The Gamma function

I attempt here a somewhat unorthodox introduction to the Gamma function and related matters of harmonic analysis. If one asks, *What is the problem for which the Gamma function is the solution?*, there are a host of good answers. One of the principal roles it plays in current mathematics is as factors of zeta and $L$-functions contributed by real completions of number fields. This topic is most often treated along the lines of Tate’s thesis, and that is the main point of view I adopt here. But the Gamma function also occurs elsewhere in number theory, so I include some of the classical theory, too.

The first part is concerned with analysis related to Tate’s functional equations. The main result is the uniqueness of distributions that transform as characters under multiplication. This includes some simple analysis not far removed from elementary calculus. I recover Tate’s functional equation, as well as a second formula related to it that involves the Laplace transform.

The second part is concerned with classical formulas involving Gamma functions, including a discussion of how to compute explicit values.

My principal references here are [Schwartz:1965] and [Tate:1950/1967].

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Part I. Functional equations
1. Calculus exercises

The fundamental theorem of calculus tells us that if \( f \) is a smooth function in an open interval \( U \) around \( 0 \) then
\[
f(x) - f(0) = \int_0^x f'(s) \, ds.
\]
If we set \( s = tx \) this equation becomes
\[
f(x) = f(0) + x \int_0^1 f'(tx) \, dt = f(0) + x f_1(x).
\]
The integral
\[
f_1(x) = \int_0^1 f'(tx) \, dt
\]
defines a smooth function of \( x \). Induction and a simple calculation give us:

1.1. Proposition. If \( f \) is smooth in an interval \( U \) around \( 0 \) in \( \mathbb{R} \) then for every \( m \)
\[
f(x) = \sum_{j<m} \frac{x^j}{j!} f^{(j)}(0) + x^m f_m(x)
\]
with \( f_m \) smooth in \( U \).

The Schwartz space \( S(\mathbb{R}^n) \) is the space of all smooth functions \( f \) on \( \mathbb{R}^n \) such that
\[
\left| \frac{\partial^n f}{\partial x^n} \right| \ll (1 + |x|)^{-N} \quad (n = (n_i))
\]
for all \( n, N \geq 0 \) or, equivalently, for which
\[
\|f\|_{N,n} = \sup_{\mathbb{R}} (1 + |x|)^N \left| \frac{\partial^n f}{\partial x^n} \right| < \infty
\]
for all non-negative integers \( N, n \). It is a Fréchet space with these semi-norms.

The space \( S(\mathbb{R}^n) \) is taken into itself by polynomial multiplication. It thus becomes a module over \( \mathbb{C}[x_1, \ldots, x_n] \).

1.2. Proposition. Let \( m \) be the ideal of \( \mathbb{C}[x] \) generated by the \( x_i \) (\( i = 1, \ldots, n \) ). Then \( f \) in \( S(\mathbb{R}^n) \) lies in \( m^m S(\mathbb{R}^n) \) if and only if all derivatives of \( f \) of order less than \( m \) vanish at the origin.

Proof. One way is immediate. For the other, apply a double induction on \( n \) and \( m \). For \( n = 1 \), if \( f^{(k)}(0) = 0 \) for \( k < m \), then by the previous result \( f(x)/x^m \) is locally smooth. But it and all its derivatives clearly vanish rapidly at infinity, so it lies in \( S(\mathbb{R}) \).

Now assume true for dimension \( n - 1 \). Let \( \varphi(y) \) be a function of compact support on \( \mathbb{R} \) identically 1 near 0. As a consequence of the case \( n = 1 \) the function \( f(x) - f(x_1, \ldots, x_{n-1}, 0) \varphi(x_n) \) is divisible by \( x_n \), and hence
\[
f(x) = f(x_1, \ldots, x_{n-1}, 0) \varphi(x_n) + x_nf_n(x),
\]
with \( f_n(x) \) in \( S(\mathbb{R}^n) \). Suppose \( m = 1 \), so we assuming that \( f(0) = 0 \). The second term vanishes at 0, and to the first we may apply induction. Hence \( f \) lies in \( mS(\mathbb{R}) \).

For \( m > 1 \) we may apply double induction without trouble.

We shall need a variation of Proposition 1.1:
1.3. **Proposition.** If \( f \) is smooth in the interval \([0, x]\) then

\[
f(x) = f(0) + xf'(0) + \cdots + \frac{x^{p-2}}{(p-2)!} f^{(p-2)}(0) + \frac{x^{p-1}}{(p-1)!} f^{(p-1)}(0) + \int_0^x \int_0^{x_1} \cdots \int_0^{x_{p-1}} f^{(p)}(x_p) \, dx_p \, \cdots \, dx_1.
\]

**Proof.** We start with

\[
f(x) = f(0) + \int_0^x f'(x_1) \, dx_1.
\]

and then we extend this by repeating the same process with \( f'(x_1) \) etc. to get

\[
f(x) = f(0) + \int_0^x f'(x_1) \, dx_1 = f(0) + x f'(0) + \int_0^x \left( \int_0^{x_1} f''(x_2) \, dx_2 \right) \, dx_1
\]

... \[
= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \int_0^x \int_0^{x_1} f'''(x_3) \, dx_3 \, dx_2 \, dx_1
\]

... \[
= f(0) + x f'(0) + \cdots + \frac{x^{p-1}}{(p-1)!} f^{(p-1)}(0) + \int_0^x \int_0^{x_1} \cdots \int_0^{x_{p-1}} f^{(p)}(x_p) \, dx_p \, \cdots \, dx_1.
\]

If \( f \) is a function on the open interval \((0, \alpha)\), I write

\[
f(x) \sim c_0 + c_1 x + c_2 x^2 + \cdots
\]

if

\[
f(x) - (c_0 + c_1 x + \cdots + c_{m-1} x^{m-1}) = O(x^m)
\]

for every \( m \).

I define \( C^\infty[0, \alpha) \) to be the space of all \( f \) in \( C^\infty(0, \alpha) \) such each \( \lim_{\varepsilon \to 0} f^{(m)}(\varepsilon) \) exists.

1.4. **Corollary.** Suppose \( f \) in \( C^\infty(0, \alpha) \). The following are equivalent:

(a) the function \( f \) lies in \( C^\infty[0, \alpha) \);

(b) it is the restriction to \((0, \alpha)\) of a smooth function defined in \( C^\infty(-\alpha, \alpha) \);

(c) it has an asymptotic expansion

\[
f(x) \sim c_0 + c_1 x + c_2 x^2 + \cdots
\]

**Proof.** For the implication from (a) to (c), apply the Proposition to \( \varepsilon < \alpha \) instead of 0 and let it tend to 0.

To go from (c) to (b): according to a well known result attributed to Émile Borel, there exists a smooth function \( \varphi \) on all of \((-\alpha, \alpha)\) whose Taylor series at 0 is \( \sum c_i x^i \). But then the function that is \( \varphi \) in \((-\alpha, 0)\) and \( f \) on \([0, \alpha)\) is also smooth.

1.5. **Corollary.** Suppose \( f \) to be a function in \( C^{r+1}(0, \rho) \), such that for some \( \kappa_{r+1} \)

\[
|f^{(r+1)}(x)| \leq \frac{\kappa_{r+1}}{x^r}
\]
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for all \(0 < x \leq \rho\). Then

\[
    f_0 = \lim_{x \to 0} f(x)
\]

exists, and

\[
    |f_0| \leq C \kappa_{r+1} + \sum_{0 \leq k \leq r} \frac{\rho^k}{k!} |f^{(k)}(\rho)|
\]

for some \(C > 0\) independent of \(f\).

In effect, the function \(f(x)\) extends to a continuous (but not necessarily differentiable) function on all of \([0, \rho]\).

As an illustration of this result, let \(\ell(x) = x \log x - x\). We have on the one hand

\[
    \ell(x) = x \log x - x
\]

\[
    \ell'(x) = \log x
\]

\[
    \ell''(x) = \frac{1}{x}
\]

\[
    \ell'''(t) = -\frac{1}{x^2}
\]

\[
    \ell^{(p)}(x) = (-1)^p \frac{(p - 2)!}{x^{p-1}}
\]

\[
    \ell^{(r+1)}(x) = (-1)^{r+1} \frac{(r - 1)!}{x^r}
\]

and on the other \(\lim_{x \to 0} \ell(x) = 0\). The function \(\ell(x)\) will play a role in the proof of Corollary 1.5.

Proof. I begin by recalling the elementary criterion of Cauchy: If \(f(x)\) is continuous in \((0, \rho]\) then \(\lim_{x \to 0} f(x)\) exists if and only if for every \(\varepsilon > 0\) we can find \(\delta > 0\) such that \(|f(z) - f(y)| < \varepsilon\) whenever \(0 < y, z < \delta\).

The cases \(r = 0, r \geq 1\) are treated differently.

- **The case \(r = 0\).** By assumption, \(f\) is \(C^1\) on \((0, \rho]\) and \(f'\) is bounded by \(\kappa_1\) on that interval. For any \(y, z\) in \((0, \rho]\),

\[
    f(z) - f(y) = \int_y^z f'(x) \, dx, \quad |f(z) - f(y)| \leq \kappa_1 |z - y|.
\]

Therefore Cauchy’s criterion is satisfied, and the limit \(f_0 = \lim_{x \to 0} f(x)\) exists. Furthermore

\[
    f_0 = f(\rho) - \int_0^\rho f'(x) \, dx
\]

\[
    |f_0| \leq |f(\rho)| + \kappa_1 \rho.
\]

- **The case \(r > 0\).** If we apply Proposition 1.3 twice with a shift of origin, we get

\[
    f(z) = f(\rho) + (z - \rho) f'(\rho) + \cdots + \frac{(z - \rho)^r}{r!} f^{(r)}(\rho) + \int_\rho^z \cdots \int_{x_r}^{x_r} f^{(r+1)}(x_{r+1}) \, dx_{r+1} \cdots dx_1
\]

\[
    f(y) = f(\rho) + (y - \rho) f'(\rho) + \cdots + \frac{(y - \rho)^r}{r!} f^{(r)}(\rho) + \int_\rho^y \cdots \int_{x_r}^{x_r} f^{(r+1)}(x_{r+1}) \, dx_{r+1} \cdots dx_1
\]

\[
    f(z) - f(y) = \sum_{k=1}^r \frac{(z - \rho)^k - (y - \rho)^k}{k!} f^{(k)}(\rho) + \int_y^z \cdots \int_{x_r}^{x_r} f^{(r+1)}(x_{r+1}) \, dx_{r+1} \cdots dx_1
\]
In order to apply Cauchy’s criterion, we must now show how to bound
\[
\left| \int_y^z \cdots \int_r^{x_{r+1}} f^{(r+1)}(x_{r+1}) \, dx_{r+1} \cdots dx_1 \right| \leq \int_y^z \cdots \int_r^{x_{r+1}} \frac{\kappa_{r+1}}{x_{r+1}^{r+1}} \, dx_{r+1} \cdots dx_1.
\]
But we have already seen the iterated integral
\[
\int \cdots \int \frac{1}{x_{r+1}^{r+1}} \, dx_{r+1} \cdots dx_1.
\]
If we set \( f(x) = \ell(x) = x \log x - x \) in the calculation above we get
\[
\ell(y) = y \log y - y
\]
\[
= \int y \cdots \int r (r-1)! \, dx_{r+1} \cdots dx_1
\]
\[
+ \frac{(y-\rho)^r}{r!} \ell^{(r)}(\rho) + \frac{(y-\rho)^{r-1}}{(r-1)!} \ell^{(r-1)}(\rho) + \cdots + \ell(\rho),
\]
so that
\[
K_{y,\rho} = \int \cdots \int \frac{\rho (r-1)!}{x_{r+1}^{r+1}} \, dx_{r+1} \cdots dx_1
\]
\[
= \ell(y) - \frac{(y-\rho)^r}{r!} \ell^{(r)}(\rho) - \frac{(y-\rho)^{r-1}}{(r-1)!} \ell^{(r-1)}(\rho) - \cdots - \ell(\rho).
\]
Since \( \ell(x) \) is continuous on \([0, \rho]\) we may now apply Cauchy’s criterion in the other direction to see that the limit \( f_0 \) exists. Furthermore, the bound on \( f^{(r+1)} \) together with the equation for \( f(z) - f(y) \) enable us to see that
\[
|f_0| \leq \kappa_{r+1} |K_{0,\rho,r+1}| + \sum_{k=0}^r \frac{\rho^k}{k!} |f^{(k)}(\rho)|.
\]

2. The multiplicative Schwartz spaces

The space \( S(\mathbb{R}) \) contains as closed subspace the Schwartz space \( S(\mathbb{R}^\times) \) of functions whose Taylor series at 0 vanish identically. A well known theorem attributed to Émile Borel asserts that the sequence
\[
0 \rightarrow S(\mathbb{R}^\times) \rightarrow S(\mathbb{R}) \rightarrow \mathbb{C}[x] \rightarrow 0
\]
is exact.

