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Essays on Coxeter groups

Kazhdan-Lusztig polynomials

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Contents

1. Review of the Hecke algebra

Let (W,S) be a Coxeter system, $\mathcal{H} = \mathcal{H}(W,S)$ the Hecke algebra over the Laurent polynomial ring $\mathbb{Z}[v^{\pm 1}]$ with generators T_w satisiying relations

$$T_w T_s = \begin{cases} T_{ws} & \text{if } ws > w \\ (q-1)T_w + qT_{ws} & \text{otherwise} \\ T_{sw} & \text{if } sw > w \\ (q-1)T_w + qT_{sw} & \text{otherwise.} \end{cases}$$

Here $q = v^{-2}$. Let $\tau_w = v^{\ell(x)} T_x$ be normalized elements of $\mathcal{H}$.

Let ι be the involutory automorphism

$$\tau_w \longmapsto \tau_{w^{-1}}$$

of $\mathcal{H}$. Thus

$$\tau_s^\iota = \tau_s + (v - v^{-1})$$

2. Kazhdan-Lusztig polynomials

My exposition will follow [Kazhdan-Lusztig:1979] closely with suggestions taken from [Soergel:1997], and [Shi:1986].

[KL-basis] **Theorem. 2.1.** *For every x in W there exists a unique element γ_x in $\mathcal{H}$ which is invariant under the involution ι and of the form*

$$\gamma_x = \sum_{y \le x} v\pi_{y,x}(v)\tau_y = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\tau_y \,,$$

where (a) $\pi_{y,x}(v)$ is a polynomial in $\mathbb{Z}[v]$ of degree at most $\ell(x) - \ell(y) - 1$ for $y < x$ and (b) $\pi_{x,x} = 1/v$.

Extend this by specifying $\pi_{y,x} = 0$ for $y \not\le x$.

What I call γ_x is the same as C'_x in [Kazhdan-Lusztig:1979]. If we set

$$\Gamma_x = v^{-\ell(x)}\gamma_x, \quad P_{y,x} = v^{-(\ell(x)-\ell(y)-1)}\, v\pi_{y,x}(v) \,,$$

the equation for γ_x means

$$\Gamma_x = T_x + \sum_{y<x} P_{y,x}\, T_y \, .$$

The Theorem will be supplemented by:

[kl-polys] **Proposition 2.2.** *The polynomial $P_{y,x}$ is a polynomial in $\mathbb{Z}[q]$ of degree at most $\big(\ell(x) - \ell(y) - 1\big)/2$. Its constant term is 1.*

The motivation for introducing these polynomials is that they are the Poincaré polynomials of local intersection cohomology on the subvarieties $B\backslash ByB \subseteq B\backslash\overline{BxB}$. The invariance of γ_x in this case is related to the self-duality of the perverse sheaves giving rise to intersection cohomology.

For the proof of the Theorem I'll follow the treatment in [Soergel:1997], which is somewhat more concise than others in the literature, although not substantially different from the original one of Kazhdan & Lusztig.

Before beginning the proof, let's look at some examples.

(a) It is trivial that $\gamma_1 = \tau_1$ (which I'll usually write just as 1 from now on).

(b) We must have

$$\begin{aligned}\gamma_s &= \tau_s + v\pi_{1,s}(v)\\ \gamma_s^\iota &= \tau_s + (v - v^{-1}) + v^{-1}\pi_{1,s}(v^{-1}) = \tau_s + v + \big(v^{-1}\pi_{1,s}(v^{-1}) - 1\big)\end{aligned}$$

which implies that $\pi_{1,s} - 1 = 0$, so

$$\gamma_s = \tau_s + v\, .$$

(c) Suppose $s \ne t$. What is γ_{st}? As we'll see in a moment, the key to the construction of the γ_w is the observation that if γ_x is invariant under ι then so is $\gamma_x\gamma_s$. In this case

$$\begin{aligned}\gamma_s\gamma_t &= (\tau_s + v)(\tau_t + v)\\ &= \tau_s\tau_t + v\tau_s + v\tau_t + v^2\\ &= \tau_{st} + v\tau_s + v\tau_t + v^2\, ,\end{aligned}$$

which says that we can choose $\gamma_{st} = \gamma_s\gamma_t$, with $\pi_{s,st} = \pi_{t,st} = 1$, $\pi_{1,st} = v$. Uniqueness in this case is simple—if γ is another element of $\mathcal{H}$ satisfying the conditions, then $\gamma_{st} - \gamma$ will be ι-invariant and of the form

