Decomposition of symmetric powers

by Bill Casselman

This will be a very elementary talk.

It is a report on work in progress.

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1. Introduction

In *Singularités et transfert*, Bob posed the problem of decomposing at the symmetric powers of irreducible representations of $G = \mathrm{GL}_2(\mathbb{C})$ of finite dimension.

Suppose σ_{std} to be the standard representation of G on $\mathbb{C}^2$, and let

$$\sigma_k = S^k(\sigma_{\mathrm{std}})\,.$$

It is irreducible, of dimension $k+1$. Its character is

$$\mathrm{trace}(\gamma) = \alpha^k + \alpha^{k-1}\beta + \cdots + \alpha\beta^{k-1} + \beta^k \quad \left(\gamma = \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix}\right).$$

What is the decomposition of $S^m(\sigma_k)$ into irreducible representations?

There is a potential application to the trace formula, as we shall see later.

In any case, as Langland remarks, "…ce n'est pas la détermination précise des coefficients [in the decomposition] dont on aura besoin, mais leur comportement asymptotique." We shall see that there is some hope of achieving this, and even of answering the analogous question for arbitary reductive groups.

The first interesting case is the decomposition of σ_2. The answer is quite simple, and well known:

$$S^m(\sigma_2) = \sigma_{2m} + \sigma_{2m-4} \cdot \det{}^2 + \cdots + \begin{cases} \sigma_0 \cdot \det{}^n & \textbf{if } m = 2n \\ \sigma_2 \cdot \det{}^n & \textbf{if } m = 2n+1 \end{cases} .$$

Proof. **Say $\sigma = \sigma_2$ is spanned by e_2, e_0, e_{-2}. Then the weights of $S^m(\sigma)$ correspond to partitions $a_2 + a_0 + a_{-2} = m$. Sort these by the value of a_0. Those for a fixed a_0 are in bijection with partitions of $m - a_0$. Weights of irreducible components can be picked off easily.**

For example, $S^3(\sigma_2)$ as (partition: weight on SL_2) :

$$\begin{array}{llllllll}
(3,0,0): & 6 & (2,1,0): & 4 & (1,2,0): & 2 & (0,3,0): & 0 \\
(2,0,1): & 2 & (1,1,1): & 0 & (0,2,1): & -2 & & \\
(1,0,2): & -2 & (0,1,2): & -4 & & & & \\
(0,0,3): & -6 & & & & & &
\end{array}$$

leading to

$$S^3(\sigma_2) = \sigma_6 + \sigma_2 \cdot \det^2 .$$

But Bob had a fair amount of trouble with $\sigma = \sigma_3$. He remarks that at any rate the state of related investigations is not very advanced, and that "*…il n'est guère utile d'essayer de trouver une expression précise pur les coefficients …*"

Nonetheless, although σ_3 is lesss simple than σ_2, it is possible to figure out, with the help of a computer, exactly what happens for any given m. In addition, there is an exact formula that's apparently been known for along time. It's a bit complicated, but its asymptotic behaviour is very simple. The answer is suggestive and interesting and, one might hope, of eventual value in applying the trace formula.

It is easy enough to compute the decomposition of any one $S^m(\sigma_k)$ explicitly, at least if m and k are small. In general, there is a very simple if not generally practical way to compute the irreducible decomposition of any $S^m(\pi)$.

- Find the weight multiplicities for π, for which there are a number of competing algorithms.

- Traverse all the monomials of degree m in the eigenbasis, accumulating weight multiplicities as you go.

This is unavoidably slow, but works well enough in low dimensions. The program `LiE` computes these weights in *very* low dimensions, but fails lamentably in the interesting range.

Knowing weight multiplicities, one can compute the decomposition multiplicities by applying an almost trivial observation first found in one of Kostant's papers, where it is attributed to Bott (!):

- The decomposition multiplicities can be found by multiplying the weight polynomial by the denominator of a suitable form of Weyl's character formula, and restricting the output to the dominant chamber.

This very simple observation has been rediscovered many times.

For GL_2, special notation is convenient.