Let \( D \) be the multiplicatively invariant derivative \( x \, d/dx \).

2.1. Proposition. Suppose \( f \) to be a function in \( C^\infty(\mathbb{R}) \). The following are equivalent:
(a) the function \( f \) lies in \( S(\mathbb{R}^\times) \);
(b) for all \( n \geq 0, N \in \mathbb{Z}, |x|^N f^{(n)}(x) \) is bounded;
(c) for all \( n \geq 0, N \in \mathbb{Z}, |x|^N D^n f(x) \) is bounded.

Proof. This follows from Proposition 1.1.

2.2. Corollary. For any \( s \in \mathbb{C} \) multiplication by \(|x|^s\) is an isomorphism of \( S(\mathbb{R}^\times) \) with itself.

Proof. This follows from Leibniz’s formula for \((x^s f)^{(n)}\).
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The space $S(\mathbb{R}^\times)$ contains the closed subspace $S(0, \infty)$ of functions vanishing on $(-\infty, 0]$, and similarly $S(-\infty, 0)$.

The multiplicative group $\mathbb{R}^\times > 0$ of positive real numbers acts on both of the spaces $S(\mathbb{R})$ and $S(0, \infty)$, as well as on their continuous linear duals, by the formulas:

$$\mu_a f(x) = f(a^{-1}x), \quad \langle \mu_a \Phi, f \rangle = \langle \Phi, \mu_a^{-1} f \rangle .$$

The scale factor $a^{-1}$ rather than $a$ has been chosen for compatibility with linear representations of non-abelian groups. The Lie algebra of $\mathbb{R}^\times > 0$ is spanned by the differential operator $D = x d/dx$, and the representations are smooth in the sense that

$$\lim_{h \to 0} \frac{\mu_{1+h} f - f}{h} = -Df$$

is again in $S(\mathbb{R})$. The $-$ sign here comes about because of the choice of $a^{-1}$ rather than $a$. It will continue to annoy.

According to Corollary 2.2, for every $s$ in $\mathbb{C}$ the integral

$$\langle \Phi_s, f \rangle = \int_0^\infty x^s f(x) \frac{dx}{x}$$

defines a continuous linear functional on $S(0, \infty)$—in effect a kind of distribution.

2.3. Proposition. The distribution $\Phi_s$ on $S(0, \infty)$ is an eigendistribution for $\mu_a$ with eigencharacter $a^{-s}$. Furthermore $D\Phi_s = s\Phi$.

Proof. We have

$$\langle \mu_a \Phi_s, f \rangle = \langle \Phi_s, \mu_a^{-1} f \rangle$$

$$= \int_0^\infty x^s f(ax) \frac{dx}{x}$$

$$= \int_0^\infty (y/a)^s f(y) \frac{dy}{y}$$

$$= a^{-s} \int_0^\infty y^s f(y) \frac{dy}{y}$$

$$= a^{-s} \langle \Phi_s, f \rangle$$

so that $\mu_a \Phi_s = a^{-s} \Phi_s$ as a distribution (as well as a function).

For the second claim:

$$\langle D\Phi_s, f \rangle = -\langle \Phi_s, Df \rangle$$

$$= -\int_0^\infty x^s f'(x) \, dx$$

$$= s \int_0^\infty x^{s-1} f(x) \, dx$$

$$= s \langle \Phi_s, f \rangle$$

There is a converse to this claim, and there is also a uniqueness theorem for eigendistributions. If $f$ lies in $S(0, \infty)$, its Mellin transform is

$$\hat{f}(s) = \langle \Phi_s, f \rangle .$$

It is uniformly bounded on any horizontally bounded strip $|\text{RE}(s)| \leq C$. It is also holomorphic in all of $\mathbb{C}$, and

$$D\hat{f} = s\hat{f}.$$
2.4. Proposition. The map \( f \mapsto \hat{f} \) is an isomorphism of \( \mathcal{S}(0, \infty) \) with \( \mathcal{M}(0, \infty) \).

Proof. One way is because \( D\Phi_s = s\Phi_s \). The other way involves a simple shift of contours of the inverse integral

\[
\hat{f}(s) = \frac{1}{2\pi i} \int_{\text{Re}(s) = 0} \hat{f}(s) x^{-s} ds.
\]

For any fixed \( a \) the image in \( \mathcal{M}(0, \infty) \) of multiplication by \( s - a \) is the subspace of \( F \) such that \( F(a) = 0 \), which is of codimension one. Hence the quotient \( \mathcal{S}(0, \infty)/(D - a)\mathcal{S}(0, \infty) \) is isomorphic to \( \mathbb{C} \). The space of distributions \( \Phi \) such that \( D\Phi = a\Phi \) is the dual of \( \mathcal{S}(0, \infty)/(D + a)\mathcal{S}(0, \infty) \), hence:

2.5. Corollary. The space of distributions on \( (0, \infty) \) such that \( D\Phi = s\Phi \) is spanned by \( \Phi_s \).

It is therefore also true that the space of distributions on \( (0, \infty) \) such that \( \mu_a\Phi = a^{-s}\Phi \) for all \( a \) in \( \mathbb{R}_+^\times \) is spanned by \( \Phi_s \).

Now suppose \( \chi \) to be a character of \( \mathbb{R}_+^\times \), which is to say a continuous homomorphism from \( \mathbb{R}_+^\times \) to \( \mathbb{C}_+^\times \). It will be of the form \( x \mapsto \text{sgn}(x)|x|^s \). The space \( \mathcal{S}(\mathbb{R}_+^\times) \) is the direct sum of \( \mathcal{S}(0, \infty) \) and \( \mathcal{S}(-\infty, 0) \), and \( \chi \) defines a \( \chi \)-equivariant distribution on \( \mathcal{S}(\mathbb{R}_+^\times) \):

\[
\langle \Phi_{\chi}, f \rangle = \int_{\mathbb{R}_+^\times} \chi(x)f(x) \frac{dx}{|x|}.
\]

The group \( \mathbb{R}_+^\times \) is the direct product \( \{ \pm 1 \} \times (0, \infty) \). The space of functions \( f \) in \( \mathcal{S}(\mathbb{R}_+^\times) \) such that \( \mu_c f = c^n f \) for \( c = \pm 1 \) is isomorphic to \( \mathcal{S}(0, \infty) \). Therefore Corollary 2.5 implies:

2.6. Corollary. The space of distributions on \( \mathcal{S}(\mathbb{R}_+^\times) \) such that \( \mu_t \Phi = \chi^{-1}(t)\Phi \) is spanned by \( \Phi_{\chi} \).

Something similar happens for \( \mathbb{C}_+^\times \). Here \( \mathcal{S}(\mathbb{C}_+^\times) \) is the subspace of functions in \( \mathcal{S}(\mathbb{C}) \) whose Taylor series at \( 0 \) vanish identically. On \( \mathbb{C} \) I use the invariant measure \( |dz \wedge d\overline{z}| \), which is \( 2\, dz\, dy \), and define the norm

\[
\|z\| = z\overline{z}.
\]

The space \( \mathcal{S}(\mathbb{C}_+^\times) \) is isomorphic to \( \mathbb{S} \times (0, \infty) \). The space of \( f \) in \( \mathbb{C}_+^\times \) such that \( \mu_c f = c^n f \) for \( c \) in \( \mathbb{S} \) is isomorphic to \( \mathcal{S}(0, \infty) \). Hence:

2.7. Corollary. If \( \chi \) is a character of \( \mathbb{C}_+^\times \), then the space of distributions on \( \mathcal{S}(\mathbb{C}_+^\times) \) such that \( \mu_t \Phi = \chi^{-1}(t)\Phi \) is spanned by the distribution

\[
\langle \Phi_{\chi}, f \rangle = \int_{\mathbb{C}_+^\times} \chi(z)f(z) \frac{|dz \wedge d\overline{z}|}{\|z\|}.\]
3. Multiplicatively equivariant distributions on \( \mathbb{R} \) and \( \mathbb{C} \)

In this section, let \( F \) be \( \mathbb{R} \) or \( \mathbb{C} \). In both cases, the norm \( \| c \| \) is the modulus of multiplication by \( c \), so that \( \text{meas}(c \Omega) = \| c \| \text{meas}(\Omega) \). If \( F = \mathbb{R} \) then \( \| c \| \) is the usual absolute value, but for \( \mathbb{C} \) it is \( c \bar{c} \).

Suppose \( \chi \) to be a unitary character of \( F^\times \), and for \( s \) in \( \mathbb{C} \) let \( \chi_s(c) = \chi(c) \| c \|^s \). For \( \Re(s) > 0 \) and \( f \) in the Schwartz space \( S(F) \) the integral

\[
\langle \chi_s, f \rangle = \int_F \chi(x) \| x \|^{s-1} f(x) \, dx
\]

converges and defines a tempered distribution that varies holomorphically in \( s \). Integration by parts allows us to see that this continues meromorphically to all of \( \mathbb{C} \) except for simple poles at a subset of non-positive half-integers.

As earlier, for \( c \) in \( F^\times \) define the operator

\[
[\mu_c f](x) = f(c^{-1} x)
\]

on functions, and dually on linear functionals

\[
\langle \mu_c \Phi, f \rangle = \langle \Phi, \mu_{1/c} f \rangle.
\]

Then

\[
\mu_c \chi_s = \chi_{-1}(c) \| c \|^{-s} \chi_s.
\]

The space \( S(F^\times) \) embeds as a closed subspace of \( S(F) \). According to Borel’s theorem, the quotient may be identified with the space of Taylor series \( \mathcal{E}_0 \) at the origin—\( \mathbb{C}[\llbracket x \rrbracket] \) if \( F = \mathbb{R} \) and \( \mathbb{C}[\llbracket \bar{z}, z \rrbracket] \) if \( F = \mathbb{C} \). We have therefore an exact sequence

\[
0 \to S(F^\times) \to S(F) \to \mathcal{E}_0 \to 0.
\]

A character \( \chi \) of \( F^\times \) defines a distribution on \( S(F^\times) \). It may not extend to an equivariant distribution on \( S(F) \). For example the integral

\[
\langle 1, f \rangle = \int_{\mathbb{R}^\times} f(x) \frac{dx}{\| x \|}
\]

defines a multiplicatively invariant distribution on \( \mathbb{R}^\times \), but it does not extend to one on all of \( \mathbb{R} \). Nonetheless, there is a multiplicatively invariant distribution on \( \mathbb{R} \), namely

\[
\delta_0: f \mapsto f(1).
\]

In this section, I shall explain how this illustrates a general fact:

**3.1. Theorem.** If \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \), then for each character \( \chi \) the space of distributions \( \Phi \) on \( S(F) \) such that

\[
\mu_t \Phi = \chi^{-1}(t) \Phi
\]

has dimension one.

The proof will take a while.

In his thesis Tate proved that for generic \( s \) the Fourier transform of \( \chi_s \) is a scalar multiple of \( \chi_{-1}^{-1(s)} \). For him, this was an explicit calculation, but [Weil:1966] pointed out that this can be deduced more conceptually from this Theorem, together with an observation about how the Fourier transform and the multiplicative groups interact. This idea has now become a standard tool in representation theory.
THE SCHWARTZ SPACE OF THE NON-NEGATIVE REALS. Define $S(0, \infty)$ to be the space of all functions $f$ in $C^\infty((0, \infty))$ for which (a) all $f^{(n)}$ decrease more rapidly at infinity than any $x^{-n}$, and (b) all derivatives $f^{(n)}(x)$ possess a limit as $x \to 0$.

