Essays on representations of real groups

The real Jacquet module

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If \((\pi, V)\) is an admissible representation of a \(p\)-adic reductive group \(G\) and \(P = MN\) is a parabolic subgroup with unipotent radical \(N\), its Jacquet module \(V_N\) is the universal \(N\)-trivial quotient \(H_0(N, V)\) of \(V\) by the span of vectors \(\pi(n)v - v\) \((n \in N)\). One fundamental property is that \(V \rightarrow V_N\) is an exact functor. This construction plays an important role in homomorphisms into representations induced from parabolic subgroups, the asymptotic behaviour of matrix coefficients, formulas for characters, and the classification of admissible representations. It is computed in practice by means of the Bruhat filtration of representations induced from a parabolic subgroup.

For admissible representations of a real group, the corresponding functor is somewhat more complicated, but also more interesting. Suppose

\[
\begin{align*}
G &= \text{a real reductive group defined over } \mathbb{R} \text{ whose complex group is connected} \\
g &= \text{its complex Lie algebra} \\
K &= \text{a maximal compact subgroup of } G \\
p &= \text{the Lie algebra of a minimal parabolic parabolic subgroup } P \\
n &= \text{the Lie algebra of the unipotent radical of } P
\end{align*}
\]

Admissible representations of \(G\) are not in fact representations of \(G\) at all, but are interpreted to be representations of the pair \((g, K)\) satisfying certain conditions. They are at any rate finitely generated over the Lie algebra \(n\). The \(p\)-adic example leads us to look for a functor \(V \rightarrow V[n]\) that satisfies these conditions: (a) it is well defined for every finitely generated module \(V\) over \(U(n)\); (b) it is exact; (c) it is related to \(H_0(n, V)\) and hence homomorphisms into parabolically induced representations; (d) it is related to matrix coefficients when \(V\) is an admissible representation of \((g, K)\); and (e) it may be calculated explicitly for generic principal series representations in terms of the Bruhat decomposition.

One feature that distinguishes real from \(p\)-adic groups is that in the \(p\)-adic case every admissible representation of the minimal parabolic subgroup \(P\) is trivial on \(N\), whereas for real groups it need only be nilpotent. This observation motivates the definition of the real Jacquet module \(V[n]\) as the projective limit of the quotients

\[
\text{Tor}^{U(n)}_0(U(n)/n^kU(n), V) = V/n^kV.
\]

This definition, although apparently—and perhaps unfortunately—somewhat technical, has much to be said for it. For some, the complexity might be ameliorated by the fact that \(V[n]\) is the linear dual of a certain finitely generated \((g, P)\)-module, a kind of Verma module.

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In this section I’ll illustrate by an example how the asymptotic behaviour of matrix coefficients is related to the action of the unipotent radical of a parabolic subgroup.

For this section let $G = \text{SL}_2(\mathbb{R})$, $K = \text{SO}_2$, and $P = AN$ the subgroup of upper triangular matrices. The group $G$ acts on the upper half plane $\mathcal{H}$ in the usual way by fractional linear transformations, 

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}.$$ 

The group $G$ takes the real line $\mathbb{P} = \mathbb{R}$ into itself and $P$ fixes $\infty$. The Cayley transform $z \mapsto (z - i)/(z + i)$ takes $\mathcal{H}$ to the unit disk $\mathbb{D}$, so by conjugation $G$ acts on it as well, taking the unit circle $\partial \mathbb{D}$ to itself. We may thus identify $\mathbb{P}$ with its image $\partial \mathbb{D}$. Since the Cayley transform of $\infty$ is $1$, the group $P$ fixes it.

It will be useful to have explicit expressions for the action of certain subgroups of $G$ on $\mathbb{D}$, as well as certain elements of its Lie algebra $\mathfrak{g}$. The action of the Lie algebra is most simply expressed in terms of holomorphic functions—I identify a complex number with a real vector, so that the complex-valued function $f(z)$ with real and imaginary parts $a(z)$, $b(z)$ corresponds to the vector field

$$a(z) \frac{\partial}{\partial x} + b(z) \frac{\partial}{\partial y} = f(z) \frac{\partial}{\partial z} + \overline{f(z)} \frac{\partial}{\partial \bar{z}} \left( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \right).$$

The subgroup $K$ acts through the homomorphism

$$k_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mapsto Ck_\theta C^{-1} = \begin{bmatrix} 1/\zeta(\theta) \\ \circ \end{bmatrix} \zeta(\theta),$$

hence through multiplication by $\zeta(\theta)^{-2}$ on $\mathbb{D}$. Its Lie algebra is spanned by

$$\kappa = \begin{bmatrix} \circ & -1 \\ 1 & \circ \end{bmatrix}.$$ 

The calculation of vector fields is most easily done by using the group over the nilring $\mathbb{C}[\varepsilon]$ in which $\varepsilon^2 = 0$. Thus, since $\zeta(\varepsilon) = 1 + i \varepsilon$ and $(1 + i \varepsilon)^{-1} = (1 - i \varepsilon)$, the vector field corresponding to the action of $\kappa$ is 

$$\frac{k_\varepsilon z - z}{\varepsilon} = \frac{1}{\varepsilon} \left( \frac{1 - i \varepsilon}{1 + i \varepsilon} z - z \right) = \frac{(1 - i \varepsilon)(1 + i \varepsilon)z - z}{\varepsilon} = \frac{(1 - 2i \varepsilon)z - z}{\varepsilon} = -2iz.$$
The unipotent matrix
\[ n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \]
acts as
\[ Cn(x)C^{-1} = \frac{1}{2i} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + w & -w \\ w & 1 - w \end{bmatrix} \]
\((w = ix/2)\).

The element
\[ \nu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]
of the Lie algebra corresponds to the vector field
\[
\frac{n_\epsilon(z) - z}{\epsilon} = \frac{1}{\epsilon} \left( \frac{(1 + i\epsilon/2)z - i\epsilon/2}{i\epsilon z/2 + (1 - i\epsilon/2)} - z \right) \\
= \frac{1}{\epsilon} \left( \frac{z + (i\epsilon/2)(z - 1)}{1 + (i\epsilon/2)(z - 1)} - z \right) \\
= \frac{1}{\epsilon} \left( z + (i\epsilon/2)(z - 1) \right) \left( 1 - (i\epsilon/2)(z - 1) \right) - z \\
= -(i/2)(z - 1)^2.
\]
one can either compute similarly, or apply the equation \( \kappa = -\nu + \nu_+ \), to see that
\[
\nu_+ \mapsto -(i/2)(z + 1)^2.
\]

The diagonal matrix
\[
a_x = \begin{bmatrix} x & \circ \\ \circ & 1/x \end{bmatrix}
\]
acts as
\[
Ca_x C^{-1} = \frac{1}{2i} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} x & \circ \\ \circ & 1/x \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} x + x^{-1} & x - x^{-1} \\ x - x^{-1} & x + x^{-1} \end{bmatrix} \sim \begin{bmatrix} x^2 + 1 & x^2 - 1 \\ x^2 - 1 & x^2 + 1 \end{bmatrix}
\]

The element
\[
\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
of the Lie algebra therefore corresponds to the vector field
\[
\frac{a_{1+x} z - z}{\varepsilon} = \frac{1}{\varepsilon} \left( \frac{(1 + \varepsilon)z + 2\varepsilon}{\varepsilon z + (1 + \varepsilon)} - z \right)
\]
\[
= \frac{z + \varepsilon(z + 1)}{\varepsilon} \left(1 - \varepsilon(z + 1)\right) - z
\]
\[
= 1 - z^2.
\]

The group \( G \) acts geometrically on the left on \( \mathbb{D} \), so we define its left action on functions according to the formula
\[
[T \varphi](z) = \varphi(T^{-1} z).
\]

It takes the space
\[
V = \text{harmonic functions on } \mathbb{D} \text{ having smooth boundary values on } \mathbb{P}
\]
to itself. Its Lie algebra acts correspondingly by the negatives of the associated geometric vector fields, as derivations. Explicitly, acting on holomorphic functions:
\[
\kappa \mapsto 2iz \partial/\partial z
\]
\[
\nu \mapsto (i/2)(1 - z)^2 \partial/\partial z
\]
\[
\nu_+ \mapsto (i/2)(z + 1)^2 \partial/\partial z
\]
\[
\alpha \mapsto (z^2 - 1) \partial/\partial z.
\]
The space $V$ therefore contains as $K$-eigenfunctions the powers of $z$ and $\overline{z}$, and its $K$-finite vectors are the holomorphic and anti-holomorphic polynomials in $z$ and $\overline{z}$.

