Essays on the structure of reductive groups

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Finite root systems

This essay discusses some elementary points concerning finite integral root systems. My approach is slightly different from the standard one in [Bourbaki:1968], because that does not distinguish carefully between roots and coroots.

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1. Introduction

Root systems are a crucial tool in representation theory, because they describe the structure of reductive groups in all characteristics. I’ll motivate what’s to come by a simple example.

Let

\[ G = \text{GL}_3(\mathbb{C}) \]
\[ T = \text{subgroup of diagonal matrices in } G \]
\[ \mathfrak{g}, \mathfrak{t} = \text{their Lie algebras} . \]

The Lie algebra \( \mathfrak{g} \) is \( M_3(\mathbb{C}) \), and \( \mathfrak{t} \) is isomorphic to \( \mathbb{C}^3 \). The character group \( X^*(T) \) is the lattice of algebraic homomorphisms \( T \to \mathbb{C}^\times \), the cocharacter group that of algebraic homomorphisms \( \mathbb{C}^\times \to T \). Thre two are canonically dual. I wrote them additively. If \( x \) has diagonal entries \( x_i \), we have the characters

\[ x \mapsto x^{i_j} = x_i . \]

The group \( T \) acts by conjugation on \( \mathfrak{g} \), which decomposes into the direct sum of \( \mathfrak{t} \) and six eigenspaces of dimension one, on which \( T \) acts by the multiplicative characters

\[ x^{i_j} = x_i / x_j \quad (1 \leq i, j \leq 3, \ i \neq j) . \]

These make up the set \( \Sigma \) of roots \( \alpha_{i,j} = \varepsilon_i - \varepsilon_j \) of \( (G, T) \), which lie in \( X^*(T) \). Corresponding to each of these pairs is an embedding \( \varphi_{i,j} \) of \( \text{SL}_2(\mathbb{C}) \) into \( G \). For example, if \( i = 1, j = 3 \):

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\mapsto
\begin{bmatrix}
a & 0 & b \\
0 & 1 & 0 \\
c & 0 & d \\
\end{bmatrix} .
\]
If
\[ \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]
then its image \( \sigma_{i,j} \) with respect to \( \varphi_{i,j} \) lies in the normalizer \( N_G(T) \) of \( T \). The quotient \( N_G(T)/T \) is isomorphic to \( \mathfrak{S}_3 \). This embedding of \( \text{SL}_2 \) also gives rise to embeddings of \( \mathbb{C}^\times \) through composition
\[ x \mapsto \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \rightarrow T \subset G. \]
These make up the set \( \Sigma^\vee \) of \textbf{coroots} \( x^{\alpha_i^\vee} = x^{\alpha_i^\vee - \epsilon_j^\vee} \), lying in \( X^* (T) \). For every root \( \alpha \),
\[ \langle \alpha, \alpha^\vee \rangle = 2, \]
in effect since
\[ \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}^{-1} = \begin{bmatrix} 1 & x^2 \\ 0 & 1 \end{bmatrix}. \]
The matrices \( \sigma_{i,j} \) act as reflections on \( X^*(T) \):
\[ \sigma_{i,j}: \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i. \]
If \( V = \mathbb{R} \otimes X^*(T) \) and \( V^\vee = \mathbb{R} \otimes X_*(T) \) its dual, the quadruple \((V, \Sigma, V^\vee, \Sigma^\vee)\) is a root system.

I now look at this notion systematically.

2. Reflections

A \textbf{reflection} in a finite-dimensional vector space is a linear transformation that fixes vectors in a hyperplane, and acts on a complementary line as multiplication by \(-1\). Every reflection can be written as
\[ s_{f,f^\vee}: v \mapsto v - \langle v, f^\vee \rangle f \]
for some linear function \( f^\vee \neq 0 \) and vector \( f \) with \( \langle f, f^\vee \rangle = 2 \). This fixes points on the hyperplane \( f^\vee = 0 \) and takes \( f \) to \(-f\). The function \( f^\vee \) and the vector \( f \) are unique up to non-zero scalar.

I'll abbreviate \( s_{f,f^\vee} \) to \( s_f \) in circumstances when \( f \) determines \( f^\vee \).