$$P = vp_s(v)\tau_s + vp_t(v)\tau_t + vp_1(v)\, .$$

But then

$$\begin{aligned}P^\iota &= v^{-1}p_s(v^{-1})\tau_s^\iota + v^{-1}p_t(v^{-1})\tau_t^\iota + v^{-1}p_1(v^{-1})\\ &= v^{-1}p_s(v^{-1})\big(\tau_s + (v - v^{-1})\big) + v^{-1}p_t(v^{-1})\big(\tau_t + (v - v^{-1})\big) + v^{-1}p_1(v^{-1})\\ &= v^{-1}p_s(v^{-1})\tau_s + v^{-1}p_t(v^{-1})\tau_t\\ &\qquad + \big(v^{-1}p_s(v^{-1})(v - v^{-1}) + v^{-1}p_t(v^{-1})(v - v^{-1}) + v^{-1}p_1(v^{-1})\big)\end{aligned}$$

which implies that both $vp_s(v)$, $vp_t(v)$ are invariant under the involution $v \mapsto v^{-1}$, hence must vanish. This in turn implies that $vp_1(v)$ vanishes.

Proof of the Proposition.

In the proof to follow we shall find useful these formulas for multiplication on the right by γ_s:

$$\tau_x\gamma_s = \begin{cases}\tau_{xs} + v\tau_x & \text{if } xs > x\\ v^{-1}\tau_x + \tau_{xs} & \text{if } xs < x.\end{cases}$$

For w in W, its **Bruhat closure** $\overline{w}$ is the set of all $x \le w$ in the Bruhat order. We shall repeatedly use this well known observation:

[s-stability] **Lemma 2.3.** (s-stability of the closure) *Suppose x in W, s in S, $x > xs$. Then $\overline{x}$ is stable under right multiplication by s. If $ys < y$ then $y < x$ if and only if $ys < xs$.*

Equivalent formulations, under the hypothesis that $xs < x$: *If $w > ws$ and $x \ne ws$ then $x < w$ if and only if $xs < w$.* Or: *The map $w \mapsto ws$ is a bijection of*

$$\{w \le x \,|\, ws > w\} \longrightarrow \{w \le x \,|\, ws < w\}\,.$$

Another variation: *If $sx < x$, $sy < y$, $y < x$, then $sy < sx$.* XXX

We are also going to need this:

[h-gamma-inversion] **Lemma 2.4.** *Suppose that for every $y \le x$*

$$\gamma_y = \tau_y + \sum_{z<y} v p_{z,y}(v)\tau_z\,.$$

with $p_{z,y} \in \mathbb{Z}[v]$. Then we also have

$$\tau_y = \gamma_y + \sum_{z<y} v q_{z,y}(v)\gamma_z$$

for suitable $q_{z,y}(v)$ in $\mathbb{Z}[v]$.

Proof. By an easy induction. This amounts to solving a system of equations with a unipotent triangular matrix. ▯

Step 1. *Existence.* We are going to prove by induction on $\ell(x)$ that *there exists a self-dual γ_x such that*

$$\gamma_x = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\tau_y\,.$$

with $\pi_{y,x}(v) \in \mathbb{Z}[v]$.

We have already defined $\gamma_1 = 1$, $\gamma_s = \tau_s + v$. Now assume $\ell(x) > 1$ and assume the Proposition true for $z < x$. Choose s with $z = xs < x$. We know the claim to be true for z so that

$$\gamma_z = \tau_z + \sum_{w<z} v\pi_{w,z}(v)\tau_w\,.$$

Multiplying on the right by γ_s:

$$\begin{aligned}\gamma_z\gamma_s &= \tau_z\gamma_s + \sum_{w<z} v\pi_{w,z}(v)\tau_w\gamma_s \\ &= \tau_x + v\tau_z + \sum_{\substack{w<z\\ ws>w}} v\pi_{w,z}(v)(\tau_{ws} + v\tau_w) + \sum_{\substack{w<z\\ ws<w}} v\pi_{w,z}(v)(v^{-1}\tau_w + \tau_{ws})\,.\end{aligned}$$

This is of the form

$$\tau_x + \sum_{y<x} p_{y,x}\tau_y\,.$$

Nearly all the terms are divisible by v, as required. The exceptions are the terms with v^{-1}, in which $ys < y$. Its easy to fix this problem by defining

$$\gamma_x = \gamma_z \gamma_s - \sum_{\substack{y<z \\ ys<y}} \pi_{y,z}(0)\,\gamma_y = \tau_x + \sum_{y<xa]topys<y} \big(p_{y,x}(v) - p_{y,x}(0)\big)\tau_y - p_{y,x}(0)\big(\gamma_y - \tau_y\big)\,.$$ ∎

Step 2. *Uniqueness.* We now have for each x in W an element γ_x in $\mathcal{H}$ which is fixed by ι and of the form

$$\gamma_x = \tau_x + \sum_{y<x} v p_{y,x}(v)\tau_y\,.$$

It remains to see that it is unique with these properties. This will follow from the claim: *If $\sum_y v\pi_y(v)\tau_v$ is fixed by the involution ι, it is* 0.