Every function in $V$ possesses a Taylor series at 1, which lies in the space $V$ of harmonic formal power series at 1. This is the formal power series in powers of $z - 1$ and $\overline{z} - 1$ (but no mixed terms). Let $V_k$ be the ideal generated by the $(z - 1)^m$, $\overline{V}_k$ that generated by the $(\overline{z} - 1)^m$, with $m \geq k$. Since $P$ fixes 1, it takes the ideal $m$ of functions in $V$ vanishing at 1 to itself, hence $V$ and each $V_k$, $\overline{V}_k$ also. Explicitly:

\[ n_z: (z - 1) \mapsto \frac{(z - 1)}{1 - w(z - 1)} \]
\[ a_z: (z - 1) \mapsto \frac{x^2(z - 1)}{1 - (x^2 - 1)(z - 1)/2} \]
\[ \nu: (z - 1) \mapsto (i/2)(z - 1)^2 \]
\[ (z - 1)^m \mapsto (mi/2)(z - 1)^{m+1} \]
\[ \alpha: (z - 1) \mapsto (z - 1)(2 + (z - 1)) \]

The diagonal matrix $a_z$ acts as $z^{2k}$ on $V_k/V_{k+1}$, while the unipotent element $n_z$ acts trivially on $V_k/V_{k+1}$, and $\nu$ takes $V_k$ onto $V_k$. Hence $V_{k+1} = n^k V$. In other words, the filtration of $V$ by the $V_k$ and $\overline{V}_k$ is essentially the same as that by powers of $n$. This is the relationship that will be generalized for all admissible $(g, K)$-modules.

It is not only the Lie algebra of $P$, but also the full Lie algebra $g$ of $G$ that acts on the space of harmonic series at 1—the projective limit of the $V/V_k$. Explicitly:

\[ \nu_-: (z - 1)^m \mapsto (im/2)(4(z - 1)^{m-1} + 4(z - 1)^m + (z - 1)^{m+1}) . \]

What kind of a representation is this? To put it in a familiar context, consider the continuous dual of $V$, those linear functionals on it that vanish on some $V_k$. Since

\[ \langle X^m F, f \rangle = \langle F, (-1)^m X^m f \rangle \]

for $F$ in this dual and $n^k$ takes $V$ into $V_k$, every one of these is annihilated by some $n^k$. A basis for this dual is made up of the functionals that take a series to the coefficients of either $(z - 1)^m$ or $(\overline{z} - 1)^m$, say $F_m$ and $\overline{F}_m$. If $f = \sum f_m(z - 1)^m$ then

\[ \nu f = (i/2)(z - 1)^2 \frac{\partial f}{\partial z} = (i/2) \sum m f_m(z - 1)^{m+1} \]

so

\[ [\nu f]_m = \begin{cases} 0 & \text{if } m = 0 \\ (m-1)i/2 f_{m-1} & \text{if } m \geq 1 \end{cases} \]

which implies that

\[ \nu F_m = \begin{cases} 0 & \text{if } m = 0 \\ (m-1)/2 F_{m-1} & \text{otherwise,} \end{cases} \]

and similarly for $\overline{F}_m$. This tells us among other things that $F_0$, $F_1$ and $\overline{F}_1$ span the subspace annihilated by $\nu$. One can also see that

\[ [\alpha f] = \sum_{m \geq 0} m(2 + (z - 1)) f_m(z - 1)^m \]

leading to

\[ \alpha F_m = \begin{cases} 0 & \text{if } m = 0 \\ 2m F_m + (m-1) F_{m-1} & \text{if } m \geq 1, \end{cases} \]
so that \( F_0, F_1, \) and \( \overline{F}_1 \) are all eigenvectors of \( \alpha \). A little more concretely, with basis \( F_m \) we obtain for \( \alpha \) the matrix
\[
\begin{bmatrix}
\circ & \circ & \circ & \circ & \ldots \\
2 & 1 & \circ & \circ & \ldots \\
\circ & 4 & 2 & \circ & \ldots \\
\circ & \circ & 6 & 3 & \ldots \\
\circ & \circ & \circ & 8 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}.
\]
I leave it as an exercise to find its eigenvectors. Finally
\[
\nu_m F_m = \begin{cases} 
2i(F_1 - \overline{F}_1) & \text{if } m = 0 \\
\frac{i}{2} (4(m+1)F_{m+1} + 4mF_m + (m-1)F_{m-1}) & \text{otherwise.}
\end{cases}
\]
This implies that the dual representation \( M \) has the trivial representation as a quotient, with kernel equal to the direct sum of two copies of the same irreducible module. This submodule has the property that the subspace annihilated by \( n \) has dimension one, and this is an eigenspace of weight 2 for \( \alpha \).

Now the Verma module associated to a one-dimensional character \( \chi \) of \( p \) is
\[
U(g) \otimes_{U(p)} \mathbb{C}_\chi
\]
which has the universal property that
\[
\text{Hom}_g(U(g) \otimes_{U(p)} \mathbb{C}_\chi, U) = \text{Hom}_p(\mathbb{C}_\chi, U).
\]
I leave it as an exercise to verify that what our calculations show is that \( \hat{V} \) is the image of the sum of two Verma modules, the one \( M_0 \) corresponding to \( \alpha \mapsto 0 \), the other \( M_2 \) to \( \alpha \mapsto 2 \). This is consistent with what we know, since \( M_2 \) is a submodule of \( M_0 \) and this gives us an extension
\[
0 \to M_2 \oplus M_2 \to M \to \mathbb{C} \to 0.
\]
It is a classical result that the map from the space \( V \) to \( C^\infty(\mathbb{P}) \) taking \( f \) to \( f|\partial D \) is a \((g, K)\)-isomorphism. We shall understand this exact sequence better when we understand the Bruhat filtration of \( C^\infty(\mathbb{P}) \) and the filtration of \( V \) by \( mV \).

2. Completions

In the next few sections, suppose \( n \) to be any nilpotent Lie algebra defined over \( \mathbb{C} \), say of dimension \( r \). The right ideal
\[
(n) = nU(n)
\]
of \( U(n) \) generated by \( n \) is the kernel of the augmentation homomorphism—the trivial representation—hence two-sided. Thus for every \( m \geq 1 \) the right ideal \( n^m U(n) \) generated by products of \( m \) elements of \( n \) is also two-sided, and in fact the \( m \)-th power of \( n \). For some Lie algebras this construction doesn’t mean much—if \( g \) is semi-simple, for example, then \([g, g] = g\) and powers of \((g)\) are just the same as \((g)\). But for nilpotent ones it is very important.

If \( V \) is a module over \( U(n) \), then set \( n^n V \) to be \((n)^n V \). It is a \( U(n)\)-submodule of \( V \), that generated over \( U(n) \) by the
\[
\nu_1 \ldots \nu_n v \quad (\nu_i \in n).
\]
Recall the descending series of \( n \):
\[
\mathcal{D}^0 n = n, \quad \mathcal{D}^{k+1} n = [n, \mathcal{D}^k n].
\]
Since $n$ is nilpotent, $D^k n = 0$ for $k$ large enough.

