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Essays on Coxeter groups

Bruhat closures

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This essay presents new and, I believe, simpler proofs of the principal results concerning Bruhat closure in a Coxeter group.

My sources have been §§5.8–5.11 of [Humphreys:1990] and [Dixmier:1974], but my approach is somewhat different, at least at th ebeginning. In the last section my treatment is novel only in so far as it is accompanied by pictures that I believe make the argument clearer.

Contents

1. The closure of an element

I recall first some basic facts I need about Coxeter groups.

Suppose (W, S) to be a Coxeter system with Coxeter matrix $(m_{s,t})$ (possibly with $m_{s,t} = \infty$). It is the group defined by generators in S and relations

$$s^2 = 1, \quad (st)^{m_{s,t}} = 1 .$$

Let $V = \mathbb{R}^S$ with basis (α_s). Define on V the inner product

$$\alpha_s \bullet \alpha_t = -\cos(\pi/m_{s,t}) .$$

For each s let

$$\rho_s\colon\ v \longmapsto v - 2(\alpha_s \bullet v)\alpha_s .$$

be the orthogonal reflection in the hyperplane $\alpha \bullet v = 0$. The map $s \mapsto \rho_s$ extends to a embedding of W in $\mathrm{GL}(V)$. Let C be the open cone

$$C = \{v \in V \mid \alpha_s \bullet v > 0 \text{ for all } s \in S\}$$

and let $\mathcal{C}$ (called by some the **Tits cone** and by others the **Vinberg cone**) be the interior of the union of the closures $\overline{C}$. The group W acts discretely on $\mathcal{C}$ and $\overline{C}$ is a strict fundamental domain for this action.

If H is a half space bounded by a wall of C and containing C, then I'll call $\mathcal{C} \cap H$ a **simple root** of the system, and the W-transforms of these I'll call the **roots**. If λ is a root, let $-\lambda$ be the opposite half of the hyperplane it lies in. For an arbitrary Coxeter group, it is this geometric (not algebraic) notion of 'root' that is significant.

A root λ is **positive** if it contains C, negative if $-\lambda$ does. A root is either positive or negative. For w in W and $\lambda > 0$, $w\lambda < 0$ if and only if C and $w^{-1}C$ are on opposite sides of the boundary of λ. Let Δ be the set of simple roots, Σ the set of all roots, and Σ^+ the set of positive roots.

For every root λ, let s_λ be the reflection in the boundary of λ. If $\lambda = w\alpha_s$ with $\alpha = \alpha_s$ in Δ, then $s_\lambda = ws_\alpha w^{-1}$.

A **word** in the alphabet S is the concatenation $s_1 \bullet \ldots \bullet s_n$ of elements of S. If ω is a word, let $\overline{\omega}$ be the corresponding element $s_1 \ldots s_n$ of W. For w in W, let $\ell(w)$ be the length of the shortest word representing it (which is called a **reduced** word). Thus for s in S, $\ell(sw) = \ell(w) \pm 1$.

The basic result relating the combinatorics and geometry of W is that if s lies in S then $\ell(sw) = \ell(w) + 1$ if and only if $w^{-1}\alpha_s > 0$. In this case, I'll write $sw > w$. This is the simplest case of a more general fact. For any w in W, let

$$L_w = \{\lambda > 0 \,|\, w^{-1}\lambda < 0\}\,.$$

Then $\ell(xy) = \ell(x) + \ell(y)$ if and only if L_{xy} is the disjoint union of L_x and xL_y. AS one consequence $\ell(w) = |L_w|$.

For $T \subseteq S$ let W_T be the subgroup of W generated by T. It also is a Coxeter group, and the length functions in W_T and $W = W_S$ are the same.

> *Every w in W can be represented uniquely as xy with (a) $x \in W_T$, (b) $ty > y$ for all t in T, and (c) $\ell(w) = \ell(x) + \ell(y)$.*

If $T = \{s, t\}$ and $m_{s,t} < \infty$ then W_T contains $2m_{s,t}$ elements. All but one of them has a unique expression as a word. The exception is the longest element, which is

$$w_{\ell,T} = st\ldots = ts\ldots \quad (m_{s,t} \text{ terms on each side})\,.$$

This is called a **braid relation**. This longest element w is also singled out in W_T by the conditions $sw < w$, $tw < w$.

The following is a special case of a result of [Tits:1968].