♣ **[h-gamma-inversion]** *Proof* of the claim. By Lemma 2.4 it is equivalent to the claim that if

$$\sum_x v p_x(v)\gamma_x \quad \big(p_x(v) \in \mathbb{Z}[v]\big)$$

is invariant under ι then it vanishes. This is immediate. ∎

To summarize: first of all, for every z we have

$$\gamma_z = \tau_z + \sum_{w<z} v\pi_{w,z}(v)\tau_w\,.$$

Second, we have a recursive formula for γ_x, which I'll recall in a moment. From now on let

$$\mu(w,z) = \begin{cases} \pi_{w,z}(0) & \text{if } w < z \\ \mu(z,w) & \text{if } z < w \\ 0 & \text{otherwise.} \end{cases}$$

Define $w \prec z$ to mean $w < z$ and $\mu(w,z) \neq 0$.

From the definition of γ_x in the proof, the following is immediate:

$$\begin{aligned}\gamma_x = \tau_x + v\tau_z + \sum_{\substack{y<z \\ ys>y}} v\pi_{y,z}(v)\,\tau_{ys} + \sum_{\substack{y<z \\ ys>y}} v^2\pi_{y,z}(v)\,\tau_y \\ + \sum_{\substack{y<z \\ ys<y}} \big(\pi_{y,z}(v) - \mu(y,z)\big)\,\tau_y + \sum_{\substack{y<z \\ ys<y}} v\pi_{y,z}(v)\,\tau_{ys} \\ - \sum_{\substack{w \\ y<w\prec z \\ ws<w}} \mu(w,z)\,v\pi_{y,w}(v)\tau_w\,.\end{aligned}$$

The map $y \mapsto ys$ is a bijection of $\{y < z\}$ with $\{y < x \,|\, ys < y\}$. Since $\pi_{y,z} = 0$ if $y \not\leq z$ this expression becomes

$$\begin{aligned}\gamma_x = \tau_x + v\tau_z + \sum_{\substack{y<x \\ ys<y}} v\pi_{ys,z}(v)\,\tau_y + \sum_{\substack{y<z \\ ys>y}} v^2\pi_{y,z}(v)\,\tau_y \\ + \sum_{\substack{y<z \\ ys<y}} \big(\pi_{y,z}(v) - \mu(y,z)\big)\,\tau_y + \sum_{\substack{y<x \\ ys>y}} v\pi_{ys,z}(v)\,\tau_y \\ - \sum_{\substack{w \\ y<w\prec z \\ ws<w}} \mu(w,z)\,v\pi_{y,w}(v)\tau_w\,,\end{aligned}$$

leading to this:

[gamma-recursion] **Proposition 2.5.** *Suppose $xs < x$. Then*

$$\begin{aligned} \pi_{x,x} &= 1/v \\ \pi_{xs,x} &= 1 \\ \pi_{y,x} &= \frac{\pi_{y,xs}(v) - \mu(y,xs)}{v} + \pi_{ys,xs}(v) - \sum_{\substack{w \\ y<w\prec xs \\ ws<w}} \mu(w,xs)\pi_{y,w}(v) \quad (ys<y) \\ \pi_{y,x} &= \pi_{ys,xs}(v) + v\pi_{y,xs}(v) - \sum_{\substack{w \\ y<w\prec xs \\ ws<w}} \mu(w,xs)\pi_{y,w}(v) \quad (ys>y)\,. \end{aligned}$$

Two of the most important facts about the $\pi_{y,x}$:

[pixy] **Corollary 2.6.** *Suppose $xs < x$, $y < x$. Then*

(a) if $ys > y$, $\pi_{y,x} = v\pi_{ys,x}$;
(b) if $ys < y$ but $y \not\leq xs$, $\pi_{y,x} = \pi_{ys,xs}$.

