned} S_\gamma = \sum_{k\ge1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} &= -\sum_{k\le0} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} \\ &= \sum_{k\ge0} \big((\mu+k(-\gamma))\bullet(-\gamma)\big) \dim V_{\mu+k(-\gamma)} \\ &= -\mu\bullet\gamma + S_{-\gamma} \,. \end{aligned}$$

Property (b) leads immediately to the best known version of Freudenthal's formula:

2.2. Theorem. (Freudenthal) *For $V = V^\lambda$*

$$\big(\|\lambda+\rho\|^2 - \|\mu+\rho\|^2\big) \dim V_\mu = 2\sum_{\gamma>0}\sum_{k\ge1} \big((\mu+k\gamma)\bullet\gamma\big) \dim V_{\mu+k\gamma} \,.$$

This can be used to compute $\dim V_\mu$ recursively for dominant weights μ, starting with $\dim V_\lambda = 1$. In the process of doing this, it may very well happen that $\nu = \mu + k\gamma$ is not dominant. In that case, there exists α with $\langle \nu, \alpha^\vee\rangle < 0$. In these circumstances $s_\alpha\nu$ has greater height than ν. A sequence of such reflections will produce an element in the fundamental domain, leaving the weight multiplicity invariant and continually raising heights. That is to say, we can find w in W such that $\mu + k\gamma$ lies in the fundamental domain $\mathcal{D}$. The multiplicity $\dim V_{w(\mu+k\gamma)}$ is the same as $\dim V_{\mu+k\gamma}$. Induction therefore remains effective, if we have stored values of $\mu{\bullet}\gamma$ for all μ in $\mathcal{D}$ and positive roots γ, as well as values of $\gamma{\bullet}\gamma$.

This result has one immediate and useful consequence. Let Ω be the set of weights of π, Ω^{++} that of dominant weights in Ω. The consequence is:

2.3. Corollary. *If $\mu \neq \lambda$ lies in Ω, then there exists a root $\gamma > 0$ such that $\mu + \gamma$ also lies in Ω.*

Proof. What the formula implies immediately is that both μ and some $\mu + k\gamma$ with $k \geq 1$ both lie in Ω. But the weights in a γ-string are the weights in a representation of a copy of $\mathfrak{sl}_2$, and there are therefore no gaps in it. So $\mu + \gamma$ is also in Ω.

3. Constructing the dominant weights

Continue to let Ω be the set of weights of π, Ω^{++} that of dominant weights in Ω. The first requirement in the computation of weight multiplicities is to be able to scan through Ω^{++}. This is based on:

3.1. Lemma. *If $\mu \neq \lambda$ is in Ω^{++}, there exists a root $\gamma > 0$ such that $\mu + \gamma$ is also in Ω^{++}.*

The point is that one can construct all of Ω^{++} by starting with λ and descending from there, without ever leaving it.

Proof. The proof is constructive. Suppose $\mu \neq \lambda$ lies in Ω^{++}. By Corollary 2.3 there exists $\gamma > 0$ with $\mu + \gamma$ in Ω. If $\mu + \gamma$ is dominant, there is nothing more to prove.

Otherwise, suppose $\mu + \gamma$ not to be dominant. Then

$$\langle \mu + \gamma, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle + \langle \gamma, \alpha^\vee \rangle < 0$$

for some α in Δ. Since μ is dominant the first term is non-negative and hence $\langle \gamma, \alpha^\vee \rangle < 0$. Consider

$$s_\alpha(\mu + \gamma) = \mu + \gamma - \langle \mu + \gamma, \alpha^\vee \rangle \alpha \, .$$

The coefficient in the second term is positive. This reflection also lies in Ω. The sum $\mu + \gamma + \alpha$ also lies in Ω, because of convexity. Since $\langle \gamma, \alpha^\vee \rangle < 0$ and s_α takes all positive roots except α to positive roots, $s_\alpha(\gamma)$ is a positive root, and again by convexity $\gamma + \alpha$ is also a root. Thus we can replace γ by $\gamma + \alpha$, and repeat the argument. Sooner or later the repetition has to stop.

The norms $\|\mu\|^2$ and dot products $\mu{\bullet}\gamma$ can be computed by descending induction on height:

$$(\mu - \gamma){\bullet}\delta = (\mu{\bullet}\delta) - (\gamma{\bullet}\delta), \quad \|\mu - \gamma\|^2 = \|\mu\|^2 - 2(\mu{\bullet}\gamma) + (\gamma{\bullet}\gamma) \, ,$$

given pre-computed values of $\gamma{\bullet}\delta$. So we compute the values of $\lambda{\bullet}\gamma$ for all $\gamma > 0$ directly, and then compute other values as we construct the dominant closure of λ.