If σ is an irreducible representation of GL_2, I define its trace polynomial to be

$$\tau_\sigma(q) = \mathrm{trace}\ \pi(\gamma) \quad \left(\gamma = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\right),$$

which is a polynomial in q. Thus the trace polynomial of σ_k is

$$\tau_k = 1 + q + \cdots + q^k = \frac{q^{k+1} - 1}{q - 1}.$$

Given the central character of π this determines π completely, since

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & b/a \end{bmatrix}.$$

The highest weight of $S^m(\sigma_k)$ is km, and all components will have the same central character. The decomposition will be of the form

$$\sum_0^{\lfloor km/2 \rfloor} c_i \cdot \sigma_{km-2i} \cdot \det{}^i \,.$$

The trace polynomial will then be

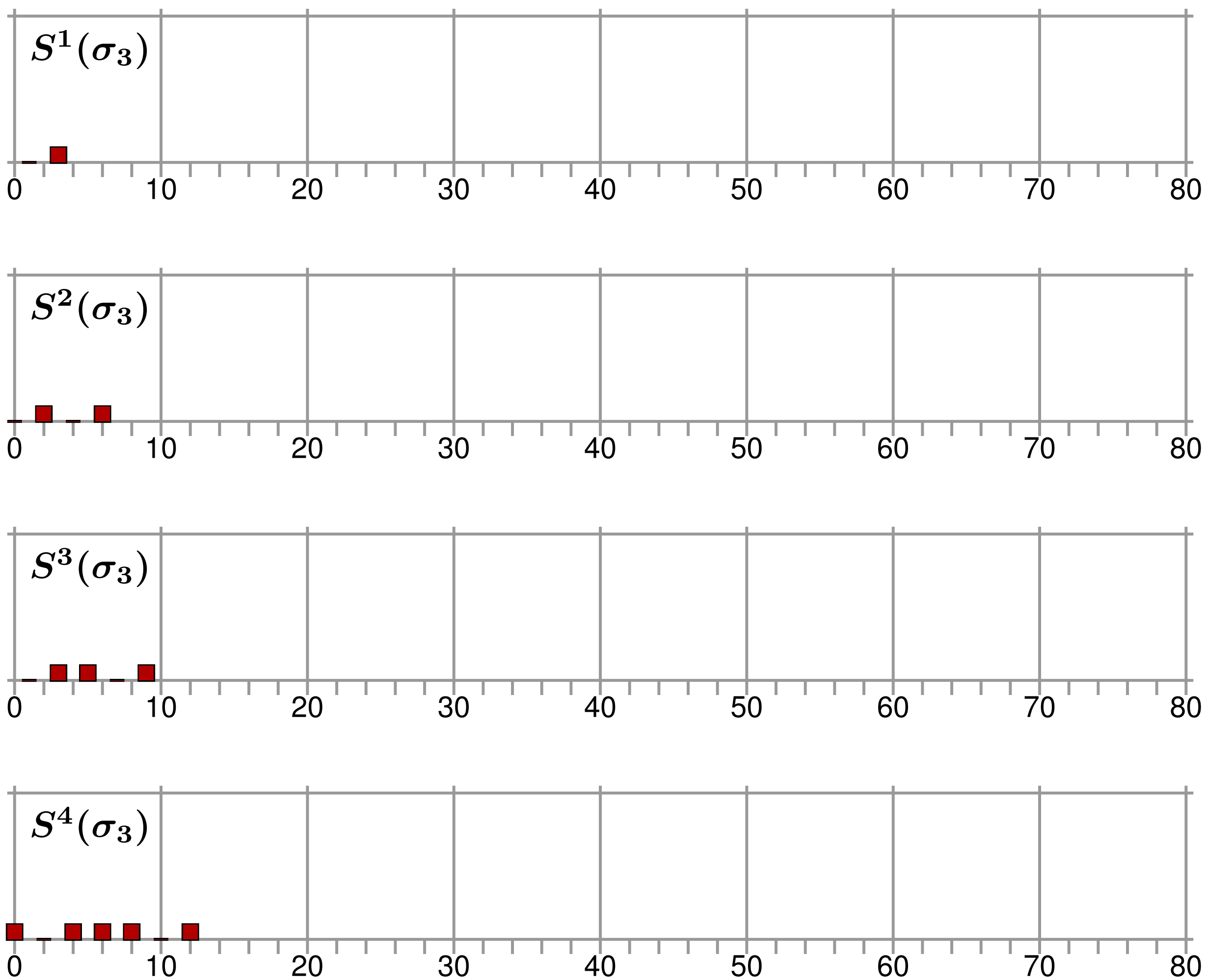
$$\tau_k^m = \sum c_i \cdot q^i \cdot \frac{1 - q^{km-2i}}{1-q}$$