[braids] **Lemma 1.1.** *If*

$$w = s_1 \ldots s_n = t_1 \ldots t_n$$

are two reduced expressions for w then one may be obtained from the other by a sequence of braid relations.

Proof. The proof is by induction on n. The cases $n = 1$ or 2 are trivial. So assume $n > 1$, and that

$$s_1 \ldots s_n = t_1 \ldots t_n$$

are reduced. If $s_1 = t_1$ we can cancel the common left factor and apply induction. Otherwise suppose $s = s_1 \neq t = t_1$, and in particular $n > 1$. Let

$$x = s_2 \ldots s_n, \quad y = t_2 \ldots t_n\,.$$

Then $sw < w$ so $w\alpha_s < 0$, and $tw < w$ so $w\alpha_t < 0$. If we write $w = uv$ with $u \in W_{s,t}$ and v such that $v\alpha_s > 0$, $v\alpha_t > 0$ then $su < u$, $tu < u$ hence $u = w_{\ell,s,t}$. We can write

$$w = sw_{s,t}z = tw_{t,s}z$$

where $sw_{s,t} = tw_{t,s} = w_{\ell,s,t}$. Since $sw_{s,t}z = ss_2 \ldots s_n$, we may cancel s and by induction obtain $w_{s,t}x$ from $s_2 \ldots s_n$ by a sequence of braid relations. Similarly for $w_{t,s}z$ and $t_2 \ldots t_n$. But then we can also obtain $sw_{s,t}z$ from $tw_{t,s}z$ by a single braid relation, so the Lemma is proved. □.

If $\omega = s_1 \bullet \ldots \bullet s_n$ is a word in S, then I define the **word-closure** $\mathrm{cl}(\omega)$ of ω to be the set of all words obtained by deleting some of its letters or, equivalently, those obtained by successively deleting one letter:

$$\omega \mapsto s_1 \bullet \ldots \bullet s_{i-1} \bullet s_{i+1} \bullet \ldots \bullet s_n\,.$$

It is immediate that $\mathrm{cl}(\chi \bullet \omega) = \mathrm{cl}(\chi) \bullet \mathrm{cl}(\omega)$.

[subexpressions] **Proposition 1.2.** *If/ w in W is represented by the reduced word ω, then the image in W of the words in $\mathrm{cl}(\omega)$ depends only on w.*

This is called the the **Bruhat closure** of w. Another way to phrase it is that *the elements represented by subexpressions of a given reduced expression in W does not depend on the particular reduced expression.*

♣ [braids] *Proof.* By Lemma 1.1, it suffices to prove that it is true for two reduced expressions interchanged by a braid relation. But this will follow from the simple case of the Proposition in which S has two elements. In this case, in can be easily checked that if ω is a reduced word for w, then $\overline{\mathrm{cl}(\omega)}$ consists simply of all elements x with $\ell(x) \leq \ell(w)$. □

2. Root reflections

If r is a root reflection, so is wrw^{-1}. If $w = urv$ then uv is what we get by deleting r. But $uv = uru^{-1} \cdot urv = uru^{-1}w$. Hence:

[deletion] **Proposition 2.1.** *If $w = s_1 \ldots\ s_n$ is an expression for w as product of elements in S, then*

$$u = (s_1 \ldots\ s_{i-1}) \cdot (s_{i+1} \ldots s_n) = (s_1 \ldots\ s_{i-1}) \cdot s_i \cdot (s_{i-1} \ldots\ s_1) \cdot (s_1 \ldots\ s_n)$$

is of the form rw where r is a reflection in W.

As a partial converse:

[strong-exchange] **Proposition 2.2.** *Let w be in W, $r = r_\lambda$ a root reflection with $\lambda > 0$. Then $\ell(rw) < \ell(w)$ if and only if $w^{-1}\lambda < 0$, and if $w = s_1 \ldots s_n$ then*

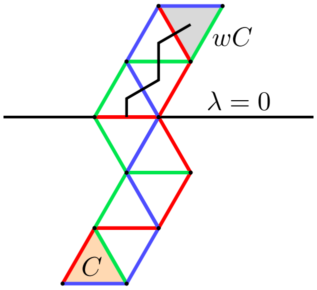
$$rw = s_1 \ldots\ s_{i-1} \cdot s_{i+1} \ldots\ s_n$$

for some intermediate s_i. If the expression for w is reduced, then s_i is unique.