Recall that $T_x = v^{-\ell(x)}\tau_x$ and $\Gamma_x = v^{-\ell(x)}\gamma_x$. In particular

$$\Gamma_s = T_s + 1\,.$$

If we multiply the equation

$$\gamma_x = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\,\tau_y$$

by $v^{-\ell(x)} = q^{\ell(x)/2}$ we get

$$\begin{aligned} \Gamma_x &= T_x + \sum_{y<x} q^{-1/2} q^{(\ell(x)-\ell(y))/2}\pi_{y,x}(v)\,T_y \\ &= T_x + \sum_{y<x} P_{y,x}\,T_y \qquad \big(P_{y,x} = q^{(\ell(x)-\ell(y)-1)/2}\pi_{y,x}(q^{-1/2})\big)\,. \end{aligned}$$

We know that $\pi_{y,x}(v)$ is a polynomial in v, hence $P_{y,x}$ is $v^{-(\ell(x)-\ell(y)-1)/2}$ times a polynomial in v. In particular, $\mu(y,x) = \pi_{y,x}(0)$ is the coefficient of $q^{(\ell(x)-\ell(y)-1)/2}$ in $P_{y,x}$. The recursion formula for γ_x leads to:

[p-formula] **Corollary 2.7.** *Suppose $x = zs > z$. Then for $y \leq x$*

$$\begin{aligned} P_{y,x} &= P_{ys,z} + qP_{y,z} - \sum_{\substack{w \\ y\leq w\prec z \\ ws<w}} \mu(w,z) q^{(\ell(x)-\ell(w))/2}\,P_{y,w} \quad (ys<y) \\ &= qP_{ys,z} + P_{y,z} - \sum_{\substack{w \\ y\leq w\prec z \\ ws<w}} \mu(w,z) q^{(\ell(x)-\ell(w))/2}\,P_{y,w} \quad (ys>y)\,. \end{aligned}$$

Here we set $P_{y,x} = 0$ if $y \not\leq x$. The formula can also be written more succinctly as Kazhdan & Lusztig do:

$$P_{y,x} = q^{1-c}P_{ys,z} + q^{c}P_{y,z} - \sum_{\substack{w \\ y\leq w\prec z \\ ws<w}} \mu(w,z) q^{(\ell(x)-\ell(y))/2}\,P_{y,w} \quad c = \begin{cases} 1 & ys<y \\ 0 & ys>y \end{cases}$$

Induction will give us:

[pyx] **Corollary 2.8.** *The expression $P_{y,x}$ is a polynomial in q. If $y < x$ it has degree at most $q^{(\ell(x)-\ell(y)-1)/2}$, and $P_{x,x} = 1$. The constant term of $P_{y,x}$ is always 1.*

3. Elementary properties

[mts] **Proposition 3.1.** *If* $zs > z$

$$\gamma_z \tau_s = -v\gamma_z + \gamma_x + \sum_{\substack{w \\ w \prec z \\ ws < w}} \mu(w,z)\gamma_w ,$$

while if $zs < z$ *we have*

$$\gamma_z \tau_s = v^{-1}\gamma_z .$$

We shall see later what role these play in the representation theory of $\mathcal{H}$.

Proof. We start with

$$\gamma_x = \gamma_z\gamma_s - \sum_{\substack{w \\ w \prec z \\ ws < w}} \mu(w,z)\gamma_w$$

which we rewrite as

$$\gamma_z \tau_s = -v\gamma_z + \gamma_x + \sum_{\substack{w \\ w \prec z \\ ws < w}} \mu(w,z)\gamma_w .$$

This is the first formula.

We do the second one by induction. Since $\gamma_s\tau_s = v^{-1}\gamma_s$ and because of the induction hypothesis, we multiply the same equation by τ_s to get

$$\begin{aligned} \gamma_x\tau_s &= \gamma_z\gamma_s\tau_s - \sum_{\substack{w \\ w \prec z \\ ws < w}} \mu(w,z)\gamma_w\tau_s \\ &= v^{-1}\gamma_z\gamma_s + \sum_{\substack{w \\ w \prec z \\ ws < w}} \mu(w,z)v^{-1}\gamma_w \\ &= v^{-1}\gamma_x . \end{aligned}$$