so that

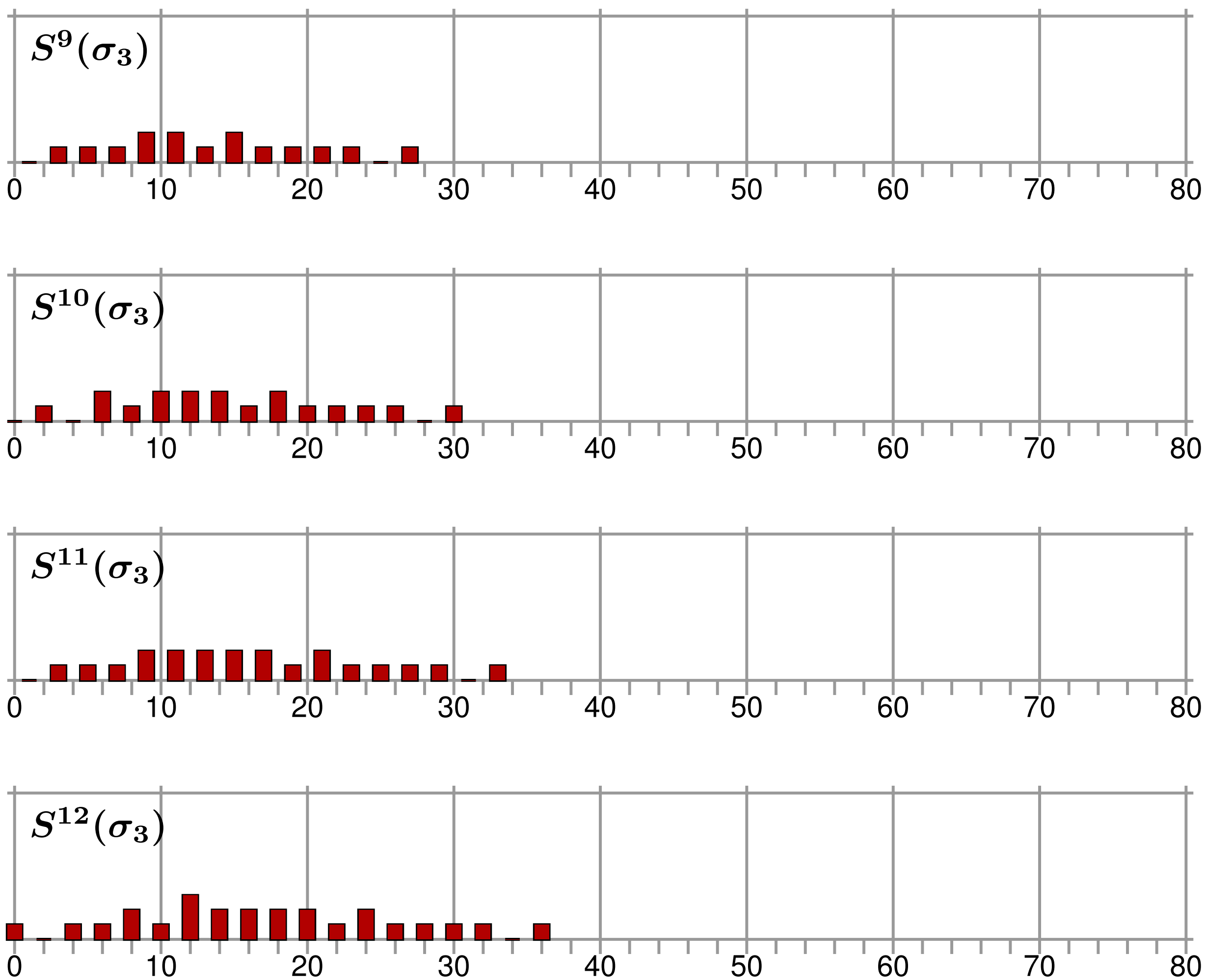
$$(1-q)\tau_k^m = \sum c_i q^i \cdot (1 - q^{km-2i}) \,.$$

The decomposition polynomial $\delta_k^m = \sum c_i q^i$ *will be* $(1-q)\tau_k^m$ *truncated beyond degree* $\lfloor km/2 \rfloor$.