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (2,2) node[midway, above] {\( \mathbb{R} \)};
  \draw[->] (0,0) -- (2,-2) node[midway, below] {\( \mathbb{S} \)};
  \draw[->] (0,0) -- (0,2) node[midway, right] {\( \mathbb{R} \)};
  \draw[->] (0,0) -- (0,-2) node[midway, left] {\( \mathbb{S} \)};
\end{tikzpicture}
\end{center}

2.1. \textbf{Lemma}. Suppose \( s \) to be a linear transformation fixing points on a hyperplane \( H \) through the origin. It is a reflection if and only if the induced transformation on \( V/H \) is scalar multiplication by \(-1\).

\textbf{Proof}. If \( s(v) = -v + h \) then \( s(v - h/2) = -(v - h/2) \).

Suppose now that \( V \) is given a Euclidean norm.

2.2. \textbf{Lemma}. The reflection \( s_{f,f^\vee} \) is orthogonal if and only if \( f \) is perpendicular to the hyperplane \( f^\vee = 0 \). In this case the function \( f^\vee \) is determined by the equation
\[ \langle v, f^\vee \rangle = 2 \left( \frac{v \cdot f}{f^\vee} \right). \]
Proof. The first assertion is immediate from the previous Lemma. Now suppose the reflection to be orthogonal. The orthogonal projection of \( v \) onto the line through \( f \) is

\[
v_{\parallel} = \left( \frac{v \cdot f}{f \cdot f} \right) f
\]

Then \( v_{\perp} = v - v_{\parallel} \) lies in the reflection hyperplane, \( v = v_{\perp} + v_{\parallel} \) is an orthogonal decomposition, and

\[
(2.3) \quad s(v) = v_{\perp} - v_{\parallel} = v - 2 \left( \frac{v \cdot f}{f \cdot f} \right) f
\]

is the reflection of \( v \).

2.4. Lemma. Suppose \( s \) and \( t \) to be two reflections fixing the same hyperplane, say with \( f^s = f^t = f^\vee \). Then the product \( st \) is the shear taking

\[
v \mapsto v + \langle v, f^\vee \rangle (f_s - f_t)
\]

Proof. Suppose \( s \) and \( t \) to be two reflections fixing the hyperplane \( f^\vee = 0 \). Then

\[
t(v) = v - \langle v, f^\vee \rangle f_t
\]

\[
st(v) = v - \langle v, f^\vee \rangle f_s - \langle v, f^\vee \rangle f_t + \langle f_t, f^\vee \rangle \langle v, f^\vee \rangle f_s
\]

\[
= v + \langle v, f^\vee \rangle (f_s - f_t),
\]

since \( \langle f_t, f^\vee \rangle = 2 \).

3. Definition of root systems

A \textbf{root system} in a real vector space \( V \) is specified by a finite subset \( \Sigma \) of \( V \) and a map \( \lambda \mapsto \lambda^\vee \) from \( \Sigma \) to the linear dual space \( V^\vee \) satisfying these conditions:

- for each \( \lambda \) and \( \mu \) in \( \Sigma \), \( \langle \lambda, \mu^\vee \rangle \) lies in \( \mathbb{Z} \);
- for each \( \lambda \) in \( \Sigma \), \( \langle \lambda, \lambda^\vee \rangle = 2 \);
- for each \( \lambda \) the reflection \( s_\lambda: v \mapsto v - \langle v, \lambda^\vee \rangle \lambda \)

takes \( \Sigma \) to itself.

That \( s_\lambda \) is in fact a reflection is a consequence of the second condition.

There are many different definitions of root systems in the literature, differing in most cases only in emphasis. Sometimes—notably when dealing with Kac-Moody algebras—the condition of finiteness is dropped, and what I call a root system in these notes would be called a \textbf{finite root system}. Sometimes the extra condition that \( \Sigma \) span \( V \) is imposed, but often in the subject one is interested in subsets of \( \Sigma \) which again give rise to root systems that do not possess this property even if the original does. Sometimes the vector space \( V \) is assumed to be Euclidean and the reflections orthogonal. The definition I have given is one that arises directly from the theory of reductive groups, but there is some justification in that theory for a Euclidean structure as well, since a semi-simple Lie algebra possesses a canonical invariant quadratic form (its Killing form). One virtue of not starting off with a Euclidean structure is that it allows one to keep in view generalizations relevant to Kac-Moody algebras, where the root system is not finite and no good inner product, let alone a Euclidean one, exists.