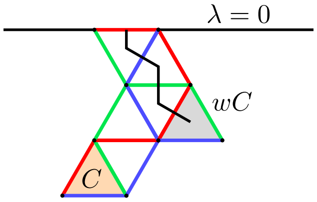
Proof. Suppose the gallery $C, s_1C, \ldots\ , wC$ crosses the hyperplane $\lambda = 0$ in a wall labeled s_i. Then

$$rw = s_1 \ldots\ \widehat{s}_i \ldots\ s_n$$

for the usual geometric reasons, and $\ell(rw) < \ell(w)$.



$w = s_2 s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_1$



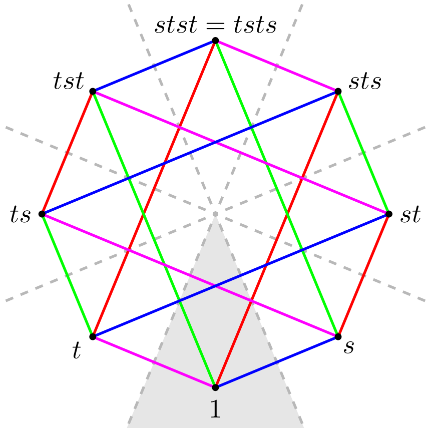
$s_\lambda w = s_2 s_1 s_3 s_2 s_3 \widehat{s}_1 s_3 s_2 s_1$

If we start with a reduced expression, the gallery crosses $\lambda = 0$ exactly once, guaranteeing uniqueness. If $w^{-1}\lambda > 0$, then $w^{-1}r^{-1}\lambda < 0$, so we can apply this argument to rwC. □

If r is not in S, the reduced expression $s_1 \dots \widehat{s}_i \dots s_n$ may collapse further, as it does in the diagrams above.

Set $x \Leftarrow y$ if $\ell(x) < \ell(y)$ and $rx = y$ for some r in R, and define $x \le y$ to mean we can reach y from x by 0 or more such reflections. Since $wr = wrw^{-1} \cdot w$, it doesn't matter whether we use left or right multiplications by reflections in this definition. This order is called the **strong Bruhat order**. I define the **strong Bruhat graph** to be that with elements of W as nodes and oriented edges $x \Leftarrow y$. The **closure** of y is the set of all $x \le y$, and if $x \le y$ the **interval** $[x, y]$ is the set of w with $x \le w \le y$.

Example. Let (W, S) be the dihedral group of order 8, with generators s, t. The following figure exhibits the strong Bruhat graph, with the orientation of every edge pointing up.



All dihedral groups exhibit the same behaviour—for these groups, $x \le y$ if and only if $\ell(x) \le \ell(y)$.

Example. Let $\Xi = \{\xi_1 < \ldots < x_n\}$ be an ordered set of n elements and W the symmetric group $\mathfrak{S}_\Xi$. Let S be the subset of elementary transpositions interchanging ξ_i and ξ_{i+1}. A permutation σ of Ξ is expressed by the array $\big(\sigma(\xi_i)\big)$. The reflections are the swaps of two item in the array. The definition says that $x \prec y$ if y is obtained from x by swapping x_j and x_k in the array (x_i), where $j < k$ and $x_j < x_k$. For example, $[2, 4, \boldsymbol{1}, \boldsymbol{5}, 3] \prec [2, 4, \boldsymbol{5}, \boldsymbol{1}, 3]$. [Humphreys:1990] (on p. 119) attributes to Deodhar a simple criterion. First some notation: if $(x_1, \dots, x_m)$ is any array, let $\langle x_1, \dots, x_m\rangle$ be the same array sorted from smallest to largest. If $x \le y$ if and only if

$$\langle x_1, \dots, x_k\rangle \preceq \langle y_1, \dots, y_k\rangle$$

for each k, in the sense that after sorting corresponding entries are less than or equal. This is clearly a necessary condition, and sufficiency may be proved by induction on n.

Of course $x \le y$ if and only if $x^{-1} \le y^{-1}$.

Strong Exchange implies that if $x = ry \Leftarrow y$ then a product expression for x may be obtained from a reduced expression for y by a single deletion. Repeating:

[lt-deletion] **Proposition 2.3.** *If $x < y$ then a product expression may be obtained for x by making one or more deletions in a reduced expression for y as a product of elements of S.*

We shall see later that the converse is also true.