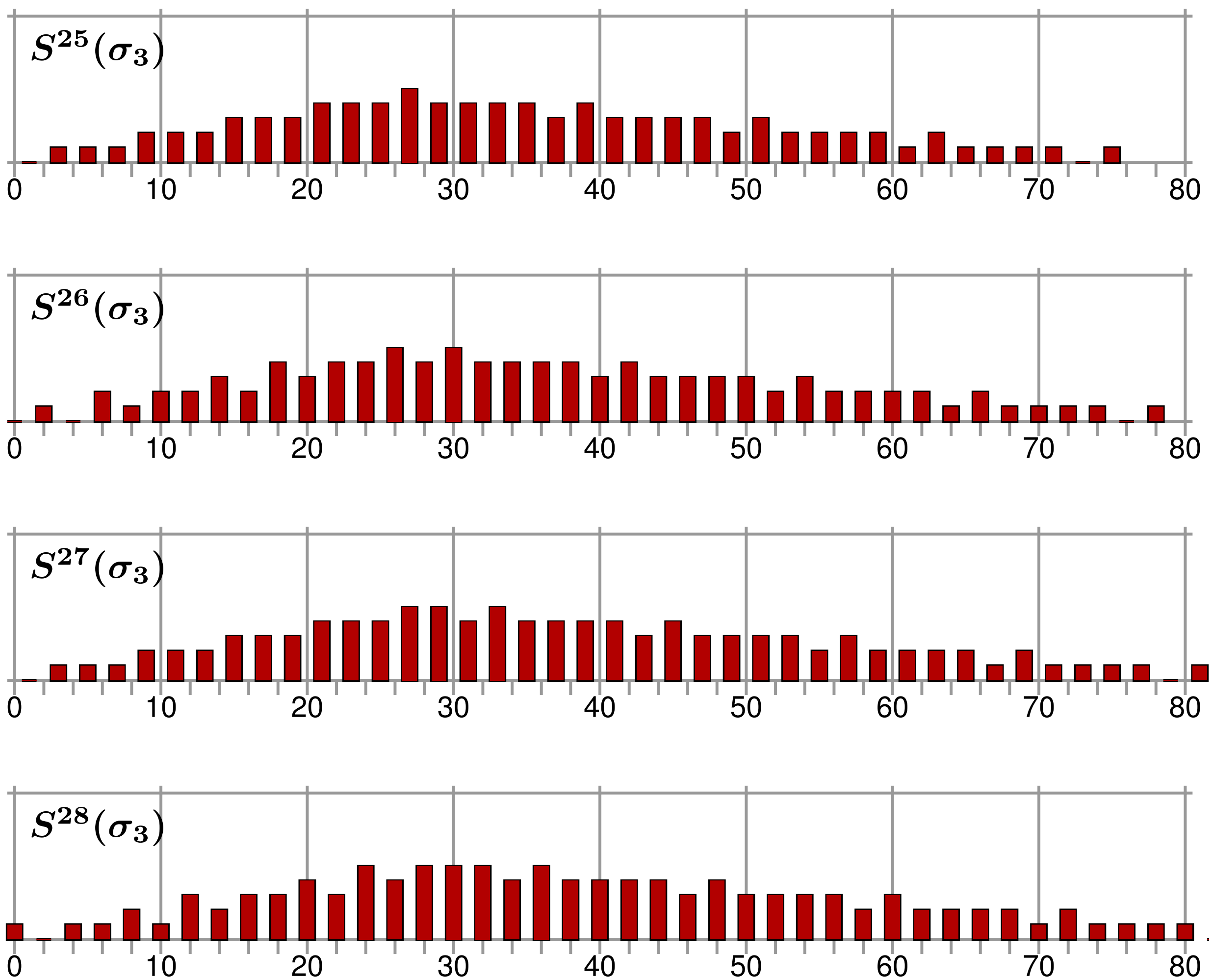
Here are the first few decompositions for σ_3:



Things look somewhat better for larger m ...