The elements of \( \Sigma \) are called the \textbf{roots} of the system, those of \( \Sigma^\vee \) its \textbf{coroots}. The \textbf{rank} of the system is the dimension of \( V \), and the \textbf{semi-simple rank} is that of the subspace \( V(\Sigma) \) of \( V \) spanned by \( \Sigma \). The system is called \textbf{semi-simple} if \( \Sigma \) spans \( V \).
The Weyl group of the system is the group $W$ generated by the reflections $s_{\lambda^\vee}$. We shall see that it is finite.

The root system is said to be reducible if $\Sigma$ is the disjoint union of two subsets $\Sigma_1$ and $\Sigma_2$ with $\langle \lambda, \mu^\vee \rangle = 0$ whenever $\lambda$ and $\mu$ belong to different components. Otherwise it is irreducible. We shall see that every system is a direct sum of irreducible ones.

I’ll write a root system symmetrically as a quadruple $(V, \Sigma, V^\vee, \Sigma^\vee)$, for reasons that will become apparent.

4. Simple properties

The simplest property of a root system is immediate, but worth noting:

4.1. Proposition. If $\lambda$ is a root so is $-\lambda$.

Proof. Since $s_{\lambda^\vee} \lambda = -\lambda$.

Next something slightly more interesting. Let $U = V(\Sigma)$.

If $\lambda$ is in $\Sigma$, its image $\lambda^\vee$ in $V^\vee$ may be restricted to $U$.

4.2. Lemma. This map from $\Sigma$ to the dual of $U$ is injective.

Thus $\Sigma^\vee$ may be identified with a subset of $U^\vee$.

Proof. Suppose $\langle \nu, \lambda^\vee \rangle = \langle \nu, \mu^\vee \rangle$ for all $\nu$ in $\Sigma$. Let $r = s_{\mu^\vee} s_{\lambda^\vee}$, which fixes points on the hyperplane $\lambda^\vee = \mu^\vee = 0$. According to Lemma 2.4 $r^n \lambda = \lambda + 2n(\mu - \lambda)$. If $\lambda \neq \mu$ the shear $r$ has infinite order and cannot take the finite set $\Sigma$ to itself.

There are two corollaries.

4.3. Proposition. The map $\lambda \mapsto \lambda^\vee$ is injective.

4.4. Proposition. The quadruple $(U, \Sigma, U^\vee, \Sigma^\vee)$ is a root system.

It is called the semi-simple root system associated to the original.

The group $W$ takes $U$ to itself. The image of $W$ in $GL(U)$ is finite, since it embeds into the group of permutations of $\Sigma$. Choose a Euclidean inner product $u \cdot v$ on $U$ left invariant by $W$. Corresponding to this is an isomorphism of $U$ with $U^\vee$. Let $\lambda^\circ$ be the element of $U$ that maps to $\lambda^\vee$. By definition

\begin{equation}
\mu \cdot \lambda^\circ = \langle \mu, \lambda^\vee \rangle
\end{equation}

for all $\mu$ in $\Sigma$.

4.6. Proposition. For every $\lambda$ in $\Sigma$

$$\lambda^\circ = \frac{2\lambda}{\lambda \cdot \lambda}.$$

Proof. This is Lemma 2.2.

Let

$$\text{RAD}(V, \Sigma) = \{ v \in V \mid \langle v, \lambda^\vee \rangle = 0 \text{ for all } \lambda \in \Sigma \} = (\Sigma^\vee)^\perp.$$

Its dimension is $\dim V - \dim V^\vee(\Sigma^\vee)$.

4.7. Proposition. The intersection of $\text{RAD}(V, \Sigma)$ with $V(\Sigma)$ is trivial, and

$$V = \text{RAD}(V, \Sigma) \oplus V(\Sigma).$$
Proof. By (4.5), if \(v\) lies in \(V(\Sigma)\) and \(\langle v, \lambda^\vee \rangle = 0\) for all \(\lambda\) then \(v \cdot \lambda = 0\) for all \(\lambda\). This proves the first assertion.

As for the second, because of what we have just seen it suffices to prove that the dimension of \(V(\Sigma)\) is at least that of \(V^\vee(\Sigma^\vee)\). But it follows from (4.5) that a linearly independent set of coroots maps to a linearly independent set of vectors in \(V(\Sigma)\).

4.8. Corollary. The Weyl group is finite.

Proof. It fixes all \(v\) in \(\text{rad}(V, \Sigma)\) and therefore embeds into the group of permutations of \(\Sigma\).