♣ [pixy] This allows us a second proof of Corollary 2.6(b). If $xs < x$ then from the second formula above we obtain

$$\begin{aligned} \gamma_x\tau_s = v^{-1}\gamma_x &= v^{-1}\sum_{y \le x} v\pi_{y,x}(v)\tau_y \\ &= \sum_{\substack{y<x \\ ys>y}} \pi_{y,x}(v)\tau_y + \sum_{\substack{y \le x \\ ys<y}} \pi_{y,x}(v)\tau_y . \end{aligned}$$

and on the other

$$\begin{aligned} \Big(\sum_{y\le x} v\pi_{y,x}(v)\tau_y\Big)\tau_s &= \sum_{\substack{y\le x \\ ys<y}} v\pi_{y,x}(v)\tau_y\tau_s + \sum_{\substack{y<x \\ ys>y}} v\pi_{y,x}(v)\tau_y\tau_s \\ &= \sum_{\substack{y\le x \\ ys<y}} v\pi_{y,x}(v)\big((v^{-1}-v)\tau_y + \tau_{ys}\big) + \sum_{\substack{y\le x \\ ys>y}} v\pi_{y,x}(v)\tau_{ys} . \end{aligned}$$

Cancelling, we have

$$\begin{aligned}\sum_{\substack{y<x\\ ys>y}} \pi_{y,x}(v)\tau_y \sum_{\substack{y\le x\\ ys<y}} v\pi_{y,x}(v)(\tau_{ys} - v\tau_y) + \sum_{\substack{y<x\\ ys>y}} v\pi_{y,x}(v)\tau_{ys} \\ = \sum_{\substack{y\le x\\ ys<y}} v\pi_{y,x}(v)\tau_{ys} \\ = \sum_{\substack{y\le x\\ ys>y}} v\pi_{ys,x}(v)\tau_y \,. \ \square\end{aligned}$$

All these results have analogues for the left action of s as well.

We have already seen one implicit result about these:

[extremal] Proposition 3.2. *If $y \prec x$ then $R_y \subseteq R_x$ and $L_x \subseteq L_y$.*

♣ **[pixy]** *Proof.* This is Corollary 2.6(a). $\square$

For w in W, let

$$\begin{aligned} L_w &= \{s \in S \,|\, sw < w\} \\ R_w &= \{s \in S \,|\, ws < w\} \\ \mathcal{L}_w &= \{r \text{ a reflection} \,|\, rw < w\} \\ \mathcal{R}_w &= \{r \text{ a reflection} \,|\, wr < w\} \end{aligned}$$

If r is a reflection then there exists exactly one geometric root $\lambda > 0$ such that $r = s_\lambda$, and $rw < w$ if and only if $w^{-1}\lambda < 0$. Each of the sets defined above could have similarly been specified in terms of positive roots as well. The following is well known:

[descents] Lemma 3.3. *Suppose $w = xy$. The following are equivalent:*

(a) $\ell(w) = \ell(x) + \ell(y)$;
(b) $\mathcal{L}_{xy}$ is the disjoint union of $\mathcal{L}_x$ and $x\mathcal{L}_y x^{-1}$;
(c) $\mathcal{R}_{xy}$ is the disjoint union of $y^{-1}\mathcal{R}_x y$ and $\mathcal{R}_y$.

Define $x \to_R y$ if $\mu(x,y) \ne 0$ and $R_y \not\subseteq R_x$. These oriented links define the **right W-graph** of W. To each link is associated also the label $R_y - R_x$, so *the right W-graph is an oriented, labelled graph*. Similarly define $x \to_L y$ and the left W-graph. For example, if $x < y = xs$ then $s \in R_y - R_x$ and $\mu(x,y) = 1$, so $x \to_R xs$. These turn out to be the only edges in the W-graph going up:

[going-up] Proposition 3.4. *Suppose $x \to_R y$ with $x < y$. Then $y = xs$ for some s in S.*

Proof. Choose s in $R_y - R_x$, so $ys < y$ but $xs > x$. Then $\pi_{x,y} = v\pi_{xs,y}$. If $y \ne xs$ then both π are polynomials, so $\mu(x,y) = 0$. $\square$