4. The formula of Moody-Patera

Moody and Patera modify Freudenthal's formula by taking into account a certain redundancy in it. In some situations, this speeds up things considerably.

It starts with the formula Proposition 2.1:

$$(\|\lambda + \rho\|^2 - \|\rho\|^2) \dim V_\mu = \|\mu\|^2 \dim V_\mu + \sum_{\gamma \in \Sigma} \sum_{k \geq 1} ((\mu + k\gamma){\bullet}\gamma) \dim V_{\mu + k\gamma} \, .$$

By property (a) above the sum

$$S_\gamma = \sum_{k \geq 1} ((\mu + k\gamma){\bullet}\gamma) \dim V_{\mu + k\gamma}$$

depends only on the W_T-orbit of γ. We can therefore convert the sum into one over orbits instead of over roots. If we pick one element $\gamma_{\mathcal{O}}$ for every orbit $\mathcal{O}$, the sum on the right hand side becomes

$$\sum\nolimits_{\mathcal{O}} |\mathcal{O}| \sum_{k\ge 1} ((\mu + k\gamma_{\mathcal{O}}) \bullet \gamma_{\mathcal{O}}) \dim V_{\mu+k\gamma_{\mathcal{O}}} \,.$$

If the number of μ fixed by elements of the weyl group is fairly large, this can lead to a noticeably more efficient calculation.

For the final statement, we need to consider how W_T acts on roots. First of all, the region

$$\{v \mid \langle v, \alpha^\vee\rangle \ge 0 \text{ for } \alpha \in T\}$$

is a fundamental domain for W_T, so in every orbit $\mathcal{O}$ there is a unique root $\gamma_{\mathcal{O}}$ in this region. Also, the group W_T takes Σ_T into itself, and each of the two sets $\Sigma^\pm - \Sigma_T^\pm$ into themselves. Let

$$\|\mathcal{O}\| = \begin{cases} \;|\mathcal{O}| & \text{if } \mathcal{O} \subset \Sigma_T \\ 2|\mathcal{O}| & \text{if } \mathcal{O} \subset \Sigma^+ - \Sigma_T^+. \end{cases}$$

4.1. Theorem. Moody-Patera) *Suppose V to be the irreducible representation with highest weight λ, μ a weight of V, and T to be chosen so that W_T is the subgroup of W fixing μ. Then*

$$\big(\|\lambda+\rho\|^2 - \|\mu+\rho\|^2\big) \dim V_\mu = \sum_{\mathcal{O}} \|\mathcal{O}\| \sum_{k\ge 1} \big((\mu + k\gamma_{\mathcal{O}}) \bullet \gamma_{\mathcal{O}}\big) \dim V_{\mu+k\gamma_{\mathcal{O}}} \,.$$

In this, the sum is over the orbits $\mathcal{O}$ of W_T in $\Sigma_T \cup \big(\Sigma^+ - \Sigma_T^+\big)$.

Proof. There are three kinds of orbits: (i) those in Σ_T; (ii) those in $\Sigma^+ - \Sigma_T^+$; (iii) those in $\Sigma^- - \Sigma_T^-$.

If $\mathcal{O}$ is of type (ii), then the corresponding orbit $-\mathcal{O}$ is of type (iii), and according to property (b) for γ in $\mathcal{O}$ we have $S_{-\gamma} = \mu \bullet \gamma + S_\gamma$. If γ lies in in Σ_T ((i.e. is of type (i)) then $\mu \bullet \gamma = 0$. So the sum over orbits becomes the one in Theorem 4.1.

It is worthwhile noting that if γ is in Σ_T and in the positive chamber for W_T then it has to be positive, since it $\langle \gamma, \alpha^\vee\rangle$ must be positive for at least one α. Thus $\gamma_{\mathcal{O}}$ is positive in all cases. It is perhaps also worth noticing that in case (i) the orbits are parametrized by the lengths of roots.

If many multiplicities are to be calculated it is best to compute data for all possible $\mathcal{O}$ in advance. The same technique remarked on in connection with the original Freudenthal formula can be used for dealing with the case that $\nu = \mu + k\gamma_{\mathcal{O}}$ is not dominant.

[Bremner:1986] contains practical advice (regarding an implementation in the now forgotten but once well loved programming language Pascal). [Moody-Patera:1982] discusses some examples for $G = E_8$, where there are indeed a relatively large number of singular weights.

5. *Appendix.* The Casimir element

In this section I'll recall a number of properties of the Casimir element of $Z(\mathfrak{g})$, including in particular how it acts on irreducible finite-dimensional representations of $\mathfrak{g}$.