3. Structure of the graph

Multiplication by s is an involution of the group. *How does this involution relate to the closure graph?* Very nicely. All possibilities are shown in the figure of the Bruhat graph of the dihedral group above. Multiplication takes takes edges to edges, if we neglect orientation, and in in a very simple way:

[xsys] **Proposition 3.1.** *Suppose s in S, $x \Leftarrow y$. Then exactly one of the following occurs:*

(a) $sx = y$, so that s reverses the edge in the strong Bruhat graph between them;
(b) s maps the edge $x \Leftarrow y$ to the edge $sx \Leftarrow sy$.

In other words, applying s to the edge doesn't reverse the orientation of the edge, unless it just exchanges its endpoints.

Proof. Suppose $x \Leftarrow y$, say $x = r_\lambda y$ with $\ell(x) < \ell(y)$, $\lambda > 0$. If $s = r$ then multiplication by s clearly reverses this edge, so suppose $r \neq s$.

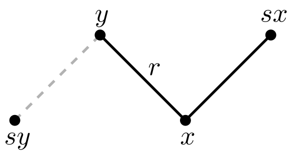
♣ [strong-exchange] Since $r_\lambda y < y$, Proposition 2.2 implies that $y^{-1}\lambda < 0$. But then

$$sx = sr_\lambda y = ss_\lambda s \cdot sy = s_{s\lambda} sy\,.$$

Since $r \neq s$, $s\lambda > 0$, so that $sx < sy$ if and only if $(sy)^{-1}s\lambda < 0$. But

$$(sy)^{-1}s\lambda = y^{-1}\lambda < 0\,. \ \square$$

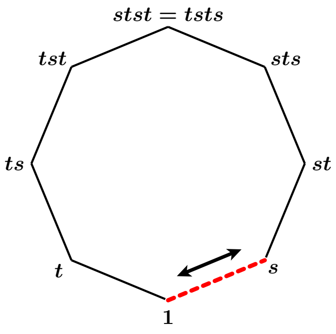
Let me analyze the situation a bit more closely. Since $sx = sry = srs \cdot sy$, it suffices to show that if $s \neq r$ then $\ell(sx) < \ell(sy)$. (a) If $sy > y$ then $\ell(sy) > \ell(y) \geq \ell(sx)$, so this case is trivial. (b) If $\ell(sx) < \ell(x)$ the conclusion is also trivial. (c) If $\ell(x) < \ell(y) - 1$ then since $\ell(y) - \ell(x)$ has to be odd, $\ell(x) \leq \ell(y) - 3$, and again $\ell(sx) < \ell(sy)$ is easy. So it is the case $sy < y$, $sx > x$, $\ell(x) = \ell(y) - 1$, in other words the configuration in the following diagram, that has to be ruled out.

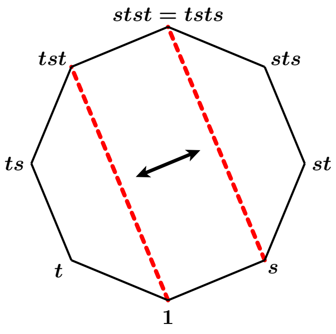


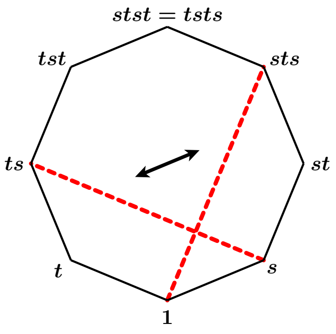
This is useful to keep in mind.

The content of the Proposition is that multiplication by s preserves the links in the strong Bruhat graph, but will reverse orientation in exactly one case, that of an edge between x and sx.

There are thus essentially three kinds of edge-swaps, given and edge $x < y$: (a) an edge reverses itself; (b) $sx < y$ and $sx < x$; or (c) $sx < x$, $sy > y$. All three cases occur already for dihedral groups:



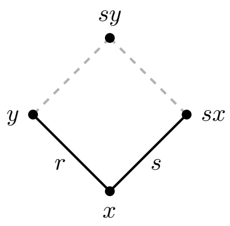


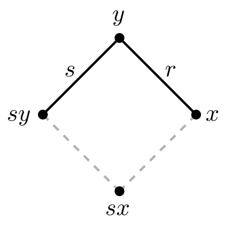


[dixmier] **Corollary 3.2.** *Suppose $x \Leftarrow y$ with $\ell(y) = \ell(x) + 1$. Then*

(a) if $sx > x$ then either $y = sx$ or $sx \Leftarrow sy$;
(b) if $sy < x$ then either $y = sx$ or $sx \Leftarrow sy$.