4.9. Corollary. The canonical map from \(V(\Sigma)\) to the dual of \(V^\vee(\Sigma^\vee)\) is an isomorphism.

4.10. Corollary. The set \(\Sigma\) is contained in a lattice of \(V(\Sigma)\).

Proof. Because it is contained in the lattice of \(v\) such that \(\langle v, \lambda^\vee \rangle\) is integral for all \(\lambda^\vee\) in some linearly independent subset of \(\Sigma^\vee\).

4.11. Corollary. The subset \(\Sigma^\vee\) of \(V^\vee\) defines a root system in \(V^\vee\).

It is called the dual root system.

As a group, the Weyl group of the root system \((V, \Sigma, V^\vee, \Sigma^\vee)\) is isomorphic to the Weyl group of its dual system, because:

4.12. Proposition. If \(\lambda, \lambda^\vee\) are any vectors in \(V, V^\vee\) with \(\langle \lambda, \lambda^\vee \rangle = 2\), the contragredient of \(s_{\lambda^\vee, \lambda}\) is \(s_{\lambda, \lambda^\vee}\).

Proof. It has to be shown that

\[\langle s_{\lambda^\vee, \lambda} u, v \rangle = \langle u, s_{\lambda^\vee, \lambda} v \rangle.\]

The first is

\[\langle u - \langle u, \lambda^\vee \rangle \lambda, v \rangle = \langle u, v \rangle - \langle u, \lambda^\vee \rangle \langle \lambda, v \rangle\]

and the second is

\[\langle u, v - \langle \lambda, v \rangle \lambda^\vee \rangle = \langle u, v \rangle - \langle \lambda, v \rangle \langle u, \lambda^\vee \rangle.\]

4.13. Proposition. For all roots \(\lambda\) and \(\mu\)

\((s_{\lambda^\vee, \lambda})^\vee = s_{\lambda^\vee, \lambda^\vee}^\vee.\)

Proof. It must be checked that

\[\langle \nu, (s_{\lambda^\vee, \lambda})^\vee \rangle = \langle \nu, s_{\lambda^\vee, \lambda^\vee}^\vee \rangle\]

for all \(\nu\) in \(\Sigma\). Apply Proposition 4.6 a couple of times.

4.14. Corollary. For any roots \(\lambda, \mu\) we have

\[s_{s_{\lambda^\vee, \lambda}} = s_{\lambda^\vee, \lambda} s_{\lambda^\vee, \lambda}^\vee.\]

Proof. The algebra becomes simpler if one separates this into two halves: (a) both transformations take \(s_{\lambda^\vee, \lambda}\) to \(-s_{\lambda^\vee, \lambda}\); (b) if \(\langle \nu, (s_{\lambda^\vee, \lambda})^\vee \rangle = 0\), then both take \(v\) to itself. Verifying these, using the previous formula for \((s_{\lambda^\vee, \lambda})^\vee\), is straightforward.

Any set of roots \(\Sigma\) is the union of mutually orthogonal subsets \(\Sigma_i\), each irreducible. The following is straightforward.

4.15. Proposition. If \(\Sigma\) is the union of mutually orthogonal subsets \(\Sigma_i\), then \(V(\Sigma) = \oplus V(\Sigma_i)\).
5. More about orthogonality

TITS’ CRITERION FOR CORoots. Proposition 4.6 characterizes $\lambda^\vee$ in terms of an invariant inner product. But there is another way to specify $\lambda^\vee$ in terms of $\lambda$, one which works even for infinite root systems with no invariant inner product. The following observation can be found in [Tits:1966].

5.1. Corollary. The coroot $\lambda^\vee$ is the unique element of $V^\vee$ satisfying these conditions:

(a) $\langle \lambda, \lambda^\vee \rangle = 2$;
(b) it lies in the subspace of $V^\vee$ spanned by $\Sigma^\vee$;
(c) for any $\mu$ in $\Sigma$, the sum $\sum_{\nu} \langle \nu, \lambda^\vee \rangle$ over the $\lambda$-string $(\mu + \mathbb{Z} \lambda) \cap \Sigma$ vanishes.

Proof. The necessity of (a) and (b) is immediate. As for (c), it is true because the reflection $s_\lambda$ preserves $(\mu + \mathbb{Z} \lambda) \cap \Sigma$ and $\langle s_\lambda \nu, \lambda^\vee \rangle = -\langle \nu, \lambda^\vee \rangle$.