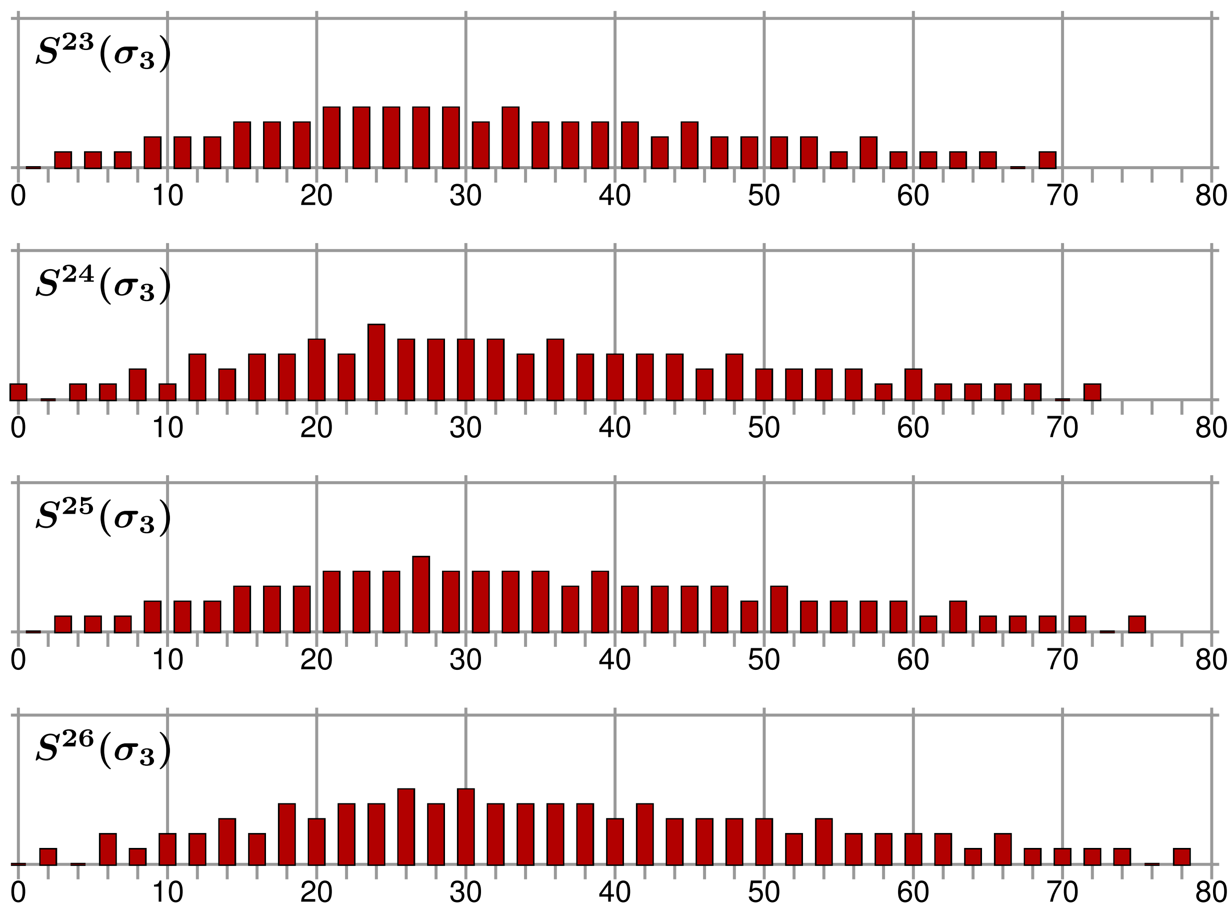


. . . much better for even larger m . . .



. . . **and much, much better for very large m. (Show `A1_3.pdf`.)**

The data clearly exhibit a simple pattern and suggest a conjecture.



The highest weight of σ_3 is 3. The highest weight of $S^m(\sigma_3)$ is $3m$, and it occurs with multiplicity 1. The multiplicity of σ_{km} in S^m is therefore also 1. The other highest weights are of the form $3m-2i$, for $i \leq \lfloor 3m/2 \rfloor$. Let $\mu^m_{3,i}$ be the multiplicity of σ_{3m-2i} in $S^m(\sigma_3)$. Define also the arrays

$$\begin{aligned}
\alpha &= [1,0,1,1,1,1] \\
\beta &= [0,1,1] \\
\gamma_0 &= [1,0,1,0,1,0] \\
\gamma_1 &= [0,1,0,1,0,1]
\end{aligned}$$

Conjecture: *If $j = \lfloor 3m/2 \rfloor - i$ then*

$$\mu^m_{3,i} = \begin{cases} \lfloor i/6 \rfloor + \alpha[i \bmod 6] & \text{if } i \leq m \\ \lfloor j/3 \rfloor + \beta[j \bmod 3] & \text{if } i > m \text{ and } m \equiv 1 \ (2) \\ \lfloor j/3 \rfloor + \gamma_0[j \bmod 6] & \text{if } i > m \text{ and } m \equiv 0 \ (4) \\ \lfloor j/3 \rfloor + \gamma_1[j \bmod 6] & \text{if } i > m \text{ and } m \equiv 2 \ (4) \end{cases}$$

We can at least begin to understand this.