Define the **depth** function on $n$:

$$d(x) = k + 1 \text{ for } x \in D^k n - D^{k+1} n .$$

The term ‘weight’ is used in [Birkhoff:1937], which seems to be the first use of the notion, and it is also used in subsequent literature. Choose a basis $x_1, \ldots, x_r$ of $n$ compatible with the filtration

$$D^0 n \supset D^1 n \supset \cdots,$$

so that $d(x_i) \geq d(x_j)$ whenever $i \geq j$. Thus for each $k$ there exists $i_k$ such that $x_{i_k}, x_{i_k+1}, \ldots, x_r$ is a basis of $D^k n$. As a consequence, $[x_i, x_j]$ is always a linear combination of the $x_k$ with $k > i, j$—if $j > i$ then

$$[x_i, x_j] = \sum_{k > j} c^k_{i,j} x_k$$

for certain structure constants $c^k_{i,j}$. In particular, $x_r$ is in the centre of $n$. For each $i \leq r$ let $d_i = d(x_i)$.

For each $n$ in $\mathbb{N}^r$ let $x^n = x_1^{n_1} \cdots x_r^{n_r}$ in $U(n)$. By the theorem of Poincaré-Birkhoff-Witt the $x^n$ form a basis of $U(n)$. Define the depth of the monomial $x^n$:

$$d(x^n) = \sum n_i d_i$$

It is not immediately obvious to what extent this definition is independent, say, of the particular choice of the $x_i$. This is answered by the following intrinsic characterization of depth:

**Proposition 2.1.** The ideal $n^m U(n)$ has as basis the $x^n$ with $d(x^n) \geq m$.

**Proof.** First to be shown is that if $d(x^n) \geq m$ then $x^n$ lies in $n^m U(n)$. This follows easily from the special case where $x^n$ is the singleton $x_i$, in which case it means that $x_i$ lies in $n^m U(n)$. But this is easy to see, since if $d(x_i) = k + 1$ then $x_i$ lies in $D^k n$, and by definition each element of $D^k n$ is a linear combination of $k + 1$-fold brackets of elements of $n$.

It remains to be seen that any element of $n^m U(n)$ is a linear combination of the $x^n$ with $d(x^n) \geq m$. For this, the proof proceeds by induction on $m$. The case $m = 0$ is trivial. We now assume the claim true for $m \geq 0$ and attempt the case $m + 1$. Since every element of $n^{m+1} U(n)$ is a linear combination of terms $x y$ with $x$ in $n$ and $y$ in $n^m$, using the induction hypothesis we need only to show that for any $n$ with $\sum n_k d_k \geq m$ and for any $j, x_j x^n$ is a linear combination of $x^m$ with $\sum m_k d_k \geq m + 1$. The canonical form of the product

$$x_j \cdot x_i^{n_i} x_{i+1}^{n_{i+1}} \cdots x_r^{n_r}$$

in which $j < i$ is just $x_j x_i^{n_i} x_{i+1}^{n_{i+1}} \cdots x_r^{n_r}$ so this case is trivial. Therefore what we want is a direct consequence of the following:

**Lemma 2.2.** For $j \geq i$ the product $x_j \cdot x_i^{n_i} x_{i+1}^{n_{i+1}} \cdots x_r^{n_r}$ is a linear combination of monomials $x_i^{m_i} \cdots x_r^{m_r}$ with $\sum m_k d_k \geq d_j + \sum n_k d_k$.

**Proof.** By descending induction on $j$. The case $j = r$ is trivial, since $x_r$ is in the centre of $n$. For $j < r$:

$$x_j \cdot x_i^{n_i} \cdots x_r^{n_r} = x_i^{n_i} x_j^{n_j+1} \cdots x_r^{n_r} - [x_j, x_i^{n_i} \cdots x_{j+1}^{n_{j+1}} \cdots x_r^{n_r}] x_{j+1}^{n_{j+1}} \cdots x_r^{n_r}.$$ 

Now

$$[x_j, x_i^{n_i} \cdots x_j^{n_j}] = \sum_{k=1}^{j} \sum_{\ell=0}^{n_k-1} x_i^{n_i} \cdots x_k^{[x_j, x_k]} x_k^{n_k-\ell-1} \cdots x_j^{n_j}.$$
Since $[x_j, x_k]$ is a linear combination of the $x_n$ with $n > k$, $j$ one may apply the induction hypothesis to

$$[x_j, x_k] x_k^{n_k-\ell-1} \ldots x_j^{n_j} \ldots x_r^{n_r},$$

using the fact that $d_m \geq d_j$ for $m \geq j$.

This also concludes the proof of the Proposition.

For any $U(n)$ module $V$ one has a decreasing sequence $V \supset nV \supset n^2V \ldots$, and hence one has also for $n \geq m$ a canonical projection $V/n^mV \rightarrow V/n^mV$. Define $V[n]$, the completion of $V$ with respect to the powers of $n$, to be

$$V[n] = \lim_{\rightarrow} V/n^mV.$$

It is the $n$-adic completion of $V$. In particular, let $U[[n]]$ be the ring $U(n)[[n]]$. For any $U(n)$ module $V$ the completion $V[n]$ becomes naturally a module over $U[[n]]$.

When $n$ is abelian $U[[n]]$ will be just the formal power series ring $\mathbb{C}[[x_1, \ldots , x_r]]$.

There is a canonical map from $V$ to $V[n]$. The image of $n^m$ in $U[[n]]$ generates a two-sided ideal of $U[[n]]$. If $V$ is any $U[[n]]$-module I’ll let $n^mV$ be the corresponding $U[[n]]$-module. If $V$ is a $U(n)$ module then $n^mV[n]$ is also the $U[[n]]$-module generated by the image of $n^mV$. Inclusion induces an isomorphism of the two modules $V/n^mV$ and $V[n]/n^mV[n]$.

\textbf{Proposition 2.3.} Let $V$ be any $U[[n]]$ module. The following are equivalent:

(a) $V$ is finitely generated over $U[[n]]$;
(b) $V/n^mV$ is finite-dimensional for all $m > 0$;
(c) $V/n^mV$ is finite-dimensional.

When these hold, any elements $v_1, \ldots , v_n$ whose images span $V/nV$ are generators of $V$, and furthermore $V = \lim_{\rightarrow} V/n^kV$.

I leave this as an exercise. As one consequence, if $V$ is finitely generated over $U(n)$ then $V[n]$ is finitely generated over $U[[n]]$.

\textbf{Proposition 2.4.} The ring $U[[n]]$ is Noetherian.

\textbf{Proof.} Define $GR(U[[n]])$ to be the graded ring associated to the decreasing filtration of $U[[n]]$ by the $n^mU[[n]]$. Thus for $m \geq 0$

$$GR^m(U[[n]]) = n^mU[[n]]/n^{m+1}U[[n]] \cong n^mU(n)/n^{m+1}U(n).$$

It follows from Proposition 2.3 that $GR(U[[n]])$ is a polynomial ring in the homogeneous variables $X_1, \ldots , X_r$ where $X_i$ is the image in $n^dU(n)/n^{d+1}U(n)$ of $x_i$, hence has degree $d_i$. In particular $GR(U[[n]])$ is Noetherian, so that one may apply the same argument as in the commutative case ([Bourbaki:1964], III.2.9, Corollary 2 to Proposition 12, or II.4 of [Serre:1965]).

If $V$ and $W$ are two $U(n)$ modules then any $U(n)$ morphism $f: V \rightarrow W$ induces a $U[[n]]$ homomorphism $f[n]: V[n] \rightarrow W[n]$.

\textbf{Proposition 2.5.} (Artin-Rees) If $W$ is a finitely generated $U(n)$ module and $V$ is a submodule of $W$ then there exists $d \geq 0$ such that for all $m \geq 0$

$$n^mV \cap W \subseteq n^{m-d}W.$$
Proof. This is not quite an elementary result even when \( n \) is abelian. The proof I give below is in fact modeled on the proof in that case (see, for example, [Bourbaki:1965] III.3.1 or §II.5 of [Serre:1965]). Define the direct sum

\[
U(n)(T) = \left\{ \sum_{m} f_m T^m \mid f_m \in n^m U(n) \right\}.
\]

The ring \( U(n)(T) \) is Noetherian.