It becomes a Fréchet space if assigned the norms
\[
\|f\|_n = \sup_{k, \ell \leq n} \|x^k f^{(\ell)}\| \quad (\|f\| = \sup_{x \geq 0} |f(x)|).
\]

Restriction to $(0, \infty)$ defines a continuous linear map from $S(R)$ to $S[0, \infty)$.

3.2. Lemma. The restriction map from $S(R)$ to $S[0, \infty)$ is surjective.

Proof. Suppose given $f$ in $S[0, \infty)$. It follows from Corollary 1.4 there exists a function $\varphi$ in $S(R)$ with the same Taylor series. The function that is $f$ on $[0, \infty)$ and $\varphi$ on $(-\infty, 0]$ lies in $S(R)$ and restricts to $f$ on $(0, \infty)$.

The sequence
\[
0 \to S(-\infty, 0) \to S(R) \to S[0, \infty) \to 0,
\]
is therefore exact. According to the Closed Graph Theorem, the surjection from $S(R)$ identifies $S[0, \infty)$ as a quotient topological vector space.

The space $S(0, \infty)$ is embedded as a closed subspace of $S[0, \infty)$ and the multiplicative group acts smoothly on both. The distribution $\Phi_s$ is well defined on $S(0, \infty)$. Does there exist an eigendistribution on $S[0, \infty)$ extending $\Phi_s$?

The key to understanding what happens is this result:

3.3. Lemma. For $\Re(s) > 0$, $f$ in $S(R)$
\[
\langle \Phi_s, f \rangle = \int_0^\infty x^s f(s) \frac{dx}{x} = \frac{(-1)^{n+1}}{s(s+1) \ldots (s+n)} \int_0^\infty x^{s+n} f^{(n+1)}(x) dx.
\]

Proof. Integration by parts give us
\[
\langle \Phi_s, f \rangle = \int_0^\infty x^s f(x) \frac{dx}{x} = \int_0^\infty x^{s-1} f(x) dx = \left[ \frac{f(x)x^s}{s} \right]_0^\infty - \frac{1}{s} \int_0^\infty x^s f'(x) dx = \frac{1}{s} \int_0^\infty x^{s+1} f'(x) \frac{dx}{x} = \frac{1}{s} \langle \Phi_{s+1}, f' \rangle
\]
and continuing:
\[
= \frac{1}{s(s+1)} \langle \Phi_{s+2}, f'' \rangle
\]
\[
= \frac{(-1)^{n+1}}{s(s+1) \ldots (s+n)} \langle \Phi_{s+(n+1)}, f^{(n+1)} \rangle.
\]
As a consequence, \( \Phi_s \) may be defined on \( \mathcal{S}[0, \infty) \) for all \( s \) not in \( -\mathbb{N} \). Thus for every one of these values of \( s \) we have an eigendistribution with eigencharacter \( x^{-s} \). Is it unique? What happens for \( s = -n \)? Set \( s = -n + h \) in the Lemma. We get

\[
(3.4) \quad \langle \Phi_s, f \rangle = \frac{-1}{(n-h)(n-1-h) \cdots (1-h)h} \int_0^\infty x^h f^{(n+1)}(x) \, dx.
\]

Thus \( (s+n)\langle \Phi_s, f \rangle \) as \( s \to -n \) (and \( h \to 0 \)) has limit

\[
-\frac{1}{n!} \int_0^\infty f^{(n+1)}(x) \, dx = \frac{f^{(n)}(0)}{n!}.
\]

The distribution \( \delta_0 \) is defined to take \( f \) to \( f(0) \). Its derivative \( \delta_0^{(n)} \) takes \( f \) to \((-1)^n f^{(n)}(0) \). The residue of \( \Phi_s \) at \( s = -n \) is therefore \((-1)^n \delta_0^{(n)}/n! \).

3.5. Lemma. The distribution \( \delta_0^{(n)} \) is an eigendistribution for the character \( a^n \).

Proof. Since \( f^{(n)}(ax) = a^n f^{(n)}(ax) \),

in other words, the distribution \( \Phi_s \) fails to be defined precisely when another eigencharacter arises.

3.6. Proposition. For every \( s \), the space of distributions \( \Phi \) such that \( D\Phi = s\Phi \) has dimension one.

It is possible to prove this by applying a version of the Mellin transform, but instead I’ll give a simpler cohomological argument. Consider the short exact sequence

\[
0 \to \mathcal{S}(0, \infty) \to \mathcal{S}[0, \infty) \to \mathbb{C}[[x]] \to 0.
\]

The last map is takes \( f \) to its Taylor series at 0, and is surjective by Borel’s Lemma. If \( T = D + sI \) this gives rise to a long exact sequence

\[
0 \to \mathcal{S}(0, \infty)(T) \to \mathcal{S}[0, \infty)(T) \to \mathbb{C}[[x]](T)
\]

\[
\to \mathcal{S}(0, \infty)/T \cdot \mathcal{S}(0, \infty) \to \mathcal{S}(0, \infty)/T \cdot \mathcal{S}(0, \infty) \to \mathbb{C}[[x]]/T \cdot \mathbb{C}[[x]] \to 0.
\]

Here \( V(T) \) is the subspace of \( v \) in \( V \) such that \( Tv = 0 \). The first two terms are always 0, since any function annihilated by \( D + a \) is a scalar multiple of \( x^{-a} \) and cannot decrease rapidly at \( \infty \).

The operator \( D + s \) takes

\[
c_0 + c_1 x + c_2 x^2 + \cdots \to ac_0 + c_1(1+s)x + c_2(2+s)x^2 + \cdots.
\]

When \( s \) does not belong to \( -\mathbb{N} \), \( D + s \) is an isomorphism of \( \mathbb{C}[[x]] \) with itself, and the third and sixth terms vanish. This proves the first assertion.

When \( s = -n \) the kernel of \( D + s \) is spanned by \( x^n \), and the cokernel is the space of power series with \( c_n = 0 \). To find the connecting map from third to fourth term, suppose \( f \) to be in \( \mathcal{S}(0, \infty) \) with Taylor series just \( x^n \).

The image of \( x^n \) is the image \( f - (D-n)f \) in \( \mathcal{S}(0, \infty) \) modulo \( (D-n)\mathcal{S}(0, \infty) \). But if \( f - (D-n)f = (D-n)g \) then \( f \) lies in \((D-a)\mathcal{S}(0, \infty) \) and hence has Taylor series identically 0.

Remark. There is a simple variant of the last part of the argument worth pointing out. Let’s prove that there is no distribution on \( [0, \infty) \) that is both \( \mathbb{R}^\times \)-invariant and which restricts to integration against \( 1/x \) on \( \mathbb{R}^\times \).

For this, suppose \( \Phi \) to be a distribution on \( \mathbb{R} \) that is invariant under multiplication by \( c \) in \( \mathbb{R}^\times \), and suppose that \( \Phi \) restricted to \( \mathcal{S}(0, \infty) \) is a constant times integration. Choose \( f \) to be a function in \( \mathcal{S}(\mathbb{R}) \) whose Taylor series is just the constant 1. Then \( \mu_c f = f \) lies in \( \mathcal{S}(\mathbb{R}^\times) \), and by invariance

\[
0 = \langle \Phi, \mu_c f - f \rangle = C \int_{\mathbb{R}^\times} f(1/x) - f(x) \, dx/|x|.
\]

But we can certainly find \( c \) and \( f \) for which the integral does not vanish, hence \( C = 0 \).
The full multiplicative group $\mathbb{C}^\times$ of this is called the finite part which always makes sense because the integrand is still a smooth function. For reasons we’ll see in a moment $S$ converges and defines a tempered distribution on $S(\mathbb{R})$. We can write

$$\langle \Phi_{s,m}, f \rangle = \int_{-\infty}^{\infty} f(x) |x|^s \text{sgn}^m(x) \frac{dx}{|x|}$$

for $\Re(s) > 0$ and $m = 0, 1$, which are again eigendistributions:

$$\mu_a \Phi_{s,m} = \text{sgn}^m(a)|a|^{-s}\Phi_{s,m}.$$

We can write

$$\int_{-\infty}^{\infty} f(x) |x|^s \text{sgn}^m(x) \frac{dx}{|x|} = (-1)^m \int_{-\infty}^{0} |x|^s f(x) \frac{dx}{|x|} + \int_{0}^{\infty} x^s f(x) \frac{dx}{x}$$

which means that $\Phi_{s,m}$ extends equivariantly and meromorphically to $S(\mathbb{R})$ over all of $\mathbb{C}$ with residue

$$((-1)^m + (-1)^n) \frac{\delta^{(n)}}{n!}$$

at $-n$. In particular, there is no pole if the parity of $m$ is different from the parity of $n$. In this case, because of (4.1) we get as value at $-n$

$$\langle \text{Pf}(1/x^{n+1}), f \rangle = \frac{1}{n!} \int_{0}^{\infty} \left[ \frac{f(n)(x) - f(n)\left(\frac{1}{x}\right)}{x} \right] \frac{dx}{x}$$

which always makes sense because the integrand is still a smooth function. For reasons we’ll see in a moment this is called the finite part of $1/x^{n+1}$. This defines an extension to $S(\mathbb{R})$ of the integral

$$\int_{\mathbb{R}} |x|^{-n-1} \text{sgn}^{n-1}(x) f(x) \frac{dx}{x} = \int_{\mathbb{R}} x^{-(n+1)} f(x) \frac{dx}{x}$$
on $\mathbb{R}^\times$.

The Schwartz space of the real line. The full multiplicative group $\mathbb{R}^\times$ acts on its own Schwartz space $S(\mathbb{R}^\times)$, the subspace of functions in $S(\mathbb{R})$ whose Taylor series at $0$ vanish. We now have distributions

$$\langle \Phi_{s,m}, f \rangle = \int_{-\infty}^{\infty} f(x) |x|^s \text{sgn}^m(x) \frac{dx}{|x|}$$

for $\Re(s) > 0$ and $m = 0, 1$, which are again eigendistributions:

$$\mu_a \Phi_{s,m} = \text{sgn}^m(a)|a|^{-s}\Phi_{s,m}.$$

We can write

$$\int_{-\infty}^{\infty} f(x) |x|^s \text{sgn}^m(x) \frac{dx}{|x|} = (-1)^m \int_{-\infty}^{0} |x|^s f(x) \frac{dx}{|x|} + \int_{0}^{\infty} x^s f(x) \frac{dx}{x}$$

which means that $\Phi_{s,m}$ extends equivariantly and meromorphically to $S(\mathbb{R})$ over all of $\mathbb{C}$ with residue

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at $-n$. In particular, there is no pole if the parity of $m$ is different from the parity of $n$. In this case, because of (4.1) we get as value at $-n$

$$\langle \text{Pf}(1/x^{n+1}), f \rangle = \frac{1}{n!} \int_{0}^{\infty} \left[ \frac{f(n)(x) - f(n)\left(\frac{1}{x}\right)}{x} \right] \frac{dx}{x}$$

which always makes sense because the integrand is still a smooth function. For reasons we’ll see in a moment this is called the finite part of $1/x^{n+1}$. This defines an extension to $S(\mathbb{R})$ of the integral

$$\int_{\mathbb{R}} |x|^{-n-1} \text{sgn}^{n-1}(x) f(x) \frac{dx}{x} = \int_{\mathbb{R}} x^{-(n+1)} f(x) \frac{dx}{x}$$
on $\mathbb{R}^\times$.