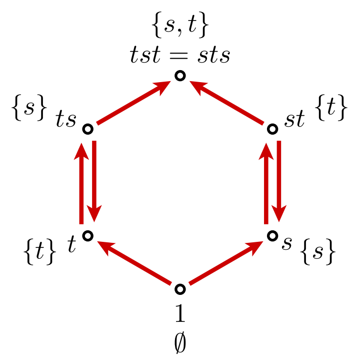
Example. Let W be the Coxeter system generated by s, t with $sts = tst$. (The group is $\mathfrak{S}_3$.) We have already seen

$$\begin{aligned} \gamma_1 &= \tau_1 \\ \gamma_s &= \tau_s + v\tau_1 \\ \gamma_t &= \tau_1 + v\tau_1 \\ \gamma_{st} &= \tau_{st} + v\tau_s + v\tau_t + v^2\tau_1 \\ \gamma_{ts} &= \tau_{ts} + v\tau_s + v\tau_t + v^2\tau_1 \end{aligned}$$

and now we calculate

$$\begin{aligned}\gamma_s\gamma_t\gamma_s &= \tau_{sts} + v\tau_{st} + v\tau_{ts} + v^2\tau_t + v^2\tau_s + \tau_s + (v+v^3)\tau_1 \\ \gamma_{sts} &= \gamma_s\gamma_t\gamma_s - \gamma_s \\ &= \gamma_s\gamma_t\gamma_s - (\tau_s + v\tau_1) \\ &= \tau_{sts} + v\tau_{st} + v\tau_{ts} + v^2\tau_t + v^2\tau_s + v^3\tau_1\,.\end{aligned}$$

All links in the W-graph are of length 1. The graph, with its nodes labelled by right descent sets, looks like this:



I define $x \equiv_R y$ to mean there is a loop in the right W graph from x through y and back to x. In $\mathfrak{S}_3$, as the figure shows, there are four equivalence classes.

[rxry] Proposition 3.5. *If $x \to_R y$ then $L_x \subseteq L_y$. and if $x \equiv_L y$ then $L_x = L_y$. Similarly for left links $x \to_L y$.*

Proof. First of all, suppose $x \to_R y$ and $x < y$. Say $y = xs > x$. Then

$$\mathcal{L}_y = \mathcal{L}_{xs} = \{xsx^{-1}\} \sqcup \mathcal{L}_x \supset \mathcal{L}_x$$
$$L_y = \mathcal{L}_y \cap S \subset L_x = \mathcal{L}_x \cap S\,.$$

That takes care of the links in the W-graph going up.

♣ [going-up] Now suppose $x \to_R y$, which means that $y \prec x$ and $R_y \not\subseteq R_x$. We want to show that $L_x \subseteq L_y$. Suppose $sx < x$, but $sy > y$. But if $sy > y$ then $L_y \not\subseteq L_x$ and $y \to_L x$. According to Proposition 3.4, $x = ty$ for some t in S. However, if $x = ty > y$ then

$$\mathcal{R}_x = \mathcal{R}_{ty} = \mathcal{R}_y \sqcup \{y^{-1}ty\}, \ \text{ hence } R_y \subseteq R_x\,,$$

which contradicts the assumption $R_y \not\subseteq R_x$. ▯

4. The star operations

Suppose s, t to be in S with $W_{s,t}$ isomorphic to $\mathfrak{S}_3$. Following [Kazhdan-Lusztig:1979], define $\mathcal{D}_R(s,t)$ to be the set of w such that $R_w \cap \{s,t\}$ consists of a single element. Another way to see this: we can write $w = xy$ with y in $W_{s,t}$ and x a distinguished coset element satisfying the condition $ys > y$, $yt > y$. Then w is in $\mathcal{D}_R(s,t)$ if and only if $y \in \{s, t, st, ts\}$. The set $\mathcal{D}$ is the domain of an operation: on the set $W_{s,t} \cap \mathcal{D}_R(s,t)$, define the map $x \mapsto x|^{s,t}$:

$$\begin{aligned} s &\longleftrightarrow st \\ t &\longleftrightarrow ts\,. \end{aligned}$$

In other words, if $R_x \cap \{s,t\} = \{u\}$ then $x|^{s,t} = xu$. These are called the 'star' operations because of the original notation ${}^*x = {}^{s,t}|x$, $x^* = x|^{s,t}$ of Kazhdan and Lusztig.

More generally, if $w = xy$ then $w|^{s,t} = x\,y|^{s,t}$. It is easy to see that $x \to_R x|^{s,t}$, so in fact $x \equiv_R x|^{s,t}$.

Similarly for left operations ${}^{s,t}|x$.

Fix s, t in the following discusion, and let $x^* = x|^{s,t}$, etc.

[kl-knuth] **Proposition 4.1.** *If x and y are in $\mathcal{D}_R(s,t)$ then*

(a) if $y \in x\,W_{s,t}$ then $y \prec x$ if and only if $x^* \prec y^*$, and then $\mu(y,x) = \mu(x^*, y^*) = 1$;
(b) if $y \notin x\,W_{s,t}$, then $x \prec y$ if and only if $x^* \prec y^*$, and then $\mu(x,y) = \mu(x^*, y^*)$.

