Analysis on SL(2)

The Schwartz space of Ehrenpreis and Mautner

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A smooth function $F$ belongs to $A_{\text{umg}}(\Gamma \backslash G)$, the space of functions of uniform moderate growth on $\Gamma \backslash G$, if and only if there exists a fixed $M$ such that $R_X F(g) = O(\|g\|^M)$ for all $X$ in $U(g)$. It belongs to the Schwartz space $S(\Gamma \backslash G)$ if $R_X F$ decreases rapidly at all cusps of $\Gamma$ for all $X$. If $F$ is a function on $\Gamma \backslash \mathcal{H}$, this definition requires lifting $F$ to a function on $\Gamma \backslash G$ by using the identification of $\mathcal{H}$ with $G/\text{SO}(2)$. It is a matter of curiosity to have a more intrinsic characterization of such functions on $\Gamma \backslash \mathcal{H}$. It seems to me possible that this question and similar ones for other arithmetic quotients are important, although I'll offer no evidence for that here.

1. Proposition. A smooth function $F$ on $\Gamma \backslash \mathcal{H}$ lies in $A_{\text{umg}}(\Gamma \backslash \mathcal{H})$ if and only if there exists a fixed $M$ such that $\Delta^k F = O(y^M)$ for all $k$.

2. Proposition. A smooth function $F$ on $\Gamma \backslash \mathcal{H}$ lies in $S(\Gamma \backslash \mathcal{H})$ if and only if $\Delta^k F$ decreases rapidly in the neighbourhood of all cusps of $\Gamma$.

These are certainly necessary conditions, since $\Delta$ is the restriction to functions on $\mathcal{H}$ of the Casimir operator in $U(g)$. The criterion in the second of these is in fact the definition of the Schwartz space in [Ehrenpreis-Mautner:1962], which I believe to have been the first to introduce the notion.

By truncating $F$ smoothly, we may assume that in the fundamental domain $F$ has support in the region $y > 1$. In that case, we may identify it with a function on $(\Gamma \cap N) \backslash \mathcal{H}$. Resolve $F$ into its Fourier series

$$F(x + iy) = \sum F_n(y)e^{2\pi i nx} \left( F_n(y) = \int_0^1 F(x + iy)e^{-2\pi i nx} \, dx \right).$$

Then $F$ satisfies the hypotheses of the theorem if and only if each of $F_0$ and $F - F_0$ separately do, and similarly for the conclusion.

Let $D = y\partial / \partial y$. It is straightforward to see that the function $F_0$ lies in $A_{\text{umg}}$ if and only if there exists $M$ such that all $D^k F_0 = O(y^M)$.

It is also easy to see that $F$ lies in $S$ if and only if all $\partial^{k+\ell} F / \partial x^k \partial y^\ell = O(y^{-N})$ for all $N$. (Remember, $F$ has support in $y > 1$.)

The proof of the Proposition therefore comes in two halves.

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1. The constant term

I’ll deal with the constant term first.

1.1. Proposition. Suppose that \( F \) is a smooth function on \((0, \infty)\) with support in \( y > 1 \) such that for some fixed \( M \) all \( \Delta^k F = O(y^M) \). Then all \( D^k F = O(y^M) \); as well.

Proof. Suppose that \( F = F_0 \).

\[
|F(y)| \leq C_0 y^M, \quad (\Delta F) = y^2 F''(y) \leq C_2 y^M.
\]

1.2. Lemma. Suppose \( F(y) \) and \( y^2 F''(y) \) are both \( O(y^M) \), and of support near \( y = \infty \). Then \( yF'(y) = O(y^M) \) as well.