The complex numbers. For $n \in \mathbb{N}$ and $s \in \mathbb{C}$ define the character

$$\chi_{s,n}(z) = \left| z^{s/(|z|)} \right|^{-n}$$
of $\mathbb{C}^\times$. Every character of $\mathbb{C}^\times$ is equal to either some $\chi_{s,n}$. For $\Re(s) > 0$ and $f$ in $S(\mathbb{C})$ the integral

$$\langle \chi_{s,n}, f \rangle = \int_{\mathbb{C}} \left| z^{s-1/(|z|)} \right|^{-n} f(z) |dz| d\bar{z}$$
converges and defines a tempered distribution on $S(\mathbb{C})$. In polar coordinates, the integral becomes

$$\int_{\mathbb{C}} \left| z^{s-1/(|z|)} \right|^{-n} f(z) |dz| d\bar{z} = 2 \int_{0}^{\infty} r^{2s-1} \left( \int_{0}^{2\pi} e^{-in\theta} f(re^{-i\theta}) d\theta \right) dr$$

$$= 4\pi \int_{0}^{\infty} r^{2s-1} J_n(r) dr \left( J_n(r) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\theta} f(re^{-i\theta}) d\theta \right).$$

Suppose $n \geq 0$; the other case is similar. The projection $J_n(r)$ has a Taylor series of the form

$$z^n(c_0 + c_2|z| + c_4|z|^2 + \ldots)$$

so Lemma 3.3 tells us that this distribution can be meromorphically continued with poles in $-n - N$.

Uniqueness remains true, by the same cohomological reasoning as in earlier sections. The residues of the distribution are those distributions taking $f$ to some coefficient in the Taylor series of $f$ at 0.
I want now to look at (3.4) again. It can be rewritten and expanded in powers of $h$:

$$\langle \Phi_s, f \rangle = -\frac{1}{n!} \cdot \frac{1}{h} \cdot (1 + h \Lambda_n + O(h^2)) \cdot \left( \int_0^∞ f^{(n+1)}(x) \, dx + h \int_0^∞ (\log x) f^{(n+1)}(x) \, dx + O(h^2) \right)$$

with

$$\Lambda_n = 1 + 1/2 + 1/3 + \cdots + 1/n .$$

We have already seen that the leading term is $f^{(n)}(0)/n!$, and now we see that the second term in the expansion is

$$-\frac{1}{n!} \cdot \left( \Lambda_n f^{(n)}(0) + \int_0^∞ (\log x) f^{(n+1)}(x) \, dx \right) .$$

The integral can be expressed also as the limit as $\varepsilon \to 0$ of

$$\int_ε^∞ f^{(n+1)}(x) \log x \, dx = \left[ f^{(n)}(x) \log x \right]_ε^∞ - \int_ε^∞ \frac{f^{(n)}(x)}{x} \, dx$$

$$= -f^{(n)}(ε) \log ε - \int_ε^∞ \frac{f^{(n)}(x)}{x} \, dx$$

$$= -f^{(n)}(0) \log ε - \int_ε^∞ \frac{f^{(n)}(x)}{x} \, dx ,$$

since $f(ε) - f(0) = O(ε)$ and $\lim_{ε \to 0} ε \log ε = 0$. The second term in the Laurent expansion of $\langle \Phi_s, f \rangle$ at $s = -n$ is therefore also

$$\text{(4.1)} \quad \frac{1}{n!} \cdot \lim_{ε \to 0} \left( f^{(n)}(0) \log ε - \Lambda_n f^{(n)}(0) + \int_ε^∞ \frac{f^{(n)}(x)}{x} \, dx \right) .$$

In order to understand the nature of certain eigenfunctions of $D$ on $\mathbb{R}$, I now recall the notion of ‘parties finies’, introduced in [Hadamard:1923] in order to understand classical techniques for solving the wave equation in high dimensions.

The first important observation is that the Dirac distributions are eigendistributions. For $n \geq 0$

$$D δ_0^{(n)} = -n δ_0^{(n)} .$$

There is, however, another distribution $Φ$ such that $D Φ = -n Φ$.

**4.2. Proposition.** We have

$$\mu_a Pf(1/x^{n+1}) = a^n \text{sgn}(a) Pf(1/x^{n+1})$$

$$D Pf(1/x^{n+1}) = -n Pf(1/x^{n+1}) .$$

**Proof.** Left as exercise.

The two distributions $δ_0^{(n)}$ and $Pf(1/x^{n+1})$ span the space of eigendistributions $Φ$ on $\mathbb{R}$ such that $D Φ = -n Φ$, or (equivalently) $μ_0 Φ = a^n Φ$, but they are distinguished by what $μ_{-1}$ does to them:

$$μ_{-1} δ_0^{(n)} = (-1)^n δ_0^{(n)} , \quad μ_{-1} Pf(1/x^{n+1}) = -(1)^n Pf(1/x^{n+1}) .$$
This has to be, of course, since a cohomological argument like the one we saw earlier shows that there at most one \( \mathbb{R}^\times \)-equivariant extension to all of \( \mathcal{S}(\mathbb{R}) \) of the distribution which on \( \mathcal{S}(\mathbb{R} \times) \) is given by the formula

\[
\int_{-\infty}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx.
\]

I’ll say more here about the construction of \emph{parties finies} distributions. Suppose \( f \) in \( \mathcal{S}(\mathbb{R}) \), and let

\[
f(x) = f_0 + f_1 x + f_2 x^2 + \cdots
\]

be its Taylor series at 0, so \( f_m = f^{(m)}(0)/m! \). Then

\[
\varphi_n(x) = \frac{f - (f_0 + x f_1 + \cdots + f_n x^n)}{x^{n+1}}
\]

is still smooth throughout \( \mathbb{R} \), although no longer in \( \mathcal{S}(\mathbb{R}) \). Then

\[
\int_{\varepsilon}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx = \int_{\varepsilon}^{1} f(x) \frac{x}{x^{n+1}} \, dx + \int_{1}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx
\]

\[
= \int_{\varepsilon}^{1} f_0 + f_1 x + \cdots + f_n x^n \frac{x}{x^{n+1}} \, dx + \int_{\varepsilon}^{1} \varphi_n(x) \, dx + \int_{1}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx.
\]

The last integral is independent of \( \varepsilon \). As \( \varepsilon \to 0 \), the second integral has a finite limit. The first integral is

\[
\left[ -\frac{f_0}{n x^n} - \frac{f_1}{(n-1) x^{n-1}} - \cdots - \frac{f_n}{x} \log x \right]_{\varepsilon}^{1}
\]

\[
= -\frac{f_0}{n} - \frac{f_1}{(n-1)} - \cdots - \frac{f_{n-1}}{n} + \frac{f_0}{n \varepsilon^n} + \frac{f_1}{(n-1) \varepsilon^{n-1}} + \cdots + f_n \log \varepsilon
\]

Therefore the limit

\[
\lim_{\varepsilon \to 0} \left( \int_{\varepsilon}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx - \left( \frac{f_0}{n \varepsilon^n} + \frac{f_1}{(n-1) \varepsilon^{n-1}} + \cdots + f_n \log \varepsilon \right) \right)
\]

exists, and agrees with \( \text{Pf}(1/x^{n+1}) \).

The distribution \( \text{Pf}(1/x^{n+1}) \) on \([0, \infty)\) does not behave equivariantly with respect to scalar multiplication, because of the \( \log \varepsilon \) term. But on \( \mathbb{R} \) the finite part is

\[
\lim_{\varepsilon \to 0} \left( \int_{-\infty}^{\varepsilon} f(x) \frac{x}{x^{n+1}} \, dx + \int_{\varepsilon}^{\infty} f(x) \frac{x}{x^{n+1}} \, dx \right) - \left( \sum_{k=1}^{n} \sum_{\substack{k \text{ odd} \& \ k \varepsilon^k}} \frac{2 f_{n-k}}{k \varepsilon^k} \right),
\]

and it does behave well, because on \((-\infty, 0] \log \varepsilon \) is replaced by \( \log |\varepsilon| \).
5. The Gamma function

One function in \( S[0, \infty) \) is the restriction of \( f(x) = e^{-x} \) to \([0, \infty)\). The Gamma function is defined to be the integral

\[
\Gamma(s) = \int_{0}^{\infty} x^{s} e^{-x} \frac{dx}{x} = \langle \Phi_s, e^{-x} \rangle.
\]

for \( \Re(s) > 0 \). The argument extending \( \Phi_s \) in the last section is classical in this case. Since here \( f'(x) = -f(x) \), we have the functional equation

\[
\Gamma(s + 1) = s \Gamma(s)
\]

and since \( \Gamma(1) = 1 \), we see by induction that if \( s \) is a positive integer \( n \)

\[
\Gamma(n) = (n - 1)!
\]

The extension formula can be rewritten as

\[
\Gamma(s) = \frac{\Gamma(s + 1)}{s}
\]

so that we can extend the definition of \( \Gamma(s) \) to the region \( \Re(s) > -1 \), except for \( s = 0 \). And so on. More explicitly we have

\[
\Gamma(s) = \frac{\Gamma(s + n + 1)}{(s + n)(s + n - 1) \ldots (s + 1)s}
\]

which allows \( \Gamma(s) \) to be defined for \( \Re(s) > -n - 1 \), except at the negative integers, where it will have simple poles (of order one).

**Proposition.** For \( n \geq 0 \) the residue of \( \Gamma(s) \) at \( -n \) is \( (-1)^n/n! \)

Another formula for \( \Gamma(s) \) can be obtained by a change of variables \( t = \pi x^2 \):

\[
\Gamma(s) = 2\pi^s \int_{0}^{\infty} e^{-\pi x^2} x^{2s} \frac{dx}{x}
\]

which can also be written as

\[
\Gamma\left(\frac{s}{2}\right) = \pi^{s/2} \int_{-\infty}^{\infty} |x|^{s} e^{-\pi x^2} \frac{dx}{|x|}
\]

or

\[
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = \int_{-\infty}^{\infty} |x|^{s} e^{-\pi x^2} \frac{dx}{|x|}
\]

This function of \( s \) is often expressed as \( \zeta_R(s) \) because of its role in functional equations of \( \zeta \) functions.
6. Tate’s functional equation

For the moment, let $F$ be either $\mathbb{R}$ or $\mathbb{C}$. Here I shall introduce the Fourier transform and its interaction with the multiplicative group. In fact I shall eventually introduce two varieties of Fourier transform. The one I work with in this section takes functions on $F$ into other functions on $F$, while the one I use in the next, for $F = \mathbb{R}$, does something slightly different.

If $F$ is $\mathbb{R}$ let $\psi$ be the additive character $e^{2\pi ix}$, while on $\mathbb{C}$ take it to be $e^{2\pi i(z + \overline{z})}$. Recall that on $\mathbb{R}$ the measure $d\omega$ we are using is $dx$, while on $\mathbb{C}$ it is $|dz \wedge d\overline{z}|$.

For $f$ in $\mathcal{S}(F)$ its Fourier transform is

$$\hat{f}(\lambda) = \int_F f(x)\psi(-\lambda x) \, d\omega$$

and this defines an isomorphism of $\mathcal{S}(F)$ with itself. The inverse is

$$f(x) = \tilde{\psi}(x) = \int_F \hat{f}(x)\psi(\lambda x) \, d\omega$$

Another way to express this is that $\hat{\tilde{f}} = \mu_{-1}f$.

How do the Fourier transform and the multiplication operators interact?

6.1. Proposition. For $a \neq 0$

$$\mu_a f = |a| \mu_{a^{-1}} \hat{f}.$$

Proof. Because

$$\mu_a f(\lambda) = \int_F [\mu_a f](x)e^{-2\pi i\lambda x} \, dx$$

$$= \int_F f(a^{-1}x)e^{-2\pi i\lambda x} \, dx$$

$$= |a| \int_F f(y)e^{-2\pi i\lambda ay} \, dy$$

$$= |a| \mu_{a^{-1}} \hat{f}(\lambda).$$

The Fourier transform $\widehat{\Phi}$ of a distribution $\Phi$ is defined by

$$\langle \widehat{\Phi}, f \rangle = \langle \Phi, \hat{f} \rangle.$$

This, as an easy calculation will show, agrees with the definition of the Fourier transform on $\mathcal{S}(\mathbb{R})$. It can be rewritten as a version of the Plancherel formula:

$$\langle \Phi, f \rangle = \langle \Phi, \mu_{-1} \hat{f} \rangle.$$

Suppose $\chi$ to be a multiplicative character. The distribution $\Phi = \Phi_\chi$ is defined by

$$\langle \Phi_\chi, f \rangle = \int \chi(x)f(x) \, d\omega$$
The Gamma function defined by convergence for certain $\chi$ and extended meromorphically. What is the Fourier transform of $\Phi$?