In diagrams:





Proof. This is just a restatement of what's forbidden. □

[rank2-diff] **Corollary 3.3.** *Suppose $x < y$, with $\ell(y) - \ell(x) = 2$, $sy < y$. Either $sx > x$ and $[x, y] = \{x, sx, sy, y\}$ or $sx < x$ and the interval $[x, y]$ is isomorphic to $[sx, sy]$.*

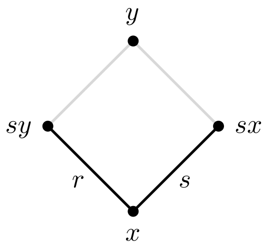
Proof. Since $x > y$, parity considerations require that the interval between x and y be filled with edges of length 1. If $[x, y] \neq \{x, sx, sy, y\}$ then there exists $x < z < y$ with $z \neq sx$, $z \neq sy$. In this case the Proposition implies that $sx < sz < sy$, and since $sy < y$ we must have $sx < x$. In particular $sx \notin [x, y]$.

Now there is a further dichotomy: either $sy \in [x, y]$ or not. In the second case, s is an isomorphism of $[x, y]$ with $[sx, sy]$. In the first case, the map $z \mapsto sz$, $sy \mapsto x$ is an isomorphism of $[x, y]$ with $[sx, sy]$. □

[interval] **Corollary 3.4.** *Suppose $x < y$ and $\ell(x) = \ell(x) - 2$. Then there exist exactly two w with $x < w < y$.*

That is to say, the Bruhat interval $[x, y]$ in this case is very simple.

Proof. By induction on $\ell(y)$. The minimum this can be is 2, in which case $x = 1$, $y = st$, and $[x, y] = \{1, s, t, st\}$.



♣ [rank2-diff] Otherwise, choose $sy < y$. If $sx < x$, then Corollary 3.3 tells us that $[x, y]$ is isomorphic to $[sx, sy]$, and we apply induction. If $sx > x$ the same result tells us $[x, y] = \{x, sx, sy, y\}$. □

[s-stability] **Corollary 3.5.** *Suppose $y < sy$. If $x < y$ then $sx < sy$.*

Proof. By induction on $\ell(y) - \ell(x)$. We can find a chain

$$x_0 = x \Leftarrow x_1 \Leftarrow \ldots \Leftarrow x_n = y < sy\,.$$

The case $n = 1$ is the Proposition (and is in fact straightforward to prove directly). Suppose $n > 1$. If $sx = x_{n-1}$ the claim is immediate. Otherwise by the Proposition $sx < sx_1$ and $sx_1 < sy$ by induction. □

[s-stability2] **Corollary 3.6.** *Suppose $sy < y$. If $x \leq y$ then $sx \leq y$.*

Proof. The proof is by induction on $\ell(x) - \ell(y)$. If it is 0, there is nothing to prove. Otherwise, we can find a chain

$$x_0 = x \Leftarrow x_1 \Leftarrow \ldots \Leftarrow x_n = y < sy\,.$$

If $n = 1$, either $x = sy$ and $sx = y$, or by the Proposition

$$sx \Leftarrow sy < y\,.$$

If $n > 1$, we have Say $x \Leftarrow x_1$ with $x_1 < y$. Induction tells us $sx_1 \leq y$. The Proposition says either $sx < sx_{n1}$ or $x_1 = sx$. Either way, $sx \leq y$. ▯

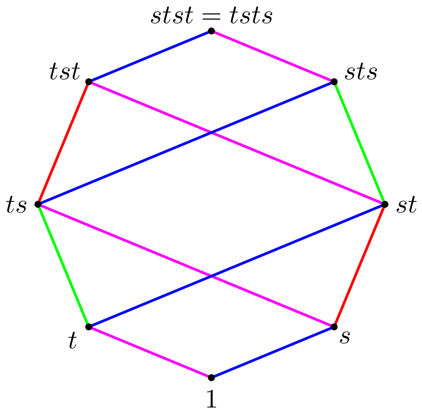
[deletion-ok] **Corollary 3.7.** *If/ x is obtained from y by one or more deletions in a reduced word for y, then $x < y$.*

Proof. What makes this not quite obvious is that one deletion might lead to a number of collapses of terms in the product, including some of the original deletions.