DUALITY AND INNER PRODUCTS. Suppose for a moment that V is any real vector space on which is assigned a non-degenerate inner product $x \bullet y$. This gives rise to an associated isomorphism φ of V with its linear dual $\widehat{V}$, defined by the formula

$$\langle u, \varphi(v)\rangle = u \bullet v \,.$$

Since the inner product is symmetric, $\varphi^\vee = \varphi$. There is an associated inner product on $\widehat{V}$:

$$\widehat{u} \bullet \widehat{v} = \varphi^{-1}(u) \bullet \varphi^{-1}(v) \,.$$

5.1. Proposition. *Suppose (v_i) to be a basis of V, $(v_i^\bullet)$ its dual with respect to the inner product. Then:*

$$\widehat{u}\bullet\widehat{v} = \sum \langle v_i, \widehat{u}\rangle\langle v_i^\bullet, \widehat{v}\rangle \,.$$

Proof. It suffices to prove this for $\widehat{u} = \widehat{v} = \varphi(v)$. We may write $v = \sum (v\bullet v_i^\bullet)\, v_i$ and then deduce

$$\begin{aligned} \|\widehat{v}\|^2 &= \|v\|^2 \\ &= v\bullet\Big(\sum (v\bullet v_i^\bullet)\, v_i\Big) \\ &= \sum (v\bullet v_i)(v\bullet v_i^\bullet \\ &= \sum \langle v_i, \widehat{v}\rangle\langle v_i^\bullet, \widehat{v}\rangle \,. \end{aligned}$$

5.2. Corollary. *The element $\sum v_i \cdot v_i^\bullet$ in $S^2(V)$ is independent of the choice of basis.*

THE KILLING FORM. The **Killing form** on $\mathfrak{g}$ is the bilinear pairing

$$x\bullet y = \mathrm{trace}\,(\mathrm{ad}_x \mathrm{ad}_y) \,.$$

It is manifestly invariant with respect to any automorphism of $\mathfrak{g}$.

Its radical certainly contains the center $\mathfrak{z}$ of $\mathfrak{g}$, so it is determined completely by its restriction to the semi-simple component $\mathfrak{g}_{\mathrm{ss}}$. For convenience, assume temporarily that $\mathfrak{g}$ itself is semi-simple. Since the trace of a nilpotent transformation vanishes, the direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus \sum_{\mu>0} (\mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}) \,.$$

is orthogonal with respect to the Killing form. For h in $\mathfrak{h}$

$$h\bullet h = \sum_\mu \langle \mu, h\rangle^2 \,,$$

so that the Killing form is positive definite on $\mathfrak{h}$.

Choose for each root μ some $x_\mu \neq 0$ in $\mathfrak{g}_\mu$ and then choose y_μ in $\mathfrak{g}_{-\mu}$ such that $\langle \mu, h_\mu\rangle = 2$, where $h_\mu = [x_\mu, y_\mu]$. Since the Killing form is invariant,

$$[x_\mu, h_\mu]\bullet y_\mu + h_\mu\bullet[x_\mu, y_\mu] = -2\, x_\mu\bullet y_\mu + h_\mu\bullet h_\mu = 0 \,.$$

Of course ad_{x_μ} and ad_{y_μ} are nilpotent. Hence:

5.3. Proposition. *For any root μ*

$$\begin{aligned} x_\mu\bullet x_\mu &= 0 \\ y_\mu\bullet y_\mu &= 0 \\ x_\mu\bullet y_\mu &= \frac{h_\mu\bullet h_\mu}{2} \,. \end{aligned}$$

Consequently, the two-dimensional space spanned by x_μ and y_μ is a hyperbolic plane, and we deduce:

5.4. Proposition. *The Killing form is non-degenerate on $\mathfrak{g}_{\mathrm{ss}}$.*

In other words, its radical is the center $\mathfrak{z}$ of $\mathfrak{g}$.

Continue to assume $\mathfrak{g}$ semi-simple. Choose a basis (h_i) of $\mathfrak{h}$, and a basis (x_μ) of the root spaces. Let $x_\mu^\bullet$ be the dual basis with respect to the Killing form. From the previous Proposition:

5.5. Proposition. *We have*

$$x^\bullet_\mu = \frac{2\,y_\mu}{h_\mu \bullet h_\mu}\,.$$

Because the Killing form is invariant under automorphisms, the restriction of the form to $\mathfrak{h}$ is invariant under the Weyl group. In particular, reflections are orthogonal, which means that

$$2\left(\frac{h_\mu \bullet v}{h_\mu \bullet h_\mu}\right) = \langle \mu, v\rangle\,.$$

This implies that

$$\varphi\left(\frac{2h_\mu}{h_\mu \bullet h_\mu}\right) = \mu$$

which implies in turn:

5.6. Proposition. *We have*

$$\lambda \bullet \mu = \frac{2\,\langle \lambda, h_\mu\rangle}{h_\mu \bullet h_\mu}\,.$$

THE CASIMIR ELEMENT. Continue to assume that $\mathfrak{g}$ is semi-simple.