Since $\mu_a \Phi, \chi = \chi^{-1}(a) \Phi, \chi$, we have

\[
\langle \mu_a \Phi, f \rangle = \langle \Phi, \mu_a^{-1} f \rangle
\]
\[
= \langle \Phi, [a]^{-1} \mu_a \hat{f} \rangle
\]
\[
= [a]^{-1} \chi(a) \langle \Phi, \hat{f} \rangle
\]
\[
= [a]^{-1} \chi(a) \langle \Phi, f \rangle
\]

so because of uniqueness $\hat{\Phi}$ must be a scalar multiple $\gamma \Phi, \chi$ where $\hat{\chi}(a) = \|a\|^{-1}$.

To calculate the scalar $\gamma$ explicitly, we calculate first the Fourier transform of some particular functions.

**THE REAL CASE.** The characters are of two kinds, depending on parity: $x \mapsto x^m \|x\|^s$ for $m = 0, 1$.

**6.2. Lemma.** The Fourier transform of $e^{-\pi x^2}$ is itself.

**Proof.** Let $f(x) = e^{-\pi x^2}$. Then

\[
\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-2\pi i \lambda x - \pi x^2} \, dx
\]
\[
= e^{-\pi \lambda^2} \int_{-\infty}^{\infty} e^{\pi \lambda^2 - 2\pi i \lambda x - \pi x^2} \, dx
\]
\[
= e^{-\pi \lambda^2} \int_{-\infty}^{\infty} e^{-\pi (x-i\lambda)^2} \, dx
\]
\[
= e^{-\pi \lambda^2} \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx
\]
\[
= e^{-\pi \lambda^2}.
\]

Let now $\chi(x) = |x|^s$, $\Phi = \Phi, \chi$. Then

\[
\langle \hat{\Phi}, e^{-\pi x^2} \rangle = \langle \Phi, e^{-\pi x^2} \rangle
\]
\[
= \int_{-\infty}^{\infty} [a]^s e^{-\pi x^2} \, dx
\]
\[
= \pi^{-s/2} \int_{-\infty}^{\infty} |x|^s e^{-\pi x^2} \, dx
\]
\[
= \pi^{-s/2} \Gamma(s/2)
\]
\[
= \zeta(s).
\]

Since $\hat{\chi} = |x|^{1-s}$:

**6.3. Proposition.** We have

\[
\hat{\Phi}(s, 0) = \gamma_s \Phi(1-s, 0)
\]

where

\[
\gamma_s = \frac{\zeta(s)}{\zeta(1-s)}.
\]
This formula isn’t quite right for values of $s$ where $\Phi_{s,0}$ or $\Phi_{1-s,0}$ have poles. The simplest way to formulate things is to observe that $\Phi_{s,0}/\zeta(s)$ is entire, and that this formula says that the Fourier transform of $\Phi_{s,0}/\zeta_R(s)$ is $\Phi_{1-s,0}/\zeta_R(1-s)$.

We can reason similarly for $|x|^s \text{sgn}(x)$ with $xe^{-\pi x^2}$.

6.4. Proposition. The Fourier transform of $xe^{-\pi x^2}$ is $-i\lambda e^{-\pi \lambda^2}$.

Proof. Differentiate the equation
\[ \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \lambda x} \, dx = e^{-\pi \lambda^2} \]
with respect to $\lambda$.

Therefore
\[ \langle \widetilde{\Phi}_{s,1}, xe^{-\pi x^2} \rangle = \langle \Phi_{s,1}, -ixe^{-\pi x^2} \rangle \]
\[ = -i \int_{\mathbb{R}} |x|^{s-1} \text{sgn}(x) xe^{-\pi x^2} \, dx \]
\[ = -i \int_{\mathbb{R}} |x|^s e^{-\pi x^2} \, dx \]
\[ = -i\zeta_R(1+s) \]
and hence:

6.5. Proposition. We have
\[ \widetilde{\Phi}_{s,1} = \lambda_s \Phi_{1-s,1} \]
where
\[ \lambda_s = -i \frac{L_R(s)}{L_R(1-s)}, \quad L_R(s) = \pi^{-(s+1)/2} \Gamma \left( \frac{s + 1}{2} \right). \]

I conclude with a useful calculation, then I examine some special cases.

6.6. Proposition. Suppose $\Phi$ to be a tempered distribution on $\mathbb{R}$. Then
(a) the Fourier transform of $\Phi'$ is $2\pi i \lambda \widetilde{\Phi}$;
(b) the Fourier transform of $x \Phi$ is $\widetilde{\Phi}' / (-2\pi i)$.

Proof. First assume $\Phi$ to be in $\mathcal{S}(\mathbb{R})$. The first assertion follows from integration by parts, the second by differentiating
\[ \int_{-\infty}^{\infty} \Phi(x) e^{-2\pi i \lambda x} \, dx = \widetilde{\Phi} (\lambda) \]
with respect to $\lambda$. Proving the assertion for distributions follows from this simpler case.

The distributions defined by integrals
\[ \int_{\mathbb{R}} x^n f(x) \, dx, \quad \int_{\mathbb{R}} x^n \text{sgn}(x) f(x) \, dx \]
are of particular importance.

6.7. Proposition. For $n \geq 0$
(a) the Fourier transform of \( x^n \) is \( \delta_0^{(n)}/(-2\pi i)^n \);
(b) the transform of \( x^n \text{sgn}(x) \) is
\[
\frac{2n!}{(2\pi i)^{n+1}} \text{Pf}(1/x^{n+1}).
\]

As for the first, calculation shows that the transform of 1 is \( \delta_0 \). But then by the previous lemma the transform of \( x^n \) is \( \delta_0^{(n)}/(-2\pi i)^n \).

For the second, we can write \( x^n \text{sgn}(x) \) as \( |x|^{n+1} \text{sgn}^{n+1}(x)/|x| \), so its transform will be an eigendistribution for \( |x|^{-n} \text{sgn}^{n+1} = x^n \text{sgn}(x) \), which means that it is a multiple of \( \text{Pf}(1/x^{n+1}) \). To compute the constant, let’s look at \( n = 0 \), where we want the Fourier transform of \( \text{sgn}(x) \) itself. Here
\[
\langle \text{sgn}, xe^{-\pi x^2} \rangle = \frac{-i}{\pi} = \frac{2}{2\pi i},
\]
so that the transform of \( \text{sgn} \) is \((2/2\pi i)\text{Pf}(1/x)\). Then
\[
\langle \text{Pf}(1/x), xe^{-\pi x^2} \rangle = 1
\]
since \( \text{Pf}(1/x)x' = -n \text{Pf}(1/x^{n+1}) \).

**THE COMPLEX CASE.** In this case the characters are of the form \( z^n \|z\|^n \) and \( z^n \|z\|^s \) with \( n \geq 0 \). Let
\[
f_n(z) = \begin{cases} 
  z^n e^{-2\pi i|x|} & \text{if } n \geq 0 \\
  \overline{z^n} e^{-2\pi i|x|} & \text{if } n < 0.
\end{cases}
\]

**6.8. Lemma.** The Fourier transform of \( f_n \) is \( i^n f_{-n} \).

**Proof.** I’ll first show this for \( f_0 \).
\[
\widehat{f_0}(\lambda) = \int e^{-2\pi i z} e^{-2\pi i(\lambda z + \overline{\lambda}\overline{z})} |dz \wedge d\overline{z}|
\]
\[
= e^{-2\pi |\lambda|} \int e^{-2\pi i z} \overline{\lambda} |dz \wedge d\overline{z}|
\]
\[
= f_0(\lambda).
\]
Note that the point of the extra factor of 2 in the measure \( |dz \wedge d\overline{z}| \) balances out to make
\[
\int e^{-2\pi i z} |dz \wedge d\overline{z}| = 2\pi \int e^{-2\pi r^2} 2\pi r dr = 1.
\]

But now apply \( D = (-1/2\pi)\partial/\partial\lambda \) to this equation to get
\[
[Df_0] = \text{the Fourier transform of } z f_0(z) \text{ evaluated at } \lambda = i\overline{\lambda} f_0(\lambda)
\]
and more generally
\[
[D^nf_0](\lambda) = \text{Fourier transform of } z^n f_0(z) \text{ evaluated at } \lambda = i^n\overline{\lambda} f_0(\lambda).
\]

Let
\[
\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s).
\]

**6.9. Proposition.** The Fourier transform of \( \Phi_\chi \) is \( \gamma(\chi)\Phi_{\chi^{-1}} \) with
\[
\gamma(\chi) = (-i)^{|n|} \frac{\Gamma_C(s)}{\Gamma_C(1 - s)}
\]
if \( \chi(z) = z^n \|z\|^n \) or \( z^n \|z\|^s \) with \( n \geq 0 \).

**Proof.** Apply the Lemma.
7. The Laplace transform

Much of this section is taken from §IX.3 of [Reed-Simon:1972].

It begins with a new version of the Fourier transform. For \( f \) in \( S(\mathbb{R}) \) and \( s \) in \( i\mathbb{R} \) define

\[
\hat{f}(s) = \int_{\mathbb{R}} f(x) e^{-sx} \, dx
\]

(7.1)

Of course \( \hat{f}(2\pi i\lambda) = \hat{f}(\lambda) \). The inverse transform is

\[
f(x) = \frac{1}{2\pi i} \int_{\text{Re}(s)=0} \hat{f}(s) e^{sx} \, ds.
\]

The point is that in certain circumstances this Fourier transform will be the boundary value of a Laplace transform holomorphic in the right half plane \( \text{Re}(s) > 0 \).

Tempered distributions with support on \([0, \infty)\) annihilate all of \( S(-\infty, 0) \) and may be identified with elements in the continuous linear dual of \( S[0, \infty) \). Suppose \( \Phi \) to have support on \([0, \infty) \). Since the restriction of \( e^{-sx} \) to \([0, \infty) \) lies in \( S[0, \infty) \) for \( \text{Re}(s) > 0 \), it makes sense to define the Laplace transform of \( \Phi \) to be the function

\[
\mathcal{L}\Phi(s) = \langle \Phi, e^{-sx} \rangle.
\]

(7.2)

This is defined for \( s \) in the positive right half plane. It is easy to see that \( \mathcal{L}\Phi \) is continuous, and from that to deduce by means of any one of several criteria that it is holomorphic. As we shall see, as \( \text{Re}(s) \to 0 \) it has the Fourier transform of \( \Phi \) as limiting value.

Recall that \( S[0, \infty) \) is a Fréchet space defined by the norms

\[
\|f\|_n = \sup_{k, \ell \leq n} \|x^k f^{(\ell)}\| \quad \left( \|f\| = \sup_{x \geq 0} |f(x)| \right).
\]

Since \( \Phi \) is a continuous linear function on \( S[0, \infty) \), there exists \( n \) such that

\[
\langle \Phi, f \rangle \ll_{\Phi, n} \|f\|_n.
\]

(7.3)

If \( f(x) = e^{-sx} \) then

\[
f^{(n)}(x) = (-s)^n e^{-sx}
\]

and the maximum value of \( |x^n e^{-sx}| \) is

\[
\left( \frac{n}{eS} \right)^n = \frac{e^{n\log n-n}}{\sigma^n} \quad (\sigma = \text{Re}(s)).
\]

(evaluated at \( x = (n/s) \)). The function \( x \log x - x \) is monotonic, and therefore we have proved:

**7.4. Proposition.** If \( \Phi \) is a tempered distribution with support on \([0, \infty) \) then

\[
\mathcal{L}\Phi(s) \ll_{\Phi} (1+|s|)^n \left( 1 + \frac{1}{\sigma^n} \right) \quad (\sigma = \text{Re}(s))
\]

for some \( m, n \).

That is to say, on any right half plane \( \text{Re}(s) \geq \sigma > 0 \) it is polynomially bounded, but it may not remain bounded near the imaginary axis. For example, the Laplace transform of the Heaviside function

\[
Y(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise.} \end{cases}
\]

is \( 1/s \).