Proof. For the moment, let \( \mathcal{R} = U(n)(T) \). Filter \( U(n) \) by the increasing order filtration

\[
U_0(n) = \mathbb{C} \subseteq U_1 \subseteq \ldots
\]

and filter \( \mathcal{R} \) accordingly. In order to prove \( U(n)(T) \) Noetherian, it suffices to show that the new associated graded ring \( \text{GR}(\mathcal{R}) \) is Noetherian.

I'll motivate the proof by looking first at the case \( n \) is the abelian Lie algebra spanned by two elements \( x \) and \( y \). We have

\[
\mathfrak{R}_n = \bigoplus_{k \geq 0} (n^k \cap \mathfrak{R}_n) T^k
\]

The ideal \( n^k \) in this case is spanned by all monomials \( x^a y^b \) with \( a + b \geq k \). Therefore if \( k > n \) the intersection \( n^k \cap \mathfrak{R}_n \) is trivial, and if \( k \leq n \) it is the space spanned by all monomials \( x^a y^b \) with \( k \leq a + b \leq n \). The quotient \( \mathfrak{R}_n / \mathfrak{R}_{n-1} \) is therefore isomorphic to the direct sum

\[
\bigoplus_{k \leq n} \langle x^a y^b \mid a + b = n \rangle T^k.
\]

Here \( \langle x^a y^b \rangle \) is the vector space spanned by the \( x^a y^b \). For example

\[
\mathfrak{R}_0 = \mathbb{C}
\]

\[
\mathfrak{R}_1 / \mathfrak{R}_0 = \langle x, y \rangle \oplus \langle x, y \rangle T
\]

\[
\mathfrak{R}_2 / \mathfrak{R}_1 = \langle x^2, xy, y^2 \rangle \oplus \langle x^2, xy, y^2 \rangle T \oplus \langle x^2, xy, y^2 \rangle T^2
\]

\[
\mathfrak{R}_3 / \mathfrak{R}_2 = \langle x^3, x^2 y, xy^2, y^3 \rangle \oplus \langle x^3, x^2 y, xy^2, y^3 \rangle T \oplus \langle x^3, x^2 y, xy^2, y^3 \rangle T^2 \oplus \langle x^3, x^2 y, xy^2, y^3 \rangle T^3
\]

It should be evident that \( \text{GR}(\mathfrak{R}) \) is isomorphic to the polynomial ring in variables \( x, y, xT, yT \).

In the general case, a bit of technical difficulty arises because \( n^k \cap \mathfrak{R}_n \) is more complicated. For example, if \( \nu \) lies in \( D^1 \) then it is in \( n^2 \cap \mathfrak{R}_1 \). More precisely for \( n \geq 1 \)

\[
n^k \cap \mathfrak{R}_n / n^k \cap \mathfrak{R}_{n-1} \cong \langle x_{i_1} \ldots x_{i_n} \mid i_j < i_{j+1}, \sum d_{i_j} \geq k \rangle
\]

For example, if \( n = \langle x, y, z \rangle \) with \( [x, y] = z \) then

\[
\mathfrak{R}_0 = \mathbb{C}
\]

\[
\mathfrak{R}_1 / \mathfrak{R}_0 = \langle x, y \rangle \oplus \langle x, y \rangle T \oplus \langle z \rangle T^2
\]

\[
\mathfrak{R}_2 / \mathfrak{R}_1 = \langle x^2, xy, y^2, xz, yz, z^2 \rangle \oplus \langle x^2, xy, y^2, xz, yz, z^2 \rangle T \oplus \langle x^2, xy, y^2, xz, yz, z^2 \rangle T^2
\]

\[
\oplus \langle xz, yz, z^2 \rangle T^3 \oplus \langle z^2 \rangle T^4
\]

We have

\[
\mathfrak{R}_1 / \mathfrak{R}_0 \cong n \oplus \bigoplus D^k n T^k
\]

and then a homomorphism from the symmetric algebra \( S^* (\mathfrak{R}_1 / \mathfrak{R}_0) \) to \( \text{GR}(\mathcal{R}) \). The list just above shows that this homomorphism is not an isomorphism (since, for example, \( xT \cdot zT = x \cdot zT^2 \)), but it is surjective.

This concludes the proof of Lemma 2.6.
In order to conclude the proof of the Artin-Rees Lemma, we need one more preliminary result. Let $M$ be a module over $U(n)$ with a filtration $M = M_0 \subseteq M_1 \subset \ldots$ compatible with the filtration of $U(R)$ by the ideals $n^kU(n)$. Then

$$\mathfrak{M} = M_0 \oplus M_1 \oplus M_2 \oplus \ldots$$

becomes a graded module over $\mathfrak{M} = U(n)\{T\}$. The following is Proposition II.5.8 of [Serre:1965], and the proof is straightforward.

**[serre-lemma] Lemma 2.7.** The following are equivalent:

1. One has $M_{n+1} = nM_n$ for $n \gg 0$;
2. For some $n$ one has $M_{n+k} = n^kM_n$ for all $k \geq 0$;
3. the $\mathfrak{M}$-module $\mathfrak{M}$ is finitely generated.

Now to conclude the proof of the Artin-Rees Lemma. This too follows the proof in the commutative case almost word for word. Given $V \subseteq W$, let $V_n = V \cap n^W \subseteq n^W$. The associated graded module $\mathfrak{M}$ is an $\mathfrak{M}$-submodule of $\mathfrak{M}$. But $\mathfrak{M}$ is finitely generated over $\mathfrak{M}$ and $\mathfrak{M}$ is Noetherian, so $\mathfrak{M}$ is also finitely generated. Apply the last Lemma.

**0**

**[CCC] Corollary 2.8.** If $V$ and $W$ are finitely generated and $f$ is injective, then so is $f_{[n]}$.

**[exact] Corollary 2.9.** For finitely generated $V$, the functor $V \mapsto V_{[n]}$ is exact.

**[tensor] Corollary 2.10.** If $V$ is a finitely generated module over $U(n)$ then the completion $V_{[n]}$ is canonically isomorphic to $U[[n]] \otimes_{U(n)} V$.

**Proof.** Again just the analogue of the one for abelian $n$. Resolve $V$ by a complex of finitely generated free $U(n)$ modules, observe that Corollary 2.10 is trivially true for them, and apply exactness. **0**

These results all extend to arbitrary two-sided ideals of $U(n)$, but proofs, given in [Gabriel-Nouazé:1967], are much more difficult.

I recall how to find the $n$-homology of an $n$-module $V$. Choose a resolution of $\mathbb{C}$ by finitely generated free $U(n)$ modules $F_n$, and then $H_n(V, \mathbb{C})$ is the homology of the complex $F_n \otimes_{U(n)} V$.

**[DDD] Lemma 2.11.** For $m > 0$, $H_m(n, U[[n]]) = 0$.

**[tensor] This follows from Corollary 2.10.**

**[EEE] Proposition 2.12.** If $V$ is a finitely generated $U(n)$ module then the canonical map from $V$ to $V_{[n]}$ induces isomorphisms

$$\text{Tor}^U_n(U(n)/n^kU(n), V) \cong \text{Tor}^U_n(U(n)/n^kU(n), V_{[n]}).$$

**Proof.** I apply the natural dévissage. Let

$$0 \rightarrow A \rightarrow F \rightarrow V \rightarrow 0$$

be a short exact sequence of finitely generated $U(n)$ modules with $F$ free. Then

$$0 \rightarrow A_{[n]} \rightarrow F_{[n]} \rightarrow V_{[n]} \rightarrow 0$$

is also exact. Temporarily, let $R = U(n)$ and $M = R/n^kR$. Since $F$ is free, $F_{[n]}$ is Tor-acyclic. Furthermore, $X/n^n \cong X_{[n]}/n^n$ for any finitely generated $X$. and , we then get from the corresponding long exact sequences the diagram

$$\begin{array}{cccccc}
0 & \rightarrow & \text{Tor}^R_1(M, V) & \rightarrow & A/n^kA & \rightarrow & F/n^kF & \rightarrow & V/n^kV & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Tor}^R_1(M, V_{[n]}) & \rightarrow & A_{[n]} & \rightarrow & F_{[n]} & \rightarrow & V_{[n]} & \rightarrow & 0
\end{array}$$
From which the isomorphism of the two $\text{Tor}_1$ is immediate. For $n > 0$ we have

$$
\begin{array}{c}
0 \rightarrow \text{Tor}^{R}_{n+1}(M, V) \rightarrow \text{Tor}^{R}_n(M, A) \rightarrow 0 \\
0 \rightarrow \text{Tor}^{R}_{n+1}(M, V_{[n]}) \rightarrow \text{Tor}^{R}_n(M, A_{[n]}) \rightarrow 0
\end{array}
$$

So the isomorphism follows by induction.
3. The Schwartz space

Now let \( N \) be a connected unipotent Lie group with Lie algebra \( n \). The exponential map is an isomorphism of \( n \) with \( N \). Pick a Euclidean metric \( \|x\| \) on \( n \), and then define \( \|\exp(x)\| = \|x\| \). Of course this norm is not uniquely determined, but all choices are equivalent in an obvious sense. The Schwartz space \( S(N) \) may be defined in many ways, but in any case it will be the space of smooth functions on \( N \) such that \( R_X f(x) = O(\|x\|^{-n}) \) as \( \|x\| \to \infty \), for all \( n \), all \( X \) in \( U(n) \).