To prove that these conditions imply uniqueness, suppose $\ell^\vee$ to be another vector satisfying the same conditions, and let $\nu^\vee = \ell^\vee - \lambda^\vee$. Suppose $\mu$ to be a root. It is immediate that $\langle \lambda, \nu^\vee \rangle = 0$. The sum over the $\lambda$-string through $\mu$ is on the one hand zero, but on the other a positive multiple of $\langle \mu, \nu^\vee \rangle$. 

To show how this can be used, I’ll reprove Proposition 4.13. According to Tits’ criterion, it must be shown that

(a) $\langle s_\lambda \mu, s_\lambda \nu^\vee \rangle = 2$;
(b) $s_\lambda \nu^\vee$ is in the linear span of $\Sigma^\vee$;
(c) for any root $\tau$ we have $\sum_{\nu} \langle \nu, s_\lambda \nu^\vee \rangle = 0$

where the sum is over roots in over $\nu$ in $(\tau + \mathbb{Z} s_\lambda \mu)$.

Condition (a) is immediate, since $s_\lambda \nu^\vee$ is the contragredient of $s_\lambda$. Condition (b) is trivial. As for (c), we know that

$$\sum_{\nu \in (\chi + \mathbb{Z} \mu) \cap \Sigma} \langle \nu, \mu^\vee \rangle = 0$$

for all roots $\chi$. But

$$\langle \nu, \mu^\vee \rangle = \langle s_\lambda \nu, s_\lambda \nu^\vee \rangle$$

and if we replace $\chi$ by $s_\lambda \tau$ we obtain equation (c).

THE CANONICAL INVARIANT NORM. Assume $(V, \Sigma)$ to be a root system. One can define a canonical semi-Euclidean structure on $V$, with respect to which the root reflections will be orthogonal. First define the linear map

$$\rho: V \rightarrow V^\vee, \quad v \mapsto \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle \lambda^\vee$$

and define a symmetric dot product on $V$ by the formula

$$u \cdot v = \langle u, \rho(v) \rangle = \sum_{\lambda \in \Sigma} \langle u, \lambda^\vee \rangle \langle v, \lambda^\vee \rangle .$$

The semi-norm

$$\|v\|^2 = u \cdot v = \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle^2$$

is positive semi-definite, vanishing precisely on $\text{rad}(V, \Sigma)$.

In particular $\|\lambda\| > 0$ for all roots $\lambda$. Since $\Sigma^\vee$ is $W$-invariant, the semi-norm $\|v\|^2$ is also $W$-invariant. That $\|v\|^2$ vanishes on $\text{rad}(V, \Sigma)$ mirrors the fact that the Killing form of a reductive Lie algebra vanishes on the radical of the algebra.
5.2. **Proposition.** For every root \( \lambda \)
\[
\|\lambda\|^2 \lambda^\vee = 2\rho(\lambda).
\]

Thus although the map \( \lambda \mapsto \lambda^\vee \) is not the restriction of a linear map, it is simply related to such a restriction.

**Proof.** For every \( \mu \in \Sigma \)
\[
\begin{align*}
    s_{\lambda^\vee} \mu^\vee &= \mu^\vee - (\lambda, \mu^\vee)\lambda^\vee \\
    (\lambda, \mu^\vee)\lambda^\vee &= \mu^\vee - s_{\lambda^\vee} \mu^\vee \\
    (\lambda, \mu^\vee)^2 \lambda^\vee &= (\lambda, \mu^\vee)\mu^\vee - (\lambda, \mu^\vee)s_{\lambda^\vee} \mu^\vee \\
    &= (\lambda, \mu^\vee)\mu^\vee + (s_{\lambda^\vee} \mu^\vee)s_{\lambda^\vee} \mu^\vee \\
    &= (\lambda, \mu^\vee)\mu^\vee + (s_{\lambda^\vee} \mu^\vee)s_{\lambda^\vee} \mu^\vee
\end{align*}
\]
But since \( s_{\lambda^\vee} \) is a bijection of \( \Sigma^\vee \) with itself, we can conclude by summing over \( \mu \) in \( \Sigma \).