The Fourier transform of a tempered distribution is defined by duality, but if \( F \) is a summable function then \( \mathcal{F}F \) is a continuous function for which (7.1) remains valid. This applies in particular to \( F \) with support in \([0, \infty) \). In these circumstances we can easily deduce a few properties of \( \mathcal{L}\Phi \):
The Gamma function

(a) the Laplace transform of \( F \) extends continuously to the imaginary axis and \( \mathcal{F}(s) = e^{s} \mathcal{F}(s) \) for \( \Re(s) = 0 \);
(b) more generally, the Fourier transform of \( F(x)e^{-\sigma x} \) is the function \( t \mapsto \mathcal{F}(\sigma + 2\pi it) \).

These properties remain valid in some sense for all distributions with support in \([0, \infty)\). It will take a while to justify this observation.

The first result is a partial converse to Proposition 7.4. Suppose \( \varphi(s) \) to be holomorphic in the region \( \Re(s) > 0 \) and satisfying the inequality

\[(7.5) \quad \varphi(s) \ll \varphi(1 + |s|)^m \left(1 + \frac{1}{\sigma^n}\right)\]

for some \( m, n \). Then the function

\[ \varphi_\sigma: t \mapsto \varphi(s + it) \]

is of moderate growth and hence defines a tempered distribution

\[ \langle \varphi_\sigma, f \rangle = \frac{1}{2\pi i} \int_\mathbb{R} \varphi(s + it) f(t) \, dt. \]

7.6. Theorem. In these circumstances
(a) the limit \( \lim_{\sigma \to 0} \varphi_\sigma \) exists and defines a tempered distribution \( \varphi_0 \);
(b) if \( \Phi \) is the inverse Fourier transform of \( \varphi_0 \) then it has support on \([0, \infty)\) and its Laplace transform is \( \varphi(s) \).

I’ll start the proof in a moment.

Proposition 7.4 tells us that for any tempered distribution \( \Phi \) with support on \([0, \infty)\) the Laplace transform \( e^s \Phi \) satisfies the hypotheses of Theorem 7.6. Hence:

7.7. Corollary. If \( \Phi \) is a tempered distribution with support in \([0, \infty)\) then for each \( \sigma > 0 \) the distribution \( e^{\sigma s} \Phi \) is the Fourier transform of \( e^{-\sigma x} \Phi \), and its limit as \( \sigma \to 0 \) is \( \Phi \).

Proof of Theorem 7.6 will take a while, and comes in a few steps. Assume \( \varphi \) given satisfying (7.5).

But now we can apply Corollary 1.5 to deduce what we want to know.

Step 1. Suppose given \( f \) in \( S(\mathbb{R}) \), and consider

\[ F(s) = \int_\mathbb{R} \varphi(s + it) f(t) \, dt \]

for \( s \) real and small. It is to be shown that the limit of \( F(s) \) exists as \( s \to 0 \), and that the limit depends continuously on \( f \). This is going to be an application of Corollary 1.5.

This involves derivatives of \( F \). We have

\[ f'(s) = \int_\mathbb{R} \frac{d}{ds} \varphi(s + it) f(t) \, dt \]
\[ = \int_\mathbb{R} \frac{1}{i} \frac{d}{dt} \varphi(s + it) f(t) \, dt \]
\[ = i \int_\mathbb{R} \varphi(s + it) f'(t) \, dt \]

and then

\[ F^{(p)}(x) = i^p \int_\mathbb{R} \varphi(s + it) f^{(p)}(t) \, dt. \]
Near \( s = 0 \) we have
\[
\varphi(s + it) \ll \frac{1 + t^n}{s^n}.
\]
Since \( f \) lies in \( S(\mathbb{R}) \) we therefore have
\[
\varphi(s + it) f^{m+1}(t) \ll \frac{1}{1 + t^n}
\]
for arbitrarily large \( N \), hence also
\[
F^{m+1}(s) \ll \frac{1}{s^n}.
\]
But this is exactly the hypothesis of Corollary 1.5, and we conclude that \( F(s) \) is continuous on \([0, \infty)\), and that the value of \( F(0) \) depends continuously on \( f \). Therefore \( \lim_{n \to 0} \varphi_\sigma \) exists and amounts to a tempered distribution \( \varphi_0 \).

**Step 2.** Let \( \Phi \) be the inverse Fourier transform of \( \varphi_0 \). It is next to be shown that \( \Phi \) has support on \([0, \infty)\), which means that
\[
\langle \Phi, f \rangle = 0
\]
whenever \( f \) is in \( C^\infty_0 \) with support on \((-\infty, 0]\).

Let \( \psi(s) \) be the Laplace transform of \( f \). It is holomorphic in all of \( \mathbb{C} \), and of uniform rapid decrease at infinity in the right half plane. Furthermore
\[
\langle \Phi, f \rangle = \frac{1}{2\pi i} \int_{\text{Re}(s) = \sigma} \varphi(s) \psi(-s) \, ds
\]
for any \( \sigma > 0 \). Using the uniform bounds and moving \( \sigma \) to \( \infty \), we see that \( \langle \Phi, f \rangle = 0 \).

**Step 3.** The last step is to show that \( \varphi_\sigma \) is the Fourier transform of \( \varphi_0 e^{-\sigma x} \). This is now easy.

8. Another formula for Tate’s factor

The Fourier transform of the tempered distribution

(8.1)
\[
\langle |\cdot|^s, f \rangle = \int \langle x \rangle^{s-1} f(x) \, dx
\]
is a scalar multiple \( \gamma(s) \) of \( |\cdot|^{1-s} \), and in a previous section we have seen Tate’s formula for it. But there is also a second formula for it, and a comparison of the two is instructive.

**8.2. Proposition.** We have
\[
\gamma(s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s).
\]

It is easy enough to derive this formula from well known formulas. As we shall see in a moment, it is equivalent to the combination of the reflection and duplication formulas for the Gamma function. But it also implies these classical formulas, and sheds some light on them. In this section I’ll elucidate this.

Because \( \hat{f}(x) = f(-x) \), we see from Proposition 8.2 that
\[
1 = \Gamma_G(1 - s) \Gamma_G(s)
\]
\[
= 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) \cdot 2(2\pi)^{s-1} \cos(\pi(1 - s)/2) \Gamma(1 - s)
\]
\[
= 2 \sin(\pi s/2) \cos(\pi s/2) \cdot \pi^{-1} \Gamma(s) \Gamma(1 - s)
\]
\[
= \sin(\pi s) \pi^{-1} \Gamma(s) \Gamma(1 - s)
\]
\[
\Gamma(s) \Gamma(1 - s) = \pi / \sin(\pi s).
\]
This is the reflection formula for $\Gamma(s)$.

Since Tate’s formula and that of Proposition 8.2 must agree:

$$2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s) = \frac{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}.$$ 

But by the reflection formula

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right) = \frac{\pi}{\sin(\pi(1-s)/2)} = \frac{\pi}{\cos(\pi s/2)}.$$ 

from which we deduce the duplication formula

$$2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s) = \pi^{1/2-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\cos(\pi s/2)\pi^{-1}$$

$$2(2\pi)^{-s}\Gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\cdot\pi^{-\frac{s}{2}}\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right).$$

**Remark.** In number theory, the duplication formula plays a role in establishing that the L-function of an induced representation is that of the inducing representation. This result for $\Gamma(s)$ has an analogue for $p$-adic fields known as the Hasse-Davenport relation, which is related to $p$-adic Gamma functions. The approach here is of interest since it reinforces the similarity of $\mathbb{R}$ and $\mathbb{Q}_p$.

Proposition 8.2 follows from the reflection and duplication formulas, but it is natural to ask whether there is a direct derivation, and this is a simple consequence of what I have said about the Laplace transform.

**Proof of Proposition 8.2.** The distribution (8.1) is the sum of two distributions, one with support on $(-\infty, 0]$ and the other with support on $[0, \infty)$. Its Fourier transform is a corresponding sum. Each of these pieces has a Laplace transform, one on the half plane $\text{RE}(s) > 0$, the other on $\text{RE}(s) < 0$, and the Fourier transform is the sum of their limits. The formula thus reduces to:

**8.3. Lemma.** The Laplace transform of the distribution

$$\langle \Phi_s, f \rangle = \int_0^\infty x^{s-1} f(x) \, dx$$

is $|\lambda|^{-s}\Gamma(s)$.

**Proof of the Lemma.** An easy calculation for $s$ real, which determines the formula everywhere since the Laplace transform is holomorphic.

There is another curious question that arises here. Formally, the Fourier transform of $|x|^{s-1}$ is given by the formula

$$G(\lambda) = \int_{\mathbb{R}} |x|^{s-1} e^{-2\pi i \lambda x} \, dx.$$ 

But the integrand converges near $0$ if and only if $\text{RE}(s) > 0$, while it apparently converges near $\infty$ if and only if $\text{RE}(s) < 0$. Still, it is natural to ask, can any meaning can be directly assigned to this expression? A peculiar answer to my question is suggested by the kind of regularization found in [Casselman:1993]. Suppose $c$ to be imaginary. The integral

$$\int_{\mathbb{R}} f(x)|x|^{s-1} e^{cx} \, dx$$

(8.4)
makes sense for all \( f \) in \( S(\mathbb{R}^\infty) \) and defines a continuous linear functional on this topological vector space. Question (1) asks whether it can be extended to allow \( f(x) = 1 \). This suggests the more general problem of extending it to all functions \( f(x) \) in \( \mathcal{C}^\infty(\mathbb{P}^1(\mathbb{R})) \). For this, express \( f \) as the sum of \( f_0 \) and \( f_\infty \), the first with support in some region \( |x| \leq R \), the second with support in some \( |x| \geq 1/S \). We must show how to evaluate both

(a) \( \int_{|x| \leq R} f_0(x)|x|^{s-1}e^{cx} \, dx \)

and

(b) \( \int_{|x| \geq 1/S} f_\infty(x)|x|^{s-1}e^{cx} \, dx \).

There is a difficulty because the first integral converges only for \( \text{RE}(s) > 0 \), the second only for \( \text{RE}(s) < 0 \). I deal with these in turn.

(a) Suppose \( f(x) \) to be in \( \mathcal{C}_c^\infty(\mathbb{R}) \), \( c \) an arbitrary complex number. The integral

\[
I_s = \int_{|x| \leq R} f(x)|x|^{s-1}e^{cx} \, dx
\]

converges for \( \text{RE}(s) > 0 \), and varies holomorphically with \( s \) in that region. But \( f(x)e^{cx} \) also lies in \( \mathcal{C}_c^\infty(\mathbb{R}) \) and by a familiar argument we have for \( \text{RE}(s) > 0 \)

\[
I_s = \left[ \frac{f(x)e^{cx}|x|^s}{s} \right]_0^R - \frac{1}{s} \int_0^R |x|^s(f(x)e^{cx})' \, dx
\]

\[
= -\frac{1}{s} \int_0^R |x|^s(f(x)e^{cx})' \, dx
\]

This defines \( I_s \) for \( \text{RE}(s) > -1 \), with a simple pole at \( s = 0 \), and we can continue in the same way to see that it is defined and holomorphic for all \( s \) except for simple poles in \( -\mathbb{N} \).

(b) This is more interesting. I adapt the argument in the introduction to [Casselman-Hecht-Miličić:2000]. Suppose \( f(x) \) to have support in \( |x| \geq 1/S > 0 \). Substituting \( y = 1/x \) and setting \( F(y) = f(1/y) \) the integral we are looking at becomes

\[
\int_{|y| \leq S} |y|^{-s-1}F(y)e^{c/y} \, dy.
\]

By assumption \( f(x) \) is smooth on \( \mathbb{P}^1(\mathbb{R}) \) is smooth, so \( F(y) \) lies in \( \mathcal{C}_c^\infty(\mathbb{R}) \). I am assuming that \( c \) is imaginary, and I now assume also that \( c \neq 0 \). The function \( e^{c/y} \) oscillates wildly near the origin but remains bounded, so it is at least plausible that integrating against a smooth function cancels out the oscillation. I’ll make this precise, again integrating by parts.