Proof. The idea is to reduce it to a variant of a well known lemma due originally to Landau:

1.3. Lemma. If

\[
g(t) = O(e^{\mu t}), \quad g''(t) = O(e^{\mu t})
\]

then so also is

\[
g'(t) = O(e^{\mu t}).
\]

Proof of Landau’s Lemma.

\[
g(t) - g(t - \varepsilon) = \varepsilon g'(t) + (\varepsilon^2/2)g''(\theta_1) \quad (t - \varepsilon < \theta_1 < t)
\]

\[
g(t + \varepsilon) - g(t) = \varepsilon f'(t) + (\varepsilon^2/2)g''(\theta_2) \quad (t < \theta_1 < t + \varepsilon)
\]

\[
g(t + \varepsilon) - g(t - \varepsilon) = 2\varepsilon f'(t) + (\varepsilon^2/2)(g''(\theta_1) + g''(\theta_2))
\]

\[
2\varepsilon f'(t) = g(t + \varepsilon) - g(t - \varepsilon) - (\varepsilon^2/2)(g''(\theta_1) + g''(\theta_2))
\]

\[
2\varepsilon f'(t) \leq P(e^{\mu(t+\varepsilon)} + e^{\mu(t-\varepsilon)}) + Q(e^{\mu t} + e^{\mu t})
\]

\[
\leq P e^{\mu t}(e^{\mu \varepsilon} + e^{-\mu \varepsilon}) + (\varepsilon^2/2)Q e^{\mu t}(e^{\mu \varepsilon} + 1)
\]

\[
|f'(t)| \leq \left( e^{\mu \varepsilon} + 1 \right) \left( \frac{P}{\varepsilon} + \frac{Q \varepsilon}{2} \right)
\]

for all \( \varepsilon > 0 \). The function of \( \varepsilon \)

\[
E(\varepsilon) = \left( e^{\mu \varepsilon} + 1 \right) \left( \frac{P}{\varepsilon} + \frac{Q \varepsilon}{2} \right)
\]

approaches \( \infty \) near 0 and also as \( \varepsilon \to \infty \), and somewhere in between takes a minimum positive value \( E_{\min} \).

Thus for all \( t \)

\[
|f'(t)| \leq E_{\min} e^{\mu t}.
\]

This concludes the proof of the two lemmas. Applied to all the \( \Delta^k F \) in turn these imply that all \( D^k F_0 = O(y^M) \), and this in turn implies that \( F_0 \) is of uniform moderate growth or rapid decrease, depending on the assumption on the \( \Delta^k F \). Q.E.D. We start with

\[
F(y) = O(y^m), \quad y^2 F''(y) = O(y^M).
\]

We can write

\[
y^2 F'' = (D^2 - D)F = (D - 1/2)^2 F - 1/4F
\]

if \( D = y\partial/\partial y \), and then deduce

\[
(D - 1/2)^2 F = O(y^M).
\]
If we set \( G(y) = y^\lambda F(y) \), then
\[
DG = y^\lambda DF + \lambda y^\lambda F \\
= y^\lambda (D + \lambda)G \\
D^2G = \lambda y^\lambda (D + \lambda)F + y^\lambda D(D + \lambda)F \\
= y^\lambda (D + \lambda)^2 F.
\]

Set \( \lambda = -1/2 \), so
\[
G = y^{-1/2}F(y) \\
= O(y^{M-1/2}) \\
D^2G = y^{-1/2}(D - 1/2)^2 F(y) \\
= O(y^{M-1/2})
\]

If we change the independent variable to \( y = e^t \) we get equations
\[
G(t) = O(e^{(M-1/2)t}) \\
G''(t) = O(e^{(M-1/2)t})
\]

2. The rest

Now I’ll deal with the rest of \( F \).

2.1. Proposition. Suppose \( F \) to be a smooth function on \( \Gamma \cap N \setminus \mathcal{H} \) with support in \( y > 1 \) and all \( \Delta^k F = O(y^M) \). If \( F_0 = 0 \) then all the \( \partial^k \Phi / \partial x^k \partial y^l = O(y^{-N}) \) for all \( N \).

Proof. I’ll start off by looking at the \( n \) individual terms in the Fourier expansion. We have
\[
(\Delta F)_n(y) = y^2 F''_n(y) - 4\pi^2 n^2 F_n(y)
\]
and we may assume \( n \neq 0 \). If \( \Delta^k F = O(y^M) \) for all \( k \) then so is \( \Delta^k F_n(y) = O(y^M) \) for all \( k \).