6. **Other characterizations of root systems**

Given a root system, we know that we can find a Euclidean structure on \( V \) such that for every \( \lambda \) in \( \Sigma \):

(a) \( 2(\lambda \cdot \mu)/(\lambda \cdot \lambda) \) is integral;
(b) the subset \( \Sigma \) is stable under the orthogonal reflection
\[
s_{\lambda^\vee}: v \mapsto v - 2 \left( \frac{v \cdot \lambda}{\lambda \cdot \lambda} \right) \lambda.
\]

Conversely, suppose that we are given a vector space \( V \) with a Euclidean norm on it, and a finite subset \( \Sigma \). For each \( \lambda \) in \( \Sigma \), let
\[
\lambda^\varphi = \left( \frac{2}{\lambda \cdot \lambda} \right) \lambda.
\]
This satisfies the equation
\[
(v \cdot \lambda^\varphi)\lambda = (v \cdot \lambda)\lambda^\varphi.
\]
For each \( \lambda \) in \( \Sigma \) define \( \lambda^\vee \) in \( V^\vee \) by the formula
\[
\langle v, \lambda^\vee \rangle = v \cdot \lambda^\varphi.
\]

6.4. **Proposition.** Suppose \( \Sigma \) to be a finite subset of \( V \), and suppose that for each \( \lambda \) in \( \Sigma \)

(a) \( \mu \cdot \lambda^\varphi \) is integral for every \( \mu \) in \( \Sigma \);
(b) the subset \( \Sigma \) is stable under the orthogonal reflection
\[
s_{\lambda^\vee}: v \mapsto v - (v \cdot \lambda^\varphi)\lambda.
\]
Then \( (V, \Sigma, V^\vee, \Sigma^\vee) \) is a root system.

Here \( \lambda^\varphi \) and \( \lambda^\vee \) are defined by (6.1) and (6.3).

This is straightforward to verify. The point is that by using the metric we can avoid direct consideration of \( \Sigma^\vee \). Here is one reward:

6.5. **Proposition.** Suppose \( (V, \Sigma) \) to be root system with invariant inner product. Suppose \( U \) to be a vector subspace of \( V \). \( \Sigma_U = \Sigma \cap U \), \( \Sigma_U^\vee = (\Sigma_U)^\vee \). Then \( (V, \Sigma_U, V^\vee, \Sigma_U^\vee) \) is a root system.
Proof. This is immediate from Proposition 6.4.

In practice, root systems can be constructed from a very small amount of data. If \( S \) is a finite set of orthogonal reflections and \( \Xi \) a finite subset of \( V \), the saturation of \( \Xi \) with respect to \( S \) is the smallest subset of \( V \) containing \( \Xi \) and stable under \( S \).

**6.6. Proposition.** Suppose \( \Xi \) to be a finite subset of a lattice \( L \) in the Euclidean space \( V \) such that \( \alpha \cdot \beta^\diamond \) is in \( \mathbb{Z} \) for every \( \alpha, \beta \) in \( \Xi \). Then the saturation \( \Sigma \) of \( \Xi \) with respect to the \( s_\alpha \) for \( \alpha \) in \( \Xi \) is finite, and (a) and (b) of Proposition 6.4 are satisfied for all \( \lambda \) in \( \Sigma \).

Proof. Let \( S \) be the set of reflections

\[
s_\alpha : v \mapsto v - (v \cdot \alpha^\diamond)\alpha
\]

for \( \alpha \) in \( \Xi \). Let \( \Xi_0 = Xi \) and \( \Xi_{n+1} = \Xi_n \cap S \cdot \Xi_n \). By induction, each \( \Xi_n \) is contained in \( L \), and all elements of \( \Xi_n \) are of bounded length. Therefore the sequence \( \Xi_n \) is stationary. Let \( \Sigma = \Xi_n \) for \( n > 0 \).

At this point we know that (1) \( s \Sigma = \Sigma \) for all \( s \) in \( S \); (2) every \( \lambda \) in \( \Sigma \) is an integral linear combination of elements of \( \Xi \); (3) \( \lambda \cdot \alpha^\diamond \) is integral for all \( \alpha \) in \( \Xi \) and \( \lambda \) in \( \Sigma \); (4) every \( \lambda \) in \( \Sigma \) is obtained by a chain of reflections in \( S \) from an element of \( \Xi \).