There is a canonical map from $\mathfrak{g} \otimes \mathfrak{g}^\vee$ to $\mathrm{End}(\mathfrak{g})$, taking $x \otimes \widehat{x}$ to the linear transfromation

$$y \longmapsto \langle y, \widehat{x}\rangle x\,.$$

It is a linear isomorphism. As I have recalled at the beginning of this section, the Killing form induces a linear isomorphism of $\mathfrak{g} \otimes \mathfrak{g}^\vee$ with $\mathfrak{g} \otimes \mathfrak{g}$. It is $\mathfrak{g}$-equivariant, since the Killing form is $\mathfrak{g}$-invariant. There is a linear map from $\mathfrak{g} \otimes \mathfrak{g}$ to the enveloping algebra $U(\mathfrak{g})$, taking $x \otimes y$ to xy. All in all these maps fit into a sequence of $\mathfrak{g}$-equivariant isomorphisms

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^\vee \xrightarrow{\text{Killing}} \mathfrak{g} \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g})$$

The identity transformation of $\mathfrak{g}$ thus gives rise to a unique element $\mathfrak{C}$ of $U(\mathfrak{g})$, the **Casimir element**. Since I commutes with elements of $\mathfrak{g}$, the Casimir does too, and since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ it lies in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

Hence:

5.7. Proposition. *If (x_i) is any a basis of $\mathfrak{g}$, $(x^\bullet)$ its dual with respect to the Killing form, then*

$$\mathfrak{C} = \sum x_i x_i^\bullet\,.$$

Proof. The linear transformation of $\mathfrak{g}$

$$y \longmapsto \sum_i \langle x_i^\bullet, y\rangle\, x_i$$

is the identity transformation, since it takes each x_i to itself. ▯

If π is an irreducible representation of $\mathfrak{g}$, the Casimir element will act on it by a scalar, which is easily described:

5.8. Theorem. *On the irreducible $\mathfrak{g}$-module V^λ with lowest weight λ, the Casimir $\mathfrak{C}$ acts as multiplication by the scalar $\|\lambda - \rho\|^2 - \|\rho\|^2$.*

For a highest weight this becomes

$$\|\lambda + \rho\|^2 - \|\rho\|^2\,.$$

Proof. Let $\mathfrak{n}$ be the nilpotent Lie algebra spanned by the x_μ with $\mu > 0$. If λ is the lowest weight, then the corresponding eigenspace may be identified with $V/\mathfrak{n}V$. A well known argument tells us that for every x in $Z(\mathfrak{g})$ there exists a unique element $\mathrm{HC}(x)$ in $U(\mathfrak{h})$ such that $x - \mathrm{HC}(x)$ lies THe element x therefore acts on $V/\mathfrak{n}V$ as $\mathrm{HC}(x)$ does. The Lie algebra $\mathfrak{h}$ acts on this quotient by the character λ. The character λ extends to a homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$: $\langle\lambda, \prod x_i\rangle = \prod\langle\lambda, x_i\rangle$. Therefore x acst by the scalar $\langle\lambda, \mathrm{HC}(x)\rangle$.

Now

$$\begin{aligned}\mathfrak{C} &= \sum h_i h_i^\bullet + \sum\nolimits_{\mu>0} x_\mu x_\mu^\bullet + \sum\nolimits_{\mu<0} x_\mu x_\mu^\bullet \\ &= \sum h_i h_i^\bullet + \sum_{\mu<0} [x_\mu, x_\mu^\bullet] + \sum\nolimits_{\mu>0} (x_\mu x_\mu^\bullet + x_{-\mu}^\bullet x_{-\mu}) \,.\end{aligned}$$

So

$$\mathrm{HC}(\mathfrak{C}) = \sum h_i h_i^\bullet + \sum_{\mu<0} [x_\mu, x_\mu^\bullet] \,.$$

and by Proposition 5.5

$$\langle\lambda, \mathfrak{C}\rangle = \sum \langle\lambda, h_i\rangle\langle\lambda, h_i^\bullet\rangle - \sum\nolimits_{\mu>0} \frac{2\,\langle\lambda, h_\mu\rangle}{h_\mu \bullet h_\mu}$$

Apply Proposition 5.1 and Proposition 5.6 to see:

$$\langle\lambda, \mathrm{HC}(\mathfrak{C})\rangle = \|\lambda\|^2 - 2\lambda \bullet \rho = \|\lambda + \rho\|^2 - \|\rho\|^2 \,.$$

6. References

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