The proof is by induction on $\ell(y)$. Suppose $y = s_1 \ldots s_n$. If s_1 is not one of the deleted items, the conclusion follows from induction. Otherwise, let $z = s_2 \ldots s_n$, so $y = s_1$ and $x = \hat{z}$ where $\hat{z}$ is obtained ♣ [s-stability] from z by deletions. By induction $\hat{z} < z$. Apply Lemma 2.3 to $\hat{z} < z < sz = y$. ▯

4. Maximal chains

There is one more thing to notice about the strong Bruhat graph of the dihedral group shown above—there is some redundancy in it. For example, the reflection sts takes t to $stst$, so $t \leq stst$. But this can be seen also by the chain t-ts-sts-$stst$. With the redundant links removed, the graph of the order looks like this:



It is easy to see for all dihedral groups that the Bruhat order is generated by pairs $x = ry$ with $\ell(x) = \ell(y) - 1$. This is a general fact, and the second of the two most important results about Bruhat order.

Define $x \prec y$ to mean $x = ry < y$ and $\ell(y) - \ell(x) = 1$.

[dist1] **Proposition 4.1.** *If/ $x < y$, then there exists a chain $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$.*

This allows a very simple algorithmic description of closures. In the proof, I follow closely [Dixmier:1974], pp. 250–252.

Proof. We may assume that $x = ry < y$. We proceed by induction on $\ell(y) + \big(\ell(y) - \ell(x)\big)$. If $\ell(x) = \ell(y) - 1$, there is nothing to be proven. So we may assume $\ell(y) \geq \ell(x) + 3$.

Choose s with $sy < y$. Then $sx = sry = srs \cdot sy$ and

$$\ell(sx) < \ell(x) + 1 \leq \ell(y) - 2 < \ell(y) - 1 = \ell(sy)\,.$$

So $sx < sy$. We may apply induction to get a chain from sx to sy:

$$sx = w_0 < w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y$$

with (say) $w_{i+1} = r_i w_i$. In particular, $r_n = s$.

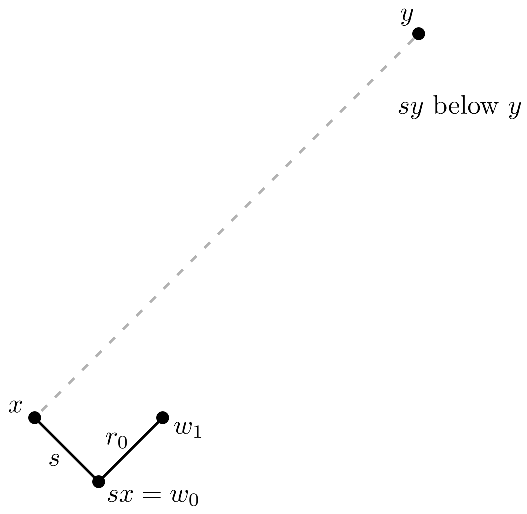
- If $x < sx$, we can just extend the chain to include x:

$$x < sx = w_0 < w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y$$

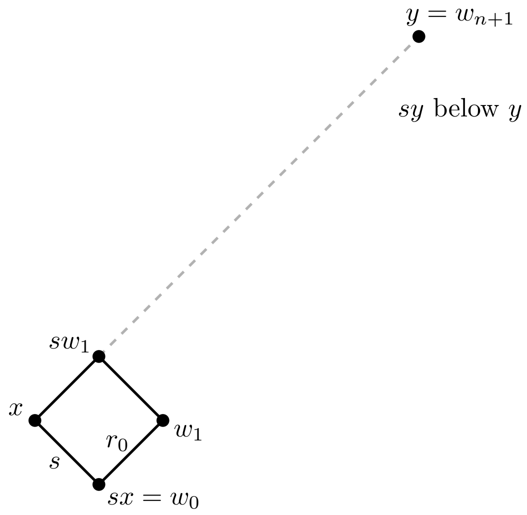
- If $x > sx$ and $w_1 = x$, the chain we want is

$$x = w_1 < w_2 < \ldots < w_n = sy < w_{n+1} = y\,.$$

- Otherwise, $sx < x$ and $w_1 \neq x$. The situation is indicated by this diagram:



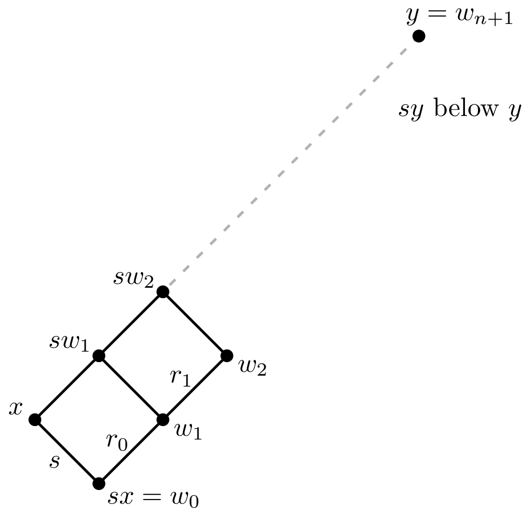
Let $t_0 = sr_0s$. Since $s \neq r_0$, we know that $sw_1 > w_1$ and that $t_0x = sw_1$, so we may fill in the diagram.



Since $\ell(y) - \ell(x) \geq 3$, $\ell(sw_1) = \ell(x) + 1 < \ell(y) - 1 = \ell(sy)$.

♣ **[s-stability]** The diagram is not deceptive. According to Lemma 2.3. since $sy < s \cdot sy$ implies that since $w_1 < sy$ we also have $sw_1 < y$. The proof can be concluded by induction. But I'll continue the proof in a way that will suggest an efficient algorithm.

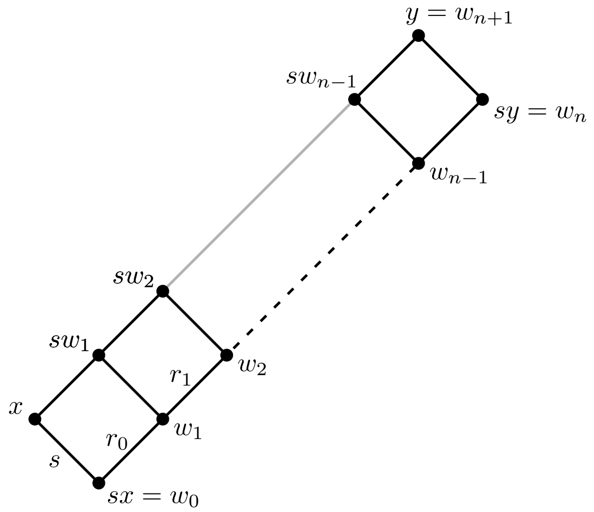
We may keep on filling in as long as $r_i \neq s$:



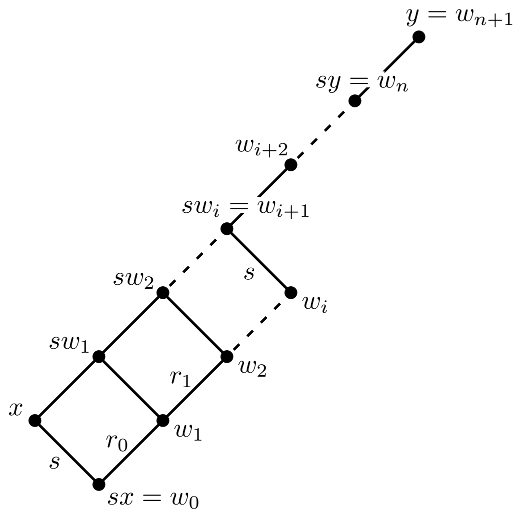
We have $r_n = s$; let i be least with $r_i = s$. So then we get a chain

$$x < sw_1 < sw_2 < \ldots < sw_i = w_{i+1} < w_{i+2} < \ldots sy < y$$

If $i = n$, the picture is this:



♣ **[dixmier]** In this case, $sw_{n-1} < y$ by Corollary 3.2. But then $x < sw_1 < sw_2 < \ldots < sw_{n-1} < y$ is the chain we want. Otherwise $i < n$, and the picture is this:



In this case, the chain is indicated in the diagram. ▯

Suppose $y = sx > x$. Then the set of $z \prec y$ is made up of (a) x together with (b) all the sw where $w \prec x$ and $sw > w$.

Hence:

[cl-construction] **Corollary 4.2.** *Suppose $y = sx > x$. The closure $\mathrm{cl}(y)$ is the union of $\mathrm{cl}(x)$ and $s\,\mathrm{cl}(x)$.*

Of course these may overlap.

5. References

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