It will not usually be the case, however, that $\pi_{y,x} = \pi_{y^*,x^*}$.

Proof. Simplify notation slightly for the moment by setting $x^* = x^*$.

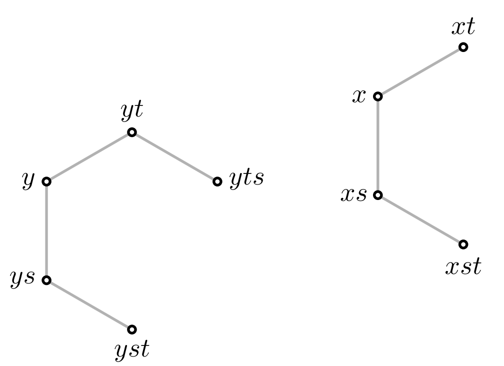
Suppose that $y = xu$, $u \in \{s,t\}$. Then $y \prec x$ if and only if $\ell(y) + 1 = \ell(x)$, and then $\mu(x,y) = 1$, $x^* = y$, and $y^* = x$. This takes care of (a).

Now for (b). Define $\pi(v) \sim \rho(v)$ to mean that $\pi(v) - \rho(v)$ is divisible by v. In particular $\pi_{y,x} \sim \mu(y,x)$.

Case 1. Suppose $x^* = xu$, $y^* = yu$. (In other words, $y^{-1}y^* = x^{-1}x^*$.) Renaming s and t if necessary, we may assume $u = s$. This case is completely symmetric, so it suffices to show that if $x \prec y$ then $x^* \prec y^*$. There are four cases to be considered: (i) $ys > y$, $xs < x$; (ii) $ys < y$, $xs > x$; (iii) $ys < y$, $xs < x$; (iv) $ys > y$, $xs > x$.

Case (i) is impossible, because y would not be extremal with respect to R_x. Case (ii) is also impossible because in this case $xt < x$, $yt > y$ (as you can see by checking the figure above) and again y is not extremal with respect to R_x.

In cases (iii) and (iv) we are going to prove that $\pi_{y,x} \sim \pi_{y^*,x^*}$, so we only have to deal with one of them,
♣ [pixy] say (iii). If $y \not\leq xs$ then by Corollary 2.6 $\pi_{ys,xs} = \pi_{y,x}$ and we are done.



So now we have $ys < y$, $xs < x$, $y < xs$. Since $y \prec x$, $\ell(x) - \ell(y)$ is odd, $\ell(xs) - \ell(y)$ is even and necessarily $\mu(y, xs) = 0$. Therefore

$$\begin{aligned}\pi_{y,x} &= \pi_{ys,xs} + \frac{\pi_{y,xs} - \mu(y,xs)}{v} - \sum_{\substack{z\\ y \prec z \prec xs\\ zs < z}} \mu(z,xs)\pi_{y,z}\\ &\sim \pi_{ys,xs} + \frac{\pi_{y,xs}}{v} - \sum_{\substack{z\\ y < z < xs\\ zs < z}} \mu(z,xs)\mu(y,z)\,.\end{aligned}$$

I claim now:

The sum over z has exactly one term $z = yt$.

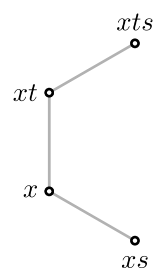
(i) $y < yt$ is apparent (from the figure, say). (ii) So is $yts < yt$; (iii) $yt < xst < xs$. Since $y > yst$ and $y \neq xst$, $y < xs$ if and only if $yt < xst$. (iv) It is the only term. This is because for any z in the sum such that $z \neq yt$, $z \neq yst$ we have $t \in R_{ys}$, hence $t \in R_z$ and then $t \in R_y$, a contradiction. Furthermore, $z = yt$ satisfies $yt < z$ while $z = yst$ does not.

Since $\mu(y, yt) = 1$, we can write

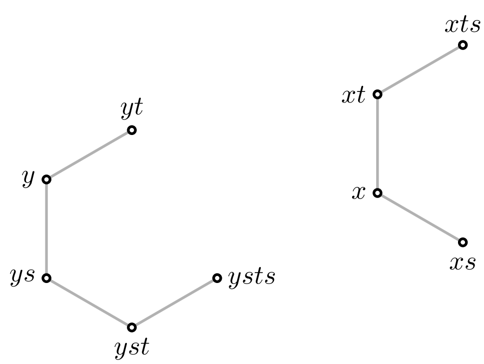
$$\pi_{y,x} \sim \pi_{ys,xs} + \pi_{y,xs}/v - \mu(yt, xs)\,.$$

Finally, $\pi_{y,xs} = v\pi_{yt,xs}$ so $\pi_{y,xs}/v - \mu(yt, xs)$ is divisible by v. So $\pi_{x,y} \sim \pi_{ys,xs}$.