Replace \( s - 1 \) by \( s \) for convenience. Set

\[
\varphi_s = |x|^se^{cx}.
\]

Since the derivative of \( |x|^s \) is \( s|x|^s/x \),

\[
\varphi'_s = (s/x - c/x^2)\varphi_s
\]

\[
x^2\varphi'_s = (sx - c)\varphi_s
\]

\[
\varphi_s = \frac{1}{c} (sx\varphi_s - x^2\varphi'_s).
\]
Therefore
\[
\int_{-S}^{S} \varphi_s(x)f(x)\,dx = -\frac{1}{c} \int_{-S}^{S} \varphi'_s(x)x^2 f(x)\,dx + \frac{1}{c} \int_{-S}^{S} s x \varphi_s f(x)\,dx
\]
\[
= -\frac{1}{c} \left[ \varphi_s(x)x^2 f(x) \right]_{-S}^{S} + \frac{1}{c} \int_{-S}^{S} \varphi_s(x)(x^2 f(x))'\,dx + \frac{1}{c} \int_{-S}^{S} s x \varphi_s f(x)\,dx
\]
\[
= \frac{1}{c} \int_{-S}^{S} \varphi_s(x)(x^2 f'(x) + (s + 2)xf(x))\,dx.
\]
Set
\[
Df = x^2 f' + (s + 2)xf.
\]
Then \(D^n f\) vanishes of order \(n\) at 0, and repeated integration by parts expresses the integral as
\[
(\frac{1}{c})^n \int_{-S}^{S} \varphi_s(x)[D^n f](x)\,dx.
\]
This converges for \(\text{RE}(s) > -n - 1\), and hence the integral defines a holomorphic function of \(s\).

Part II. Classical formulas

9. The Beta function

The Gamma function appears in a wide variety of integration formulas. Many involve the Beta function
\[
B(u, v) = \frac{\Gamma(u) \Gamma(v)}{\Gamma(u + v)}.
\]

9.1. Proposition. We have
\[
B(u, v) = \int_{0}^{\pi/2} \cos^{2u-1}(\theta) \sin^{2v-1}(\theta)\,d\theta.
\]

Proof. Start with
\[
\Gamma(s) = 2 \int_{0}^{\infty} e^{-x^2} x^{2s-1}\,dx.
\]
Moving to two dimensions and switching to polar coordinates:
\[
\Gamma(u)\Gamma(v) = 4 \int_{s \geq 0, t \geq 0} e^{-s^2 - t^2} s^{2u-1} t^{2v-1} ds\,dt
\]
\[
= 4 \int_{r \geq 0, 0 \leq \theta \leq \pi/2} e^{-r^2} r^{2(u+v)-1} \cos^{2u-1} \theta \sin^{2v-1} \theta\,dr\,d\theta
\]
\[
= 4 \int_{r \geq 0} e^{-r^2} r^{2(u+v)-1}\,dr
\]
\[
= \Gamma(u + v)B(u, v).
\]

9.2. Corollary. We have
\[
\int_{0}^{\infty} \frac{t^\alpha}{(1 + t^2)^\beta}\,dt = \frac{1}{2} \frac{\Gamma \left( \frac{\alpha + 1}{2} \right) \Gamma \left( \frac{\beta - \alpha + 1}{2} \right)}{\Gamma(\beta)}
\]
Proof. If we change variables in the Proposition to \( t = \tan(\theta) \) we get
\[
\begin{align*}
\theta &= \arctan(t) \\
d\theta &= dt/(1+t^2) \\
\cos(\theta) &= 1/\sqrt{1+t^2} \\
\sin(\theta) &= t/\sqrt{1+t^2}
\end{align*}
\]
leading to
\[
\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta} \, dt = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\beta-\frac{\alpha+1}{2}\right)}{\Gamma(\beta)}.
\]
In particular
\[
\Gamma^2(1/2) = \int_{-\infty}^\infty \frac{dr}{1+r^2} = \pi, \quad \Gamma(1/2) = \sqrt{\pi}.
\]

CAUCHY’S FORMULA. One formula very useful in representation theory happens to be one of the very last ones in the remarkable memoir [Cauchy:1830].

9.3. Proposition. For \( \text{RE}(\alpha + \beta) > 1 \)
\[
\int_{\mathbb{R}} \frac{dx}{(1+ix)^\alpha(1-ix)^\beta} = \frac{\pi 2^{1-\alpha-\beta} \Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)\Gamma(\beta)}
\]

Proof. I adapt an idea to be found in [Garrett:]—that of seeing this as a special case of the Plancherel formula to the two functions
\[
\frac{1}{(1+ix)^\alpha}, \quad \frac{1}{(1+ix)^\beta}.
\]
To do this, we must first find the Fourier transform of \( 1/(1+ix)^\alpha \).

9.4. Lemma. If \( f(x) = 1/(1+ix)^\alpha \) then
\[
\hat{f}(\lambda) = \begin{cases} 0 & \text{if } \lambda > 0 \\
e^{2\pi\lambda}(2\pi)^\alpha (-\lambda)^{\alpha-1}/\Gamma(\alpha) & \text{otherwise.}
\end{cases}
\]

Proof of the Lemma. We get
\[
\int_{\mathbb{R}} (1+ix)^{-\alpha} e^{-2\pi s x} \, dx = \frac{e^{2\pi \lambda}}{i} \int_{\text{RE}(t)=1} t^{-\alpha} e^{-2\pi t} \, dy \quad (t = 1 + ix)
\]
\[
= \frac{e^{2\pi \lambda}}{2\pi i} \int_{\text{RE}(s)=2\pi} s^{-\alpha} e^{-s} \, ds \quad (s = 2\pi t)
\]
\[
= \frac{e^{-2\pi \mu}}{2\pi i} \int_{\text{RE}(s)=2\pi} s^{-\alpha} e^{s} \, dy \quad (\mu = -\lambda)
\]
To conclude the proof of the Proposition—the integral is the inverse Laplace transform of \( s^{-\alpha} \) evaluated at \( -\lambda \). We have seen this before—it is \( \lambda^{\alpha-1}/\Gamma(\alpha) \).
Therefore by the Plancherel formula our integral is
\[
\frac{(2\pi)^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty e^{-4\pi y} y^\alpha y^\beta dy = \frac{1}{4\pi} \frac{2^{\alpha+\beta} \pi^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2^{\alpha+\beta-2} (2\pi)^{\alpha+\beta-2}} \int_0^\infty e^{-y} y^{\alpha+\beta-2} dy = \frac{\pi 2^{1-\alpha-\beta} \Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}.
\]

10. The limit product formula

The exponential function \( e^{-t} \) can be approximated by finite products.

**Lemma.** For any real \( t \) we have
\[
e^{-t} = \lim_{n \to \infty} \left( 1 - \frac{t}{n} \right)^n.
\]

This can be seen most easily by taking logarithms since for \( 0 \leq t < n \)
\[
\log \left( 1 - \frac{t}{n} \right)^n = n \log \left( 1 - \frac{t}{n} \right) = n \left( -\frac{t}{n} - \frac{1}{2} \left( \frac{t}{n} \right)^2 - \frac{1}{3} \left( \frac{t}{n} \right)^3 - \ldots \right)
\]
\[
= -t - \frac{t^2}{2n} - \frac{t^3}{3n^2} - \ldots
\]

where
\[
T = \frac{t^2}{2n} + \frac{t^3}{3n^2} + \ldots,
\]
which converges to \( 0 \) as \( n \to \infty \).

Another way of putting this is to define
\[
\varphi_n(t) = \begin{cases} (1 - t/n)^n & 0 \leq t \leq n \\ 0 & t > n \end{cases}
\]

and then define for each \( n \) an approximation \( \Gamma_n(s) \) to \( \Gamma(s) \):
\[
\Gamma_n(s) = \int_0^\infty t^{s-1} \varphi_n(t) \, dt = \int_0^n t^{s-1} \left( 1 - \frac{t}{n} \right)^n \, dt
\]

On the one hand, this can be explicitly calculated through repeated integration by parts:
\[
\int_0^n t^{s-1} \left( 1 - \frac{t}{n} \right)^n \, dt = \frac{1}{s} \frac{n-1}{n(s+1)} \frac{n-2}{n(s+2)} \cdot \cdot \cdot \frac{1}{n(s+n-1)} \int_0^n t^{s+n-1} \, dt = \frac{n! n^s}{s(s+1) \ldots (s+n)}
\]

On the other, since for all fixed \( t \) the limit of \( \varphi_n(t) \) as \( n \to \infty \) is equal to \( e^{-t} \), and both \( \varphi_n(t) \) and \( e^{-t} \) are small at \( \infty \), this is at least plausible:
Proposition. For any \( s \) with \( \Re(s) > 1 \) the limit of \( \Gamma_n(s) \) as \( n \to \infty \) is equal to \( \Gamma(s) \). In other words, for any \( s \) in \( \mathbb{C} \)

\[
\Gamma(s) = \lim_{n \to \infty} \frac{n! \, n^s}{s(s+1) \cdots (s+n)}
\]

The Euler constant \( \gamma \) is defined to be the limit

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right) - \log n.
\]

The limit product formula implies immediately a limit formula for \( 1/\Gamma(s) \):

\[
\frac{1}{\Gamma(s)} = \lim_{n \to \infty} \left[ s \left( 1 + \frac{s}{1} \right) \left( 1 + \frac{s}{2} \right) \cdots \left( 1 + \frac{s}{n-1} \right) n^{-s} \right]
\]

but

\[
n^{-s} = e^{-s \log n} = e^{-s(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}) + s \gamma_n}
\]

where \( \gamma_n \to \gamma \). Therefore:

Proposition. The inverse Gamma function has the product expansion

\[
\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{i=1}^{\infty} \left( 1 + \frac{s}{n} \right) e^{-s/n}
\]

where \( \gamma \) is Euler’s constant.

The limit product formula also implies Legendre’s duplication formula:

\[
\Gamma \left( \frac{1}{2} \right) \Gamma(s) = 2^{s-1} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s+1}{2} \right)
\]

Explicitly

\[
\Gamma \left( \frac{s}{2} \right) = \lim_{n \to \infty} \frac{2^{n+1} n! \, n^{s/2}}{s(s+2) \cdots (s+2n)}
\]

\[
\Gamma \left( \frac{s+1}{2} \right) = \lim_{n \to \infty} \frac{2^{n+1} n! \, n^{s+1/2}}{(s+1)(s+3) \cdots (s+2n+1)}
\]

so

\[
2^s \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s+1}{2} \right) / \Gamma(s)
\]

\[
= \lim_{n \to \infty} \frac{2^n \, 2^{n+1} n! \, n^{s/2}}{s(s+2) \cdots (s+2n) \, (s+1)(s+3) \cdots (s+2n+1)} \frac{2^{n+1} n! \, n^{(s+1)/2}}{(2n)! \, (2n)^s}
\]

\[
= \lim_{n \to \infty} \frac{(n!)^2 \, 2^{2n+2} \, n^{1/2}}{(2n)! \, (s+2n+1)}
\]

\[
= \lim_{n \to \infty} \frac{(n!)^2 \, 2^{2n+1}}{(2n)! \, \sqrt{n}}
\]

but this last does not depend on \( s \), and is finite since the limit on the left hand side exists, so we may set \( s = 1/2 \) to see that it is equal to \( 2\sqrt{\pi} \).
11. The reflection formula

The formula for the Beta function gives us

\[ \Gamma(s)\Gamma(1-s) = \int_0^1 u^{s-1}(1-u)^{-s} \, du \]

\[ = \int_0^\infty \frac{v^{s-1}}{1+v} \, dv \quad (u = v/1+u, v = (u/1-u), du/(1-u) = (1+v)dv) \]

We can calculate this last integral by means of a contour integral in \( \mathbb{C} \). Let \( C \) be the path determined by these four segments: (1) along the positive real axis, or just above it, from \( \epsilon \) to \( R \); (2) around the circle of radius \( R \), counter-clockwise, to the point just below \( R \); (3) along and just below the real axis to \( \epsilon \); (4) around the circle of radius \( \epsilon \), clockwise, to just above \( \epsilon \). We want to calculate the limit of the integral

\[ \int_C \frac{z^{s-1}}{1+z} \, dz \]

as \( \epsilon \to 0 \) and \( R \to \infty \).