Fix a right-invariant Haar measure on \( N \). It is unique up to a constant. Integration over \( N \) is an \( n \)-morphism from \( S(N) \) to \( \mathbb{C} \) trivial on \( nS(N) \).

\[ H_0(n, S(N)) = S(N)/nS(N) \cong \mathbb{C}. \]

For \( i > 0 \), \( H_i(n, S(N)) = 0. \)

\textbf{Proof.} For \( N \) one-dimensional it must be verified that \( H_0 = \mathbb{C} \) and that \( H_1 = 0 \), or in other words that \( nS(N) \) has codimension one in \( S(N) \) and that there are no \( n \)-invariants. The second is clear. As for the first, suppose \( \nu \) spans \( n \). With suitable coordinates, the Fourier transform translates \( R_y \) into multiplication by \( y \), and we are reduced to showing that if \( F \) lies in \( S(\mathbb{R}) \) and \( f(0) = 0 \) then \( f(y)/y \) is also in \( S(\mathbb{R}) \), which I leave as an exercise.

In general, the proof proceeds by induction on the dimension of \( N \). Let \( N_0 \) be a one-dimensional central subgroup of \( N \). If \( n^0 \) is a linear complement of \( n_0 \) in \( n \) and \( N^0 = \exp(n^0) \), then the exponential map gives us a product decomposition \( N = N_0 \times N^0 \). Hence

\[ S(N) \cong S(N_0) \hat{\otimes} S(n^0) \]
as an \( N_0 \)-space with respect to the left action. Here \( \hat{\otimes} \) means completed tensor product, which is unambiguous since all Schwartz spaces are nuclear. Since the completed tensor product is an exact functor on nuclear spaces and \( S(n^0) \cong S(n_0\backslash N) \)

\[ H_0(n_0, S(N)) \cong S(N_0\backslash N), \quad H_1(n_0, S(N)) = 0. \]

The complete Lemma then follows from the induction hypothesis after applying the Hochschild-Serre spectral sequence (see, for example, §XVI.6 of [Cartan-Eilenberg:1956]).

\textbf{Lemma 3.2.} Every quotient \( S(N)/n^k S(N) \) is finite-dimensional, and each subspace \( n^k S(N) \) is closed in \( S(N) \).

\textbf{Proof.} More generally, suppose that \( V \) is any \( U(n) \)-module with \( V/nV \) finite-dimensional, say spanned by \( v_1, \ldots, v_n \). Since \( n^k V/n^{k+1}V \) is spanned by the canonical image of the tensor product of \( V/nV \) and \( n^k/n^{k+1} \), each \( V/n^k V \) is also finite-dimensional. If \( V \) is a Fréchet space then since \( n^k V \) is the image of the space \( T^k n \otimes V \) under a continuous map, finite-dimensionality and the Closed Graph Theorem for Fréchet spaces imply closure of the image.

Define

\[ A(N) := \text{the strong topological dual of } S(N) \]
\[ A(N) := \text{the ring of polynomial functions on } N. \]

Thus \( A(N) \) is the space of tempered distributions on \( N \). The group \( N \) acts smoothly on both spaces by the right regular representation, and the \( N \)-module \( A(N) \) embeds into \( A(N) \) given a choice of Haar measure: the function \( F \) in \( A(N) \) corresponds to the distribution

\[ f \mapsto \langle F, f \rangle = \int_N F(x)f(x) \, dx. \]
The subspace of polynomials in $A(N)$ of degree less than $k$ is $N$-stable, and is an algebraic representation of $N$, so that every element in $A(N)$ is annihilated by some power of $n$.

Since $n$ acts nilpotently on algebraic representations, the space $A(N)$ is the union of its finite-dimensional subspaces $A(N)[n^k]$, its $n$-torsion elements.

**Lemma 3.3.** For each $p > 0$

(a) $H^p(n, A(N)) = 0$;
(b) $H^p(n, A(N)) = 0$.

**Proof.** The complex defining cohomology of $A(N)$ is dual to the complex defining homology of $S$, since Lemma 3.1 implies its differentials all have closed images. Thus the first claim follows from Lemma 3.1.

As for the second, the argument is essentially the one I have used in proving the last claim of Lemma 3.1—it is by induction on the dimension of $N$. If $N$ has dimension one, then $A(N) = \mathbb{C}[x]$, with $n$ acting by differentiation, and

$$H^1(n, A(N)) = \mathbb{C}[x]/(d/dx)\mathbb{C}[x] = 0.$$  

Proceed by induction. Let $N_0$ be a one-dimensional central subgroup. We can write $N$ as a product $N_0 \times N^0$ of algebraic varieties, with $N^0 \cong N_0 \setminus N$. Then $A(N) \cong A(N_0) \otimes A(N^0)$, and the Hochschild-Serre spectral sequence gives us

$$H^p(n_0 \setminus n, H^q(n_0, A(N))) \Rightarrow H^{p+q}(n, A(N)).$$

But

$$H^q(n_0, A(N)) \cong H^q(n_0, A(N_0)) \otimes A(N^0)$$

so by the case of dimension one

$$H^0(n_0, A(N)) \cong A(N_0 \setminus N)$$

$$H^1(n_0, A(N)) = 0,$$

which allows us to apply the induction hypothesis.

The following was first proved in [du Cloux:1989]. For any $n$-module $V$, let $V^{[n]}$ be the subspace of its $n$-torsion.

**Proposition 3.4.** The inclusion of $A(N)$ into $A(N)^{[n]}$ is an isomorphism.

Du Cloux’s proof depended on a much more general result, but a direct proof is not difficult.

**Proof.** We must show that $A(N)[n^k] = A[n^k]$ for each $k$, and we proceed by induction on $k$. For $k = 1$ this follows from Lemma 3.1. For $k > 1$ we have an exact sequence

$$0 \to U(n)/n^{k-1}U(n) \to U(n)/n^kU(n) \to n^k/n^{k-1} \to 0,$$

with $n$ acting trivially on the last term. Since

$$V[n^k] \cong \text{Hom}_\mathbb{C}(U(n)/n^kU(n), V), \quad \text{Hom}_\mathbb{C}(n^k/n^{k-1}, V) \cong \text{Hom}_\mathbb{C}(n^k/n^{k-1}, \mathbb{C}) \otimes V[n],$$

by Lemma 3.3 the corresponding long exact sequences associated to $A(N)$ and $A(N)$ give us two exact sequences

$$0 \to A(N)[n^{k-1}] \to A(N)[n^k] \to \text{Hom}_\mathbb{C}(n^k/n^{k-1}, \mathbb{C}) \otimes A(N)[n] \to 0$$

$$0 \to A(N)[n^{k-1}] \to A(N)[n^k] \to \text{Hom}_\mathbb{C}(n^k/n^{k-1}, \mathbb{C}) \otimes A(N)[n] \to 0$$

By the induction hypothesis, the outer two arrows are isomorphism, hence so is the middle one.
Since all quotients $S(N)/n^k S(N)$ are finite-dimensional, every linear function on $S(N)$ annihilated by $n^k$ is continuous, and the space $A(N)[n^k]$ may be identified with its dual. In other words there is a canonical isomorphism between $S(N)/n^k S(N)$ and the dual of $A(N)[n^k]$.

