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Essays on representations of p-adic groups

Cuspidal representations

Bill Casselman
University of British Columbia
cass@math.ubc.ca

In this essay, the coefficient ring $\mathcal{F}$ will be an algebraically closed field F.

1. Definition

An admissible representation (π, V) is said to be **cuspidal** (in some of the literature **absolutely cuspidal** or **super-cuspidal**) if $V_N = 0$ for every parabolic subgroup $P = MN$ other than G.

[compact-support] **Theorem 1.1.** *If (π, V) is an admissible representation of G, the following are equivalent:*

- *(a) the representation π is cuspidal;*
- *(b) the representation $\widetilde{\pi}$ is cuspidal;*
- *(c) for every v in V, $\widetilde{v}$ in $\widetilde{V}$ the matrix coefficient $\langle \pi(g)v, \widetilde{v} \rangle$ has compact support modulo the centre of G.*

♣ [global-jacquet] *Proof.* Apply Proposition 1.2(asymptotics) . In that result, all the terms parametrized by $\Theta \neq \Delta$ vanish by assumption, and all the sets parametrized by Δ are compact modulo Z_G. ▯

[cuspidal-unitary] **Corollary 1.2.** *If $\mathcal{F} = \mathbb{C}$ then every irreducible cuspidal representation with a unitary central character is unitary.*

2. A categorical property

[projective] **Theorem 2.1.** *An irreducible cuspidal representation is projective and injective in the category of smooth representations with the same central character.*

Proof. The proof is motivated by the analogous case of modules over a commutative ring with unit, in proving that a module is projective if and only if it is a summand of a free module. But in our case there are some minor difficulties because the Hecke algebra is not commutative and does not possess a unit.

Suppose (π, V) to be an irreducible cuspidal representation of G with formal degree d_π and central character ω. Suppose (σ, U) to be an arbitrary smooth representation of G with central character ω, and $F: U \to V$ a G-covariant surjection. We want to construct a splitting of F from V back to U. What we shall do is embed V as a summand of $\mathcal{H}_\omega$, and then construct a suitable map from $\mathcal{H}_\omega$ to U.

The proof is relatively straightforward, but will be clearer if I make three succinct observations:

(1) The map from V to $C^\infty(G)$ taking v to the matrix coefficient $\langle \pi(g)v, \tilde{v} \rangle$ is a G-covariant map from (π, V) to $(R, \mathcal{H}_\omega)$. Here R is the right regular representation of G.

(2) Given a vector u in U, the map taking f in $\mathcal{H}_\omega$ to $\sigma(f)u$, where $\check{f}(g) = f(g^{-1})$, is a G-covariant map from $(R, \mathcal{H}_\omega)$ to (σ, U).

(3) Schur orthogonality for a cuspidal representation π:

$$\int_{G/Z_G} \langle \pi(g)u, \tilde{u} \rangle \langle \pi(g^{-1}v, \tilde{v} \rangle \, dg = \frac{1}{d_\pi} \, \langle u, \tilde{v} \rangle \langle v, \tilde{u} \rangle \, .$$

Now for the proof.

Step 1. Fix $\tilde{v}_0$ in $\widetilde{V}$ and v_0 in V such that $\langle \tilde{v}, v\rangle = d_\pi$. For any v in V let γ_v be the matrix coefficient ♣ **[compact-support]** $\langle \pi(g)v, \tilde{v}_0\rangle$. By Theorem 1.1 the map $v \mapsto \gamma_v$ embeds V into $\mathcal{H}_\omega(G)$.

Step 2. We next want to define a G-covariant projection $\mathcal{P}$ from $\mathcal{H}_\omega \to V$. To do this, I shall pick a suitable vector $v_0 \neq 0$ in V and map f in $\mathcal{H}_\omega$ to

$$\mathcal{P}f = R_{\check{f}} v_0 \,.$$

It follows from the remark (2) above that this is a G-covariant map from $\mathcal{H}_\omega$ to V.

We now want to choose v_0 so that $\mathcal{P}\gamma_v = v$ for v in V. Well, we compute

$$\begin{aligned} \mathcal{P}\gamma_v &= R_{\check{\gamma}_v} v_0 \\ &= \int_{G/Z} \check{\gamma}_v(x)\pi(x)v_0 \, dx \\ &= \int_{G/Z} \langle \pi(x^{-1})v, \tilde{v}_0\rangle \pi(x)v_0 \, dx \,. \end{aligned}$$

This last is an element of V. But according to Schur orthogonality

$$\Big\langle \int_{G/Z} \langle \pi(x^{-1})v, \tilde{v}_0\rangle \pi(x)v_0 \, dx, \tilde{v} \Big\rangle = \int_{G/Z} \langle \pi(x^{-1})v, \tilde{v}_0\rangle \langle \pi(x)v_0, \tilde{v}\rangle \, dx = \frac{1}{d_\pi} \langle v, \tilde{v}\rangle \langle v_0, \tilde{v}_0\rangle \,.$$

so if we choose $\langle \pi(x)v_0, \tilde{v}_0\rangle = d_\pi$ then $\mathcal{P}v = v$.

Step 3. Go back to our projection $F\colon U \to V$. Choose $u_0 \in U$ with $F(u_0) = v_0$. Let Π be the map from $\mathcal{H}_\omega$ to U taking f to $\sigma(\hat{f})u_0$. Here $\hat{f}(g) = f(g^{-1})$. This is a G-morphism. The diagram

$$\begin{array}{ccc} & \mathcal{H}_{\omega^{-1}} & \\ {}^{\Pi}\swarrow & & \searrow{}^{\mathcal{P}} \\ U & \xrightarrow{\;F\;} & V \end{array}$$

is commutative, so

$$F\circ\Pi = \mathcal{P}, \quad F\circ(\Pi\circ\gamma) = (F\circ\Pi)\circ\gamma = \mathcal{P}\circ\gamma = I$$

and $\Pi\circ\gamma$ splits F.