On the one hand the integrals over the different components converge to

\[ \int_0^\infty \frac{z^{s-1}}{1+z} \, dz + 0 - e^{2\pi is} \int_0^\infty \frac{z^{s-1}}{1+z} \, dz = (1-e^{2\pi is}) \int_0^\infty \frac{z^{s-1}}{1+z} \, dz \]

But on the other there is exactly one pole inside the curves \( C \), so the integral is also equal to \( -2\pi i e^{\pi is} \).

Therefore

\[ \Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{z^{s-1}}{1+z} \, dz = \frac{-2\pi ie^{\pi is}}{1-e^{2\pi is}} = \frac{\pi}{\sin \pi s} \]

Incidentally, combined with the product formula for \( \Gamma(s) \) this gives the product formula for \( \sin \pi s \)

\[ \sin \pi s = \pi s \prod_{n=1}^\infty \left( 1 - \frac{s^2}{n^2} \right) \]

12. The Euler-Maclaurin formula

Define a sequence of polynomials

\[ B_0(x) = 1 \]
\[ B_1(x) = x - 1/2 \]
\[ B_2(x) = x^2 - x + 1/6 \]

…

recursively determined by

\[ B'_{n+1}(x) = nB_n(x), \quad \int_0^1 B_n(x) \, dx = 0. \]

These are the Bernoulli polynomials. They determine in turn functions \( \psi_n \) by extension to all of \( \mathbb{R} \) of period 1.

The following is a simple version of the much more interesting Euler-Maclaurin sum formula:
Proposition. Suppose $f$ to be a function on the interval $[k, \ell]$ which has continuous second derivatives. Then

$$f(k) + f(k+1) + \ldots + f(\ell-1) = \int_k^\ell f(x) \, dx - \frac{1}{2} (f(\ell) - f(k)) + \frac{1}{12} (f'(\ell) - f'(k)) + R_2$$

where

$$R_2 = -\frac{1}{2} \int_k^\ell f''(x) \psi_2(x) \, dx.$$

The proof is very simple, a repetition of integration by parts. Suppose $m$ to be an integer with $f$ defined and continuously differentiable on $[m, m+1]$. Then since $\psi_0 = 1$ and $\psi'_1 = \psi_0$

$$\int_{m}^{m+1} f(x) \, dx = \int_{m}^{m+1} f(x)\psi_0(x) \, dx$$

$$= [f(x)\psi_1(x)]_{m}^{m+1} - \int_{m}^{m+1} f'(x)\psi_1(x) \, dx$$

$$= \frac{1}{2} (f(m) + f(m+1)) - \int_{m}^{m+1} f'(x)\psi_1(x) \, dx$$

since $\psi'_1(x) = 1$, and of course we look at the limit of $\psi_1$ from above at $m$, the limit from below at $m+1$.

Then we sum this equation over all the unit sub-intervals of $[k, \ell]$, using the periodicity of $\psi_1$.

$$\int_k^\ell f(x) \, dx = (1/2)f(k) + f(k+1) + \ldots + f(\ell-1) + (1/2)f(\ell) - \int_k^\ell f'(x)\psi_1(x) \, dx$$

We can rewrite this and apply integration by parts successively:

$$f(k) + f(k+1) + \ldots + f(\ell-1)$$

$$= \int_k^\ell f(x) \, dx - \frac{1}{2} (f(\ell) - f(k)) + \int_k^\ell f'(x)\psi_1(x) \, dx$$

$$= \int_k^\ell f(x) \, dx - \frac{1}{2} (f(\ell) - f(k)) + \frac{1}{2} (\psi_2(\ell)f'(\ell) - \psi_2(k)f'(k)) - \frac{1}{2} \int_k^\ell f''(x)\psi_2(x) \, dx$$

$$= \int_k^\ell f(x) \, dx - \frac{1}{2} (f(\ell) - f(k)) + \frac{1}{12} (f'(\ell) - f'(k)) - \frac{1}{2} \int_k^\ell f''(x)\psi_2(x) \, dx$$

The calculations can be continued to obtain an infinite asymptotic expansion involving the polynomials and their constant terms, the Bernoulli numbers.
We can evaluate $\Gamma(s)$ where

$$\Gamma(s) = \lim_{n \to \infty} \frac{(n-1)!}{s(s+1) \ldots (s+n-1)}$$

with

$$\Gamma(s) = \lim_{n \to \infty} \frac{(n-1)!n^s}{s(s+1) \ldots (s+n-1)}$$

We have proven this for $s$ except $s = -n$ with $n$ a non-negative integer, so that by the principle of analytic continuation it must be valid wherever $\Gamma(s)$ is defined. As a consequence

$$\log \Gamma(s) = \lim_{n \to \infty} S_{n-1}(1) - S_n(s) + s \log n$$

where

$$S_n(s) = \log s + \log(s+1) + \ldots + \log(s+n-1)$$

We can evaluate $S_n(s)$ by the Euler-Maclaurin formula

$$f(0) + f(1) + \ldots + f(n-1) = \int_0^n f(x) \, dx - \frac{1}{2} (f(n) - f(0)) + \frac{\beta_2}{2} (f'(n) - f'(0)) - \frac{1}{2} \int_0^n f''(x) \psi_2(x) \, dx$$

with

$$f(x) = \log(s + x), \quad f'(x) = \frac{1}{s + x}, \quad f''(x) = -\frac{1}{(s + x)^2}$$

so

$$\log s + \log(s+1) + \ldots + \log(s+n-1) = \int_0^n \log(s + x) \, dx - \frac{1}{2} \left[ \log(s+n) - \log s \right] + \frac{1}{12} \left[ \frac{1}{s+n} - \frac{1}{s} \right] + \frac{1}{2} \int_0^n \frac{\psi_2(x)}{(s + x)^2} \, dx$$

$$= [x \log x - x]_s^{s+n} - \frac{1}{2} \left[ \log(s+n) - \log s \right] + \frac{1}{12} \left[ \frac{1}{s+n} - \frac{1}{s} \right] + \frac{1}{2} \int_0^n \frac{\psi_2(x)}{(s + x)^2} \, dx$$

$$= (s + n - 1/2) \log(s + n) - (s - 1/2) \log s - n + \frac{1}{12} \left[ \frac{1}{s+n} - \frac{1}{s} \right] + \frac{1}{2} \int_0^n \frac{\psi_2(x)}{(s + x)^2} \, dx$$

and setting $s = 1, n - 1$ for $n$:

$$\log 1 + \log 2 + \ldots + \log n$$

$$= (n - 1/2) \log n - (n - 1) + \frac{1}{12} \left[ \frac{1}{n} - 1 \right] + \frac{1}{2} \int_1^n \frac{\psi_2(x)}{x^2} \, dx$$

$$= (n - 1/2) \log n - n + 1 + \frac{11}{12n} + \frac{1}{2} \int_1^n \frac{\psi_2(x)}{x^2} \, dx$$

$$= (n - 1/2) \log n - n + C + \frac{1}{12n} - \frac{1}{2} \int_1^n \frac{\psi_2(x)}{x^2} \, dx$$
where we define the constant

\[ C = \frac{11}{12} + \frac{1}{2} \int_1^\infty \frac{\psi_2(x)}{x^2} \, dx. \]

Taking limits, therefore

\[ \log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + C - \frac{1}{12s} - \frac{1}{2} \int_0^\infty \frac{\psi_2(x)}{(s + x)^2} \, dx. \]

This is valid for all \( s \) not on the negative real axis, and gives immediately the generalization of Stirling’s formula

\[ \Gamma(s) \sim e^{C} \left( \frac{s}{e} \right)^s \]

as \( s \) goes to infinity in any region

\[ -\pi + \delta \leq \arg(s) \leq \pi - \delta \]

since the remainder will have a uniform estimate in this region. The constant \( C \) can be evaluated by letting \( t \to \pm \infty \) in the reflection formula. On the one hand

\[
\begin{align*}
\Gamma(it)\Gamma(-it) &= -\pi \\
&= -\frac{2\pi i}{it[1/e - e^{\pi}]} \\
&\sim 2\pi t^{-1}e^{-\pi t}
\end{align*}
\]

while on the other

\[
\begin{align*}
\Gamma(it)\Gamma(-it) &\sim \frac{e^C}{\sqrt{it}} \left( \frac{it}{e} \right)^it \frac{e^C}{\sqrt{-it}} \left( \frac{-it}{e} \right)^{-it} \\
&= \frac{e^{2C}}{t} \left( i \right)^it \left( -i \right)^{-it} \\
&= \frac{e^{2C}}{t} e^{(it)(\pi i)/2} e^{(-it)(-\pi i)/2} \\
&= \frac{e^{2C}}{t} e^{-\pi t}
\end{align*}
\]

I recall that

\[ x^y = e^{y \log x} \]

where log is given its principal value. This gives

\[ C = \log \sqrt{2\pi} \]

and finally the explicit version

**Proposition. (Stirling’s asymptotic formula)** As \( s \) goes to \( \infty \) in the region

\[ -\pi + \delta \leq \arg(s) \leq \pi - \delta \]

we have the asymptotic estimate

\[ \Gamma(s) \sim \sqrt{\frac{2\pi}{s}} \left( \frac{s}{e} \right)^s \]
14. The volumes and areas of spheres

If we set \( s = 1 \) in the formula for \( \zeta_{\mathbb{R}} \) at the end of the last section, we get

\[
\pi^{-1/2} \Gamma \left( \frac{1}{2} \right) = \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx.
\]

The integral on the right cannot be evaluated as an improper integral, but there is a well known trick one can use to evaluate the infinite integral. We move into two dimensions. We can shift to polar coordinates and get

\[
\left( \int_{\mathbb{R}} e^{-\pi x^2} \, dx \right)^2 = \int_{\mathbb{R}^2} e^{-\pi (x^2+y^2)} \, dx \, dy
\]

\[
= \int_0^\infty \int_0^{2\pi} 2\pi re^{-\pi r^2} \, r \, d\theta \, dr
\]

\[
= \int_0^\infty 2\pi re^{-\pi r^2} \, dr
\]

\[
= \int_0^\infty e^{-\pi r^2} (2\pi r) \, dr
\]

\[
= \int_0^\infty e^{-s} \, ds
\]

\[
= 1,
\]

so \( \pi^{-1/2} \Gamma(1/2) = 1 \), and \( \Gamma(1/2) = \sqrt{\pi} \).

We can use this formula and the same trick to find a formula for the volumes of spheres in \( n \) dimensions. Let \( S_{n-1} \) be the volume of the unit sphere in \( \mathbb{R}^n \). Then

\[
\left( \int_{\mathbb{R}} e^{-\pi x^2} \, dx \right)^n = 1
\]

\[
= \int_{\mathbb{R}^n} e^{-\pi r^2} \, dx_1 \ldots dx_n
\]

\[
= \int_0^\infty S_{n-1} r^{n-1} e^{-\pi r^2} \, dr
\]

\[
= \int_0^\infty S_{n-1} r^n e^{-\pi r^2} \, dr
\]

\[
= S_{n-1} \left( \frac{n}{2} \right) \pi^{-n/2} \Gamma(n/2).
\]

\[
S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.
\]

For example, the area of the two-sphere in \( \mathbb{R}^3 \) is

\[
S_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\pi/2} = 4\pi.
\]

The volume of the \( n \)-ball of radius \( R \) in \( \mathbb{R}^n \) is

\[
V_n(R) = \int_0^R S_{n-1} r^{n-1} \, dr = \frac{S_{n-1} R^n}{n}.
\]
Part III. References

15. References


Chapter 1 of Book III introduces ‘finite parts’ of integrals. This notion is necessary in order to interpret the fundamental solutions to the wave equation in high dimensions.


8. Laurent Schwartz, Méthodes mathématiques pour les sciences physiques, Hermann, 1965. Also available in English. This is very readable, although with few rigorous proofs.