Consider now the $n$-map from $A(N)$ to $\text{Hom}_\mathbb{C}(U(n), \mathbb{C})$ according to which a function $F$ maps $X$ to $XF(1)$—i.e. essentially taking $F$ to its Taylor’s series at 1. This is injective since the functions in $A(N)$ are analytic. And of course any image is annihilated by some power of $n$. Conversely, consider the space $\text{Hom}_\mathbb{C}(U(n)/n^k U(n), \mathbb{C})$. The Lie algebra $n$ acts on it by the contragredient of the right regular representation. Since $n$ acts nilpotently on this space, it exponentiates to an algebraic representation $R$ of $N$. If $\Lambda$ is the functional on this space taking $F$ to $F(1)$, then the function $F(n) = \Lambda(R_n F)$ is algebraic, and its Taylor’s series at 1 is just $F$ itself. In other words, this map from $A(N)[n^k]$ to the dual of $U(n)/n^k U(n)$ is surjective as well, hence an isomorphism. This is the basis of algorithms discussed in [Burde et al.2008], but also of theoretical interest.

We now have the diagram

$$S(N)/n^k S(N) \cong \text{dual of } A(N)[n^k] = \text{dual of } A(N)[n^k] \cong U(n)/n^k U(n),$$

where the restriction map is a surjection since its dual is an injection. Thus the combined map is a surjection as well, but may be split since $U(n)/n^k U(n)$ is projective in the category of $n$-modules annihilated by $n^k$. By Lemma 3.1 the image of the splitting is all of $S(N)/n^k S(N)$. Explicitly, the function $f$ in $S(N)/n^k S(N)$ is mapped to the unique $X$ in $U(n)/n^k U(n)$ such that

$$\int_N f(x) P(x) \, dx = (XP)(1)$$

for all polynomials $P$ on $N$ of degree $\leq k$. We have just proved that this is an isomorphism:

[integration-over-n] **Proposition 3.5.** The $n$-covariant map defined above from $S(N)/n^k S(N)$ to $U(n)/n^k U(n)$ is an isomorphism.

This is in some ways a rather strange isomorphism. It asserts that to any $f$ in $S(N)$ corresponds an element in the completion $U(n)[n]$, but I don’t see how to describe this correspondence except by passing through $A(N)$. This seems to be somewhat more natural in the abelian case where one can use the Fourier transform to identify the differential operators in $U(n)$ with polynomials on the dual $\mathfrak{n}$, and polynomials on $N$ with Dirac distributions on the same space. For nilpotent $n$ which are not abelian, there is as far as I know no analogue of the Fourier transform, although [du Cloux:1987] has to some extent provided a substitute mechanism. Now if $n$ is abelian, the Fourier transform and the theorem of E. Borel may be combined to prove that the map from $S(N)$ to $U(n)[n]$ is surjective. Du Cloux proves this for arbitrary $n$ as well.

[schwartz-un] **Corollary 3.6.** The completion $S(N)[n]$ is free of rank one over $U(n)[n]$.

This and Lemma 3.1 imply that the canonical map from $V = S(N)$ to $U(n)[n]$ induces an isomorphism of $\text{Tor}_U(U(n)/n^k U(n), V)$ with $\text{Tor}_U(U(n)/n^k U(n), V[n])$. 

[schwartz-free] **Corollary 3.6.** The completion $S(N)[n]$ is free of rank one over $U(n)[n]$. 

This and Lemma 3.1 imply that the canonical map from $V = S(N)$ to $U(n)[n]$ induces an isomorphism of $\text{Tor}_U(U(n)/n^k U(n), V)$ with $\text{Tor}_U(U(n)/n^k U(n), V[n])$. 

[integration-over-n]
4. Duality

Let $\mathcal{V}$ be any $U[[n]]$-module. Define $\text{Hom}_{\text{cont}}(\mathcal{V}, \mathbb{C})$ to be the space of continuous linear maps from $\mathcal{V}$ to $\mathbb{C}$—i.e. continuous in the $n$-adic topology, or equivalently trivial on some $n^m\mathcal{V}$. It is also the direct limit of the duals of the $V/n^mV$. It becomes a module over $n$ by the contragredient representation:

$$\langle x \cdot f, v \rangle = -\langle f, x \cdot v \rangle \quad (x \in n).$$

It is an $n$-torsion module—every element in it is annihilated by some $n^m$. In general, if $W$ is any $U(n)$-module then the $n$-torsion in $W$ will be expressed as $W[n]$, so this dual is $\hat{V}[n]$.

Conversely, if $W$ is an $n$-torsion module, define $\hat{W}$ to be its linear dual. Since $W = \lim_{\to} W[n^m]$, this will be the projective limit of the duals of the $W[n^m]$. It is also a module over $U[[n]]$.

For example, if $n$ is abelian, say of dimension $r$, then $U[[n]]$ is a formal power series ring and the continuous dual of $U[[n]]$ is isomorphic to the space of distributions on $\mathbb{C}^r$ with support at 0. The dual of this in turn is again $U[[n]]$.

[FFF] Proposition 4.1. We have:

(a) If $\mathcal{V}$ is any $U[[n]]$-module and $W = \text{Hom}_{\text{cont}}(\mathcal{V}, \mathbb{C})$ then $W[n^m]$ naturally isomorphic to the dual of $\mathcal{V}/n^m\mathcal{V}$.

(b) If $W$ is any torsion $n$-module and $\mathcal{V} = \hat{W}$, then $(W[n^m])$ is isomorphic to $\mathcal{V}/n^m\mathcal{V}$.

Proof. The first is elementary. If $f$ is in $W$ then $x \cdot f = 0$ if and only if for all $v$ in $\mathcal{V}$

$$\langle X \cdot f, v \rangle = \langle f, \hat{X} \cdot v \rangle = 0$$

for all $X$ in $n^mU(n)$, where $X \mapsto \hat{X}$ is the involutory anti-automorphism taking $x$ in $n$ to $-x$.

As for the second, since $n^mU(n)$ is a two-sided ideal, $f$ in $W$ lies in $W[n^m]$ if and only if $x_1 \ldots x_m \cdot f = 0$ for all $x_1, \ldots, x_m$ in $n$. Thus we have an exact sequence

$$0 \to W[n^m] \to W \to \text{Hom}(\mathbb{C} \otimes \ldots \otimes n, W).$$

The dual sequence

$$n \otimes \ldots \otimes n \otimes \hat{W} \to \hat{W} \to (W[n^m]) \to 0$$

is also exact, since $n$ is finite-dimensional.

[proj-lim] Corollary 4.2. If $\mathcal{V}$ is a finitely generated $U(n)$-module then $\hat{V}[n] = \lim_{\to} \mathcal{V}/n^m\mathcal{V}$.

[canonically] Corollary 4.3. If $\mathcal{V}$ is a finite $U[[n]]$-module and $W = \text{Hom}_{\text{cont}}(\mathcal{V}, \mathbb{C})$ then $\hat{W}$ is canonically isomorphic to $\mathcal{V}$.

[equiv] Corollary 4.4. If $W$ is an $n$-torsion module then the following are equivalent:

(a) $W[n]$ is finite-dimensional;

(b) $W[n^m]$ is finite-dimensional for all $m > 0$;

(c) $\hat{W}$ is a finitely generated module over $U[[n]]$;

When these hold, the continuous dual of $\hat{W}$ is canonically isomorphic to $W$.

Define a cofinite $n$-torsion module to be an $n$-torsion module $W$ such that $W[n]$ is finite-dimensional.

[cofinite] Proposition 4.5. Any submodule and any quotient module of a cofinite $n$-module is also cofinite.