Define the depth of \( \lambda \) to be the least \( n \) such that \( \lambda \) lies in \( \Xi_n \). To prove that \( \lambda \cdot \mu^\diamond \) is integral for all \( \lambda, \mu \) in \( \Sigma \), it suffices to prove that \( \alpha \cdot \mu^\diamond \) is integral for \( \alpha \) in \( \Xi \) and \( \mu \) in \( \Sigma \). I do this by induction on the depth of \( \mu \). For \( n = 0 \) this is the basic hypothesis. Since

\[
(s_\alpha \mu)^\diamond = \frac{2 s_\alpha \mu}{\|s_\alpha \mu\|^2} = \frac{2 \mu - (\mu \cdot \alpha^\diamond) \alpha}{\|\mu\|^2} = \mu^\diamond - (\mu^\diamond \cdot \alpha) \alpha^\diamond
\]

we have

\[
\beta \cdot (s_\alpha \mu)^\diamond = \beta \cdot \mu^\diamond - (\mu^\diamond \cdot \alpha)(\beta \cdot \alpha^\diamond),
\]

which gives the induction step.

Condition (b) follows by induction and from the identity

\[
s_{s_\lambda \mu} = s_\lambda s_\mu s_\lambda,
\]

and induction on depth.

**7. Examples**

Suppose \((V, \Sigma)\) to be a root system.

**RANK ONE SYSTEMS.** Suppose \( \lambda \) to be in \( \Sigma \). Then \(-\lambda\) also lies in \( \Sigma \). Any other root must be a multiple of \( \lambda \), say \( \mu = c\lambda \). Then

\[
\langle \mu, \lambda^\diamond \rangle = 2c
\]

so that \( 2c \) must be an integer. Applying the same reasoning to \( \mu \) instead of \( \lambda \), one sees that \( 2/c \) must also be an integer. Thus \( c = \pm 1/2 \) or \( \pm 2 \). Both cases can occur—in dimension one both

\[
\{\pm \lambda\} \text{ or } \{\pm \lambda, \pm 2\lambda\}
\]

are possible root systems. To summarize:

**7.1. Lemma.** If \( \lambda \) and \( c\lambda \) are both roots, then \( c = \pm 1, \pm 1/2, \text{ or } \pm 2 \).

A root \( \lambda \) is called **indivisible** if \( \lambda/2 \) is not a root. It is easy to see that:

**7.2. Proposition.** The indivisible roots in any root system also make up a root system.

**RANK TWO SYSTEMS.** Suppose \( \alpha, \beta \) to be linearly independent vectors in a Euclidean plane whose reflections generate a finite group preserving the lattice they span. The product \( s_\alpha s_\beta \) is a rotation preserving the lattice,
and must therefore have order 2, 3, 4, or 6. In the first case they commute. The other possibilities are shown in the following diagrams. The last is not reduced.

8. Bases

In this section I’ll summarize without proof further important developments. Suppose given a root system $(V, \Sigma, V^\vee, \Sigma^\vee)$.

The connected components of the complement in $V^\vee$ of the root hyperplanes $\lambda = 0$ for $\lambda$ in $\Sigma$ are called the chambers of the system.

8.1. Proposition. The chambers are simplicial cones.

This means that a chamber is defined by inequalities $\alpha > 0$ as $\alpha$ ranges over a set of linearly independent linear functions in $V$.

Fix one of these, say $C$. It is open in $V$. Define $\Delta$ to be the roots defining the walls of $C$. The elements of $\Delta$ are called the simple roots determined by the choice of $C$. As already noted, they are linearly independent. By definition, no root hyperplane intersects $C$. Those roots that are positive on $C$ are called positive, and the rest negative.

8.2. Proposition. Every positive root is a non-negative, integral, linear combination of simple roots.
The chamber is a strong fundamental domain for $W$. The group $W$ is generated by the elementary reflections $s_\alpha$ for $\alpha$ in $\Delta$. Because $C$ is simplicial, its closed faces of are all of the form

$$C_\Theta = \{ v \in V \mid \langle \alpha, v \rangle = 0 \text{ for all } \alpha \in \Theta \}$$

for some subset $\Theta \subseteq \Delta$. Let $C_\Theta$ be the relative interior of $C_\Theta$.

**8.3. Proposition.** The closed face $C_\Theta$ is exactly the points of $C$ fixed by the subgroup $W_\Theta$ generated by the $s_\alpha$ with $\alpha$ in $\Theta$.

Let $S$ be the set of elementary reflections $s_\alpha$. An expression

$$w = s_{\alpha_1} \ldots s_{\alpha_n} \quad (\alpha_i \in \Delta)$$

is called **reduced** if it is of minimal length among all such expressions. This minimal length $\ell(w)$ is called the **length** of $w$.