Case 2. We have $x^* = xu$, $y^* = yv$, $u \neq v$. We may as well take $u = s$, $v = t$. There are two cases: $x^* = xt < x$ and $x^* = xt > x$. Again, because the final result is symmetric with respect to x, y and x^*, y^*, we may suppose $x^* = xt > x$. Then $xs < x < xt < xts$:



We want to show that $y \prec x$ if and only if $ys \prec xt$. If $y \prec x$ then since $xs < x$ so too $ys < y$. The argument is not quite symmetric, so we have to show both directions. If $ys \prec xt$ then since $xt \cdot t = x < xt$ so too $yst < ys$. In either case we must have $yst < ys < y < yt$ as in the left of this figure:



If $y < x$ then it now follows from the two figures that $ys < xt$, whereas if we assume $ys < xt$ then of course $ys < xt$. So we may also assume $ys < xt$. This implies that $yst < x$ and $y < xts$.

Suppose $ys \not\leq x$. Since $xt \cdot t = x < xt$, we have

$$\pi_{ys,xt} = \pi_{yst,x} \quad (ys \not\leq x)$$

Since $xs < x$, $yst \cdot s > yst$ we cannot have $yst \prec x$. Therefore since $ys \not\leq x$ we must have $\pi_{ys,xt} \sim 0$ and $ys \not\prec xt$. On the other hand, suppose $y \prec x$. Then $ys < y < x$, a contradiction. So in these circumstances $y \prec x$ if and only if $ys \prec xt$, and we are through.

Otherwise, we may now assume that $ys \leq x$, so that

$$\pi_{ys,xt} \sim \pi_{yst,x} + \pi_{ys,x}/v - \sum_{\substack{z \\ ys<z<x \\ zs<z}} \mu(ys, z)\mu(z, x) \quad (ys \leq x)\,.$$

For z in this sum with $z \neq y$, $z \neq xs$ we have $s \in R_y$, hence $s \in R_z$, hence $s \in R_{ys}$, contradiction. But neither $z = y$ nor xs satisfy $zt < z$. Therefore $\pi_{ys,xt} \sim \pi_{ys,x}/v$. But since $ys \leq x$ and $s \in R_x$, $y \leq x$, $\pi_{ys,x}/v = \pi_{y,x}$. Therefore $\pi_{ys,xt} \sim \pi_{y,x}$, hence $\mu(ys, xt) = \mu(y, x)$. ▯

In the next discussion, again set $x^* = x|^{s,t}$, ${}^*x = {}^{s,t}|x$.

[xxx] **Corollary 4.2.** *Suppose x, y in $\mathcal{D}_L(s,t)$. If $x \equiv_R y$ then ${}^*x \equiv_R {}^*y$. Similarly if x, y are in $\mathcal{D}_R(s,t)$ and $x \equiv_L y$ then $x^* \equiv_R y^*$.*

Proof. It suffices to prove just one half.

f $x \in \mathcal{D}_L(s,t)$ then $x \equiv_L \mathrm{L}x$, hence $R_x = R_{\mathrm{L}x}$.

Now suppose $x \rightarrow_R y$ and $x \equiv_R y$. The second implies that $L_x = L_y$, so that $x \in \mathcal{D}_L(s,t)$ implies that $y \in \mathcal{D}_L(s,t)$. So if ${}^*x$ isdefined, so is ${}^*y$. Because $x \rightarrow_R y$, $R_{{}^*y} = R_y \not\subseteq R_{{}^*y} = R_x$. Since $x \rightarrow_R y$ we have $x \prec y$, but then by the Proposition ${}^*x \prec {}^*y$. Therefore ${}^*x \rightarrow {}^*y$.

If $x \equiv_R$, there exists a chain $x_i \rightarrow_R x_{i+1}$ from x through y and back to x. All the x_i are equivalent. The argument in the previous paragraph then tells us that ${}^*x_i \rightarrow_R {}^*x_{i+1}$, and we have a chain in the right W-graph from ${}^*x$ to ${}^*y$ and back again, so ${}^*x \equiv_R {}^*y$. ▯

5. References

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