[homology] Proposition 4.6. If $\mathcal{V}$ is any finitely generated $U[[n]]$-module and $W$ is $\text{Hom}_{\text{cont}}(\mathcal{V}, \mathbb{C})$ then $H_*(n, \mathcal{V})$ is isomorphic to the dual of the cohomology $H^*(n, W)$.

Proof. From the definition of Lie algebra cohomology in terms of the Koszul complex $\text{Hom}(\Lambda^n n, W)$, since its dual is $\Lambda^n n \otimes \hat{W}$. 

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5. Verma modules

Let \( \mathfrak{g} \) be a reductive Lie algebra over \( \mathbb{C} \) and further let

\[
\begin{align*}
\mathfrak{p} &= \text{a parabolic subalgebra of } \mathfrak{g} \\
\mathfrak{n} &= \text{the nilpotent radical of } \mathfrak{p} \\
\mathfrak{m} &= \text{a reductive complement of } \mathfrak{n} \text{ in } \mathfrak{p} \\
\mathfrak{P} &= \text{the opposite of } \mathfrak{p}, \text{ the unique parabolic subalgebra such that} \\
\mathfrak{m} \cap \mathfrak{P} &= \mathfrak{m}.
\end{align*}
\]

\[\Pi = \text{the unipotent radical of } \mathfrak{P},\]

\[\mathfrak{c} = \text{a Cartan subalgebra of } \mathfrak{m} \text{ (hence of } \mathfrak{g}),\]

\[\Sigma_{\mathfrak{n}} = \text{roots of } \mathfrak{c} \text{ with respect to } \mathfrak{c},\]

\[Z(\mathfrak{g}) = \text{the centre of the universal enveloping algebra } U(\mathfrak{g}).\]

Modifying the usual terminology a bit, I’ll call a \( \mathfrak{g} \)-module a Verma module if its restriction to \( \mathfrak{n} \) is a cofinite \( \mathfrak{m} \)-torsion module. Quotients as well as submodules are again Verma modules.

If \( V \) is a Verma module then each \( V[n^m] \) is finite-dimensional. Since each of these is \( \mathfrak{m} \)-stable and \( Z(\mathfrak{g}) \)-stable, \( V \) is \( \mathfrak{m} \)-finite and \( Z(\mathfrak{g}) \)-finite. Thus it is also \( \mathfrak{c} \)-finite. It is not necessarily \( \mathfrak{c} \)-semi-simple, but one may still define the multiplicity of any linear form \( \chi \) on \( \mathfrak{c} \) to be the dimension of the corresponding primary component, that of all \( v \) in \( V \) such that \((\mathfrak{c} - \chi(\mathfrak{c}))^m v = 0\) for some positive integer \( m \). A linear form with positive multiplicity is called a weight of \( V \).

Let \( C_\mathfrak{n} \) be the closed (generally degenerate) cone of all complex linear forms of \( \mathfrak{c} \) spanned by the elements of \( \Sigma_{\mathfrak{n}}^\ast \).

[GGG] Proposition 5.1. If \( V \) is a Verma module and \( S \) the set of weights of \( V[n] \), then the weights of \( V \) are contained in \( S - C_\mathfrak{n} \). The multiplicity of each weight is finite.

Proof. For every \( m \geq 0 \) there exists a canonical projection

\[V[n^{m+1}]/V[n^m] \longrightarrow \text{Hom}_{C}(n^m U(n)/n^{m+1} U(n), V[n])\]

namely the one under which \( v \) in \( V[n^{m+1}] \) corresponds to the map taking \( x \) to \( x \cdot v \). Thus every weight of \( V \) is of the form \(-\alpha + \beta\) where \( \alpha \) is a weight of some \( n^m U(n)/n^{m+1} U(n) \) and \( \beta \) one of \( V[n] \). Choose a basis \( x_1, \ldots, x_r \) of \( n \) as in §1 with the additional property that each \( x_i \) is an eigenvector for \( \mathfrak{c} \), say with respect to the root \( \alpha_i \). Each \( \alpha_i \) is a sum of \( d_i \) roots (in the notation of §1). According to 1.1 each weight of \( n^m U(n)/n^{m+1} U(n) \) is a sum of \( m \) roots in \( \Sigma_{\mathfrak{n}}^\ast \). The first claim of 3.2 is immediate from this. The second follows from the fact that any weight may be represented as a sum of positive roots in only a finite number of ways.

For each ideal \( I \) of \( Z(\mathfrak{g}) \) and \( m \geq 0 \) let

\[V_1^{I,m} = \{v \in V \mid I^m v = 0\}, \quad V_1^{I,\infty} = \bigcup V_1^{I,m}.\]

The maximal ideals with \( V_1^{I,\infty} \neq 0 \) I’ll call the central ideals of \( V \). Since \( V \) is \( Z(\mathfrak{g}) \)-finite, \( V \) is the direct sum of its primary modules \( V_1^{I,\infty} \). The central ideals of quotients and subspaces of \( V \) lie among those of \( V \).

[HHH] Lemma 5.2. If \( V \) is a Verma module, then there are only a finite number of maximal \( I \) with \( V_1^{I,\infty} \neq 0 \).

Proof. If \( V_1 \neq 0 \) is any \( U(n) \)-module of \( V \) then \( V_1[n] = V_1 \cap V[n] \neq 0 \). Hence each \( V_1^{I,\infty} \neq 0 \) intersects \( V[n] \) non-trivially. The Lemma follows, since \( V[n] \) is finite-dimensional and \( Z(\mathfrak{g}) \)-stable.
Examples of Verma modules are the ones generally called generalized Verma modules. Let $\xi$ be any $p$-module and let $V$ be the module $C(\xi) = C_{\mathfrak{p}, \theta}(\xi) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \xi$ coinduced from $\xi$. It has the universal property

$$\text{Hom}_p(C(\xi), V) \cong \text{Hom}_p(\xi, V)$$

for any $g$-module $V$, through the canonical inclusion of the $p$-module $\xi$ in $C(\xi)$.

[JJJ] Proposition 5.3. (a) For any finite-dimensional $p$-module $\xi$, $C(\xi)$ is a Verma module; (b) If $\xi$ is irreducible then $C(\xi)$ has a unique irreducible $g$-quotient.

I'll call the unique quotient $C^+(\xi)$. I leave the proof as an exercise.

[KKK] Proposition 5.4. Any Verma module has finite length as a $g$-module. Its composition factors are of the form $C^+(\xi)$ with $\xi$ an irreducible finite-dimensional $p$-module.

Proof. Let $V$ be given, and let $I$ be any ideal of finite-codimension in $Z(\mathfrak{g})$ such that $V = V^{1, \infty}$. According to 3.2, one might take $I$ to be the product of the central ideals of $V$. Recall the Harish-Chandra homomorphism

$$\lambda: Z(\mathfrak{g}) \longrightarrow Z(\mathfrak{m})$$

characterized by the property that $X - \lambda(X) \in U(\mathfrak{g})n$. The ideal $Z(\mathfrak{m})\lambda(I)$ generated by $\lambda(I)$ has finite codimension in $Z(\mathfrak{m})$, so that there exist only a finite number of maximal ideals of $Z(\mathfrak{m})$ containing $\lambda(I)$, hence only a finite number of irreducible modules $\xi_1, \ldots, \xi_n$ of $\mathfrak{m}$ whose restriction to $\lambda(Z(\mathfrak{g}))$ contain $\lambda(I)$ in the kernel. Define $\mu_I(V)$ to be the sum of the multiplicities of the $\xi_i$ in $V$.

[LLL] Lemma 5.5. If $V \neq 0$ is any Verma module with $V = V^{1, \infty} \neq 0$ then $\mu_I(V) \neq 0$ and some $\xi_i$ occurs as a submodule of $V$.

Proof of the Lemma. The subspace $V^n$ is non-trivial and finite-dimensional, and by definition $X$ in $Z(\mathfrak{g})$ acts as $\lambda(X)$ on it. Therefore $\lambda(I^m)$ annihilates $V^n$ for some $m > 0$. There will then exist non-trivial vectors in $V^n$ annihilated by $\lambda(I)$. The subspace of all such vectors is thus $m$-stable and non-trivial, hence has some $\xi_i$ as submodule.