There is a basic relationship between combinatorics in $W$ and geometry:

**8.4. Proposition.** If $\alpha$ is a simple root, then $\ell(ws_\alpha) = \ell(w) + 1$ if and only if $w_\alpha > 0$.

A **Cartan matrix** is an integral square matrix $C = (c_{\alpha,\beta})$, indexed by a finite set $\Delta \times \Delta$, with these properties:

(a) all diagonal entries $c_{\alpha,\alpha}$ are equal to 2;

(b) $c_{\alpha,\beta} \leq 0$ for $\alpha \neq \beta$;

(c) there exists a positive diagonal matrix $D$ such that $CD$ is positive definite symmetric.

**8.5. Proposition.** The matrix $(\langle \alpha, \beta^\vee \rangle)$ is a Cartan matrix. Every Cartan matrix is that attached to a finite root system.

If $\alpha$ and $\beta$ are two elements of $\Delta$, then the product $s_\alpha s_\beta$ is a two-dimensional rotation of finite order $m = m_{\alpha,\beta}$. Define

$$n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle .$$

It must be a positive integer, and since $W$ preserves a lattice then the only cases that occur are 0, 1, 2, or 3.

$$m = \begin{cases} 2 & \text{if } n_{\alpha,\beta} = 0; \\ 3 & n_{\alpha,\beta} = 1; \\ 4 & n_{\alpha,\beta} = 2; \\ 6 & n_{\alpha,\beta} = 3. \end{cases}$$

In other words, we recover the reduced systems of rank 2 described earlier.

Let $S$ be the set of reflections $s_\alpha$ for $\alpha$ in $\Delta$.

**8.6. Proposition.** The group $W$ is a Coxeter group with generating set $S$ and relations

$$s_\alpha^2 = I, \quad (s_\alpha s_\beta)^{m_{\alpha,\beta}} = I .$$
9. Root data

A (based) root datum is a quadruple \((L, \Delta, L^\vee, \Delta^\vee)\) in which

(a) \(L\) is a free \(\mathbb{Z}\)-module of finite rank;
(b) \(L^\vee = \text{Hom}(L, \mathbb{Z})\);
(c) \(\Delta\) is a finite subset of \(L\);
(d) \(\alpha \mapsto \alpha^\vee\) is a map from \(\Delta\) to \(L^\vee\) such that \(\langle \alpha, \beta^\vee \rangle\) is a Cartan matrix;
(e) if \(W\) is the group generated by the reflections

\[ s_\alpha: v \mapsto v - \langle v, \alpha^\vee \rangle \alpha, \]

then \(\Sigma = W \cdot \Delta\) is finite.

Root data determine reductive groups over an algebraically closed field just as root systems determine reductive Lie algebras. The simplest examples of root data are the toral data, in which \(\Delta = \emptyset\).

For a slightly more interesting example, suppose

\[ G = \text{SL}_n \]
\[ B = \text{upper triangular matrices in } G \]
\[ T = \text{diagonal matrices in } B. \]

Take \(L = X^*(T), \Delta\) the set of characters

\[ \alpha_i: \text{diag}(x_i) \mapsto x_i^{e_i - e_{i+1}} = x_i/x_{i+1} \quad (1 \leq i < n). \]

The positive roots are those of the form \(x_i/x_j\) with \(i < j\), which are those that occur in the eigenspace decomposition of the adjoint action of \(T\) on the Lie algebra of \(b\). The lattice \(L^\vee\) may be identified with the co-character group \(X^*_i(T)\).

There is an analogue of Proposition 4.7 valid for root data. Suppose \(L_1\) and \(L_2\) to be two root data of the same rank. An isogeny from the first to the second is a map from \(L_1^\vee\) to \(L_2^\vee\) such that (a) the quotient of \(L_2\) by the image of \(L_1\) is finite; (b) \(\Delta_1^\vee\) is mapped to \(\Delta_2^\vee\); (c) under the dual map, \(\Delta_2\) is taken to \(\Delta_1\).

A root datum is called simply connected if \(\Delta^\vee\) is a basis of \(L^\vee\). For example, the root datum of \(\text{SL}_n\) is simply connected.

9.1. Proposition. Any root datum is an isogeny quotient of a direct product of a toral lattice and a simply connected root datum.

10. References