2.2. Proposition. Let \( F(y) = O(y^M) \) be a smooth function on \( \mathbb{R} \) with support on \( (1, \infty) \), and suppose that for some \( \lambda \geq 1 \)
\[
y^2(F'' - \lambda^2 F) \leq Cy^M.
\]

Then \( F \leq (CC_M/\lambda)y^{M-2} \) for some constant depending only on \( M \).

Since the \( \lambda \) we are concerned with are the \( 4\pi^2 n^2 \) and the series \( \sum_{n>0} 1/n^2 < \infty \), this implies the bound we want on \( F \) itself.

Proof. Let \( G(y) = y^2(F'' - \lambda^2 F) \). Since \( F \) has support on \( (1, \infty) \) so does \( G \). Since \( E(x) = -e^{-\lambda|x|} / 2\lambda \) satisfies the distributional differential equation
\[
E'' - \lambda^2 E = \delta_0
\]
an easy calculation tells us that
\[
F(y) = c_+ e^{\lambda y} + c_- e^{-\lambda y} - \frac{1}{2\lambda} \int_1^\infty e^{-\lambda|x-y|} G(y) \, dy.
\]

Since \( G(y) = O(y^M) \) an easy calculation tells us that the integral is of moderate growth, and therefore since \( F \) is of moderate growth the coefficient \( c_+ \) has to vanish. Thus
\[
F(y) = c_- e^{-\lambda y} - \frac{1}{2\lambda} \int_1^\infty e^{-\lambda|x-y|} G(y) \, dy.
\]
Since \( F(0) = 0 \)
\[
e^{-\lambda y} = \frac{1}{2\lambda} \int_{-\infty}^{\infty} e^{-\lambda x} G(x) \, dx
\]
and hence
\[
F(y) = \frac{e^{-\lambda y}}{2\lambda} \int_{-\infty}^{\infty} e^{-\lambda x} G(x) \, dx - \frac{1}{2\lambda} \int_{1}^{\infty} e^{-\lambda|x-y|} G(x) \, dx.
\]
For the moment, let
\[
I_N = \int_{1}^{\infty} e^{-\lambda x} x^N \, dx.
\]
Integration by parts and an easy estimate gives
\[
I_N \leq e^{-\lambda y} \frac{e^{-\lambda y}}{2\lambda} \int_{1}^{\infty} e^{-\lambda x} x^{N-1} \, dx \leq e^{-\lambda y} \frac{e^{-\lambda y}}{2\lambda} N^2.
\]
if \( n = \lceil N + 1 \rceil \), keeping in mind that \( \lambda \geq 1 \). Thus the first term above is bounded by
\[
CE_{N-2} e^{-\lambda y} \frac{e^{-\lambda y}}{2\lambda}.
\]
The second term can be broken up into two parts:
\[
\int_{1}^{y} e^{-\lambda(x-y)} G(x) \, dx = \int_{1}^{y} e^{-\lambda(x-y)} G(x) \, dx + \int_{y}^{\infty} e^{-\lambda(x-y)} G(x) \, dx \leq Ce^\frac{M-2}{\lambda} + Ce^{\frac{M-2}{\lambda}}.
\]
For the first of these integrals:
\[
\int_{1}^{y} e^{-\lambda(x-y)} x^{M-2} \, dx = \left[\frac{e^{-\lambda(x-y)} x^{M-2}}{\lambda} \right]_{1}^{y} = \frac{M-2}{\lambda} \int_{1}^{y} e^{-\lambda(x-y)} x^{M-3} \, dx.
\]
which implies that the term is bounded by
\[
\frac{Cy^{M-2}}{\lambda} \leq \frac{y^{M-2}}{\lambda}.
\]
For the second:
\[
\int_y^\infty e^{-\lambda(x-y)}x^{M-2} \, dx = \int_0^\infty e^{-\lambda s}(s+y)^{M-2} \, ds
\]
\[
= \int_0^\infty e^{-\lambda s}(s+y)^m \, ds \quad (m = \lceil M \rceil).
\]
To this last expression we can apply the binomial theorem and the estimate of the first integral to see that this second term is at most \(CC^*_M y^M/\lambda\).

So now we know that \(F\) itself is rapidly decreasing. It remains to show that all its partial derivatives are also rapidly decreasing. I leave this as an exercise! \(\Box\).

3. References