Proof of 3.4. By induction on $\mu_I(V)$. By 3.5, if $\mu_I(V) = 0$ then $V = 0$ and we are done. So say $\mu_I(V) > 0$. Then some $\xi_i$ occurs as an $m$-module of $V^n$. Extend $\xi_i$ to a $p$-module trivial on $n$. There exists a non-trivial $g$-morphism from $C(\xi)$ to $V$ corresponding to the embedding of $\xi_i$ into $V$. If the image $V_i$ of $C(\xi)$ is not all of $V$ then both $V_1$ and $V_2 = V/V_1$ satisfy $\mu_I(V_1) < \mu_I(V)$ by 3.5, so that one may apply the induction hypothesis. If $V = V_i$, then $V = C(\xi_i)/V_2$ for some proper submodule $V_2$. But since there exists a unique maximal proper $g$-submodule in $C(\xi_i)$, there exists a $g$-morphism from $V$ to $C^+(\xi_i)$ through which the canonical map from $C(\xi_i)$ to $C^+(\xi_i)$ factors. Apply the induction hypothesis to the kernel of the map from $V$ to $C^+(\xi_i)$.

As one consequence:

[MMM] Corollary 5.6. The annihilator in $Z(\mathfrak{g})$ of a Verma module for $p$ has finite codimension.

6. The Jacquet module

Continue the notation of the previous section. Let $\theta$ be an involutory automorphism of $\mathfrak{g}$, and let

$$\mathfrak{k} = (+1)\text{-eigenspace of } \theta$$

$$\mathfrak{s} = (-1)\text{-eigenspace of } \theta$$

$$\mathfrak{a} = \mathfrak{m} \cap \mathfrak{k}.$$  

I'll assume that the parabolic algebra $\mathfrak{p}$ is compatible with $\theta$, which is to say that $\mathfrak{m}$ is $\theta$-stable and that $\mathfrak{g}$ has the Iwasawa decomposition $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

I define a Harish-Chandra module over $\mathfrak{g}$ to be one which is $\mathfrak{k}$-finite, $\mathfrak{k}$-semi-simple, and $Z(\mathfrak{g})$-finite.
Here I take \( n^k \) to be \( n^0 \) if \( k < 0 \).

**Proof.** By induction on \( m \). The case \( m = 0 \) is trivial. Assume the result to be true for \( m \geq 0 \) and look at \( m + 1 \). Say \( \nu \) lies in \( n_e \) with \( \delta \) in \( \Sigma^+ \), \( y \) in \( n^m \cap U(g) \). Then

\[
[x, \nu y] = [x, \nu] \cdot y + \nu \cdot [x, y] = y \cdot [x, \nu] + [x, \nu \cdot y] + \nu \cdot [x, y].
\]

By the induction assumption, \([x, y] \) is in \( n^{m-n} \cap U(g) \), which implies that the last term is in \( n^{m+1-n} \cap U(g) \). The first term lies in \( n^m \cap U(g) \), which is contained in \( n^{m+1-n} \cap U(g) \) since \( n > 0 \).

It remains to consider the second term. If \([x, \nu \cdot y] \) is in \( p \), there is no more to be done. If not, say \( \nu_1 = [x, \nu \cdot y] \) lies in \( n_e \) where \( \epsilon = \gamma + \delta \) is in \( \Sigma^- \). Since \( \gamma = \epsilon - \delta \), if we write

\[
\epsilon = -\sum e_\alpha \alpha, \quad \delta = \sum d_\alpha \alpha
\]

then \( n_{\nu_1} = e_\alpha + d_\alpha \) for all \( \alpha \) in \( \Delta \). Let \( e = \sum e_\alpha, d = \sum d_\alpha \). Then \( n = e + d \) is \( > \). Apply induction to \( \nu_1 \) and \( y \) to see that \([\nu_1, y] \) must be in \( n^{m-\epsilon} \cap U(g) \), hence in \( n^{m+1-n} \cap U(g) \).

Now let \( V \) be any finitely generated Harish-Chandra module. Recall that \( V_{[n]} \) is the projective limit of the spaces \( V/n^m V \), and that \( g = p + p \). Each \( x \) in \( p \) acts on each \( V/n^m V \), hence also on the limit. According to 4.2, there exists for every \( x \) in \( g \) an \( n \geq 0 \) such that \( x \cdot n^m V \subseteq n^{m-n} V \) for \( m \geq 0 \), so \( x \) induces an operation on the limit as well. This defines \( V_{[n]} \) as a \( g \)-module. It continuous dual \( \hat{V}_{[n]} \) becomes a \( g \)-module, also.

**Proposition 6.3.** The \( g \)-module \( \hat{V}_{[n]} \) is a Verma module.

This is because \( V \) is finite over \( U(n) \). Results from \([\S\S 1\) and 2 now give us:

**Theorem 6.4.** The functor \( V \rightarrow \hat{V}_{[n]} \) is an exact additive functor from the category of finitely generated Harish-Chandra modules to that of Verma modules modules. The homology \( H_*(n, V) \) is finite-dimensional and dual to \( H^*(n, \hat{V}_{[n]}) \).

I call the \( g \)-module \( V_{[n]} \) the Jacquet module of \( V \). It is the analogue for real groups of the Jacquet module associated to admissible representations of \( p \)-adic groups. Indeed, the construction was inspired by the \( p \)-adic construction.

It is known (for example, see [Beilinson-Bernstein:1983]) that the canonical map from \( V \) to its Jacquet module is an injection.
7. The Bruhat filtration of the minimal principal series

Suppose \( P = P_\emptyset \), \((\sigma, U)\) a finite dimensional representation of \( P_\emptyset \). Let

\[
\text{Ind}^\infty(\sigma \mid P_\emptyset, G) = \{ f \in C^\infty(G, U) \mid f(pg) = \delta_\emptyset^{1/2}(p)\sigma(p)f(g) \text{ for all } p \in P, g \in G \},
\]

the smooth induced representation of \( G \) induced from \( \sigma \), and let \( \text{Ind}(\sigma \mid P_\emptyset, G) \) be the subspace of \( K \)-finite vectors. This last is a finitely generated Harish-Chandra module over \( (G, K) \). Suppose that \( \Theta \subseteq \Delta \), and \( P = P_\Theta \) is the standard parabolic subgroup of \( G \) associated to \( \Theta \). Eventually, I want to study the representation \( \text{Ind}(\sigma) \) as a representation of \( P \). For the moment, take \( \Theta = \emptyset \), so \( P = P_\emptyset \). In this case \( \text{Ind}(\sigma) \) is finitely generated over \( U(n) \). I want now to describe a filtration of \( V|_n \) associated to the Bruhat decomposition \( G = \bigcup_{w \in W} PwN \).

In contrast to what happens for \( p \)-adic groups, there is no good filtration of \( \text{Ind}(\sigma) \) itself. Instead, the filtration of \( \text{Ind}(\sigma)|_n \) is constructed in three steps: (1) we construct a filtration of \( \text{Ind}^\infty(\sigma) \) according to the closure relations of the Bruhat decomposition; (2) we find use this to deduce a filtration of the Jacquet module \( \text{Ind}^\infty(\sigma) \); (3) we prove that for every finitely generated Harish-Chandra module \( V \), the embedding of \( V \) into its canonical Fréchet representation \( \tilde{V} \) of \( G \) induces an isomorphism of Jacquet modules. If \( V = \text{Ind}(\sigma) \) then \( \tilde{V} = \text{Ind}^\infty(\sigma) \). I’ll actually prove more, that this embedding induces an isomorphism \( \text{Tor}_{U(n)}^1(U(n)/n^kU(n), V) \) with \( \text{Tor}_{U(n)}^1(U(n)/n^kU(n), \tilde{V}) \). In a later section I’ll generalize this to deal with arbitrary \( P \).

8. The asymptotics of matrix coefficients—[stub]

As with \( p \)-adic groups, there is an intimate relationship between the Jacquet module and the asymptotic behaviour of matrix coefficients.

9. References