Let's look at the more general situation—we want to decompose $S^m(\sigma_k)$. Its highest weight is α^{km}, and it will decompose as

$$S^m(\sigma_k) = \sum_{0}^{\lfloor km/2 \rfloor} c_i \sigma_{km-2i} \cdot \det^i .$$

(Note the reversed order.)

Recall that

$$\delta_k^m = \sum c_i q^i$$

the decomposition polynomial.

If I write δ_3^m as an array in this way, I get

m	δ_3^m
1	$[1, 0]$
2	$[1, 0, 1, 0]$
3	$[1, 0, 1, 1, 0]$
4	$[1, 0, 1, 1, 1, 0, 1]$
5	$[1, 0, 1, 1, 1, 1, 1, 0]$
6	$[1, 0, 1, 1, 1, 1, 2, 0, 1, 0]$
7	$[1, 0, 1, 1, 1, 1, 2, 1, 1, 1, 0]$
8	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 1, 1, 0, 1]$
9	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 1, 1, 1, 0]$
10	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 1, 2, 0, 1, 0]$
11	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 2, 1, 1, 1, 0]$
12	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 1, 2, 1, 1, 0, 1]$
13	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 2, 2, 1, 1, 1, 0]$
14	$[1, 0, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 2, 2, 1, 2, 0, 1, 0]$

What appears here is some kind of asymptotic series whose coefficients are

$$1, 0, 1, 1, 1, 1,\ 2, 1, 2, 2, 2, 2,\ 3, 2, 3, 3, 3, 3,\ 4, 3, 4, 4, 4, 4,\ 5, 4, 5, 5, 5, 5, \ldots$$

It looks like a kind of geometric series, which is to say the Taylor series of a rational function. There is a well known technique for guessing a rational function, given enough of its Taylor series, and what is proposed here is

$$\frac{1}{(1-q^2)(1-q^3)}\ .$$

So we can make sense out of at least some of what we are looking at.

2. The classical formula

There is a classical formula for the trace polynomial of $S^m(\sigma_k)$. In order to tell you what it is, I must first recall q-analogues of familiar functions.

$$
\begin{aligned}
[n]_q &= 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \\
[n]^!_q &= [n]_q \ldots [1]_q \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]^!_q}{[k]^!_q [n-k]^!_q} \\
&= \frac{[n]_q \ldots [n-k+1]_q}{[k]_q \ldots [1]_q} \\
&= \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)} \\
&= \frac{(1 - q^n) \ldots (1 - q^{n-k+1})}{(1 - q^k) \ldots (1 - q)} .
\end{aligned}
$$

If we set $q = 1$ these evaluate to n, $n!$, and $\binom{n}{k}$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q^!}{[k]_q^! \, [n-k]_q^!} .$$

These fit into a q-analogue of Pascal's triangle:

$$\begin{array}{llllllll}
n & & & & & & & \\
0: & 1 & & & & & & \\
1: & 1 & 1 & & & & & \\
2: & 1 & 1+q & 1 & & & & \\
3: & 1 & 1+q+q^2 & 1+q+q^2 & 1 & & & \\
4: & 1 & 1+q+q^2+q^3 & 1+q+2q^2+q^3+q^4 & \ldots & 1 & & \\
5: & 1 & 1+q+q^2+q^3+q^4 & 1+q+2q^2+2q^3+2q^4+q^5+q^6 & \ldots & \ldots & 1 & \\
 & \ldots & & & & & &
\end{array}$$

There are analogues of classical formulas:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

$$\begin{bmatrix} 0 \\ k \end{bmatrix}_q = \begin{cases} 1 & \textbf{if } k = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

One consequence is that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q. This not quite obvious, just as it is not imediately obvious that $\binom{n}{k}$ is an integer.

Because of well known dimension formulas, it should not surprise you to learn that

- *the trace polynomial* λ_k^n *of* $\bigwedge^n(\sigma_{k-1})$ *is* $q^{n(n-1)/2} \cdot \begin{bmatrix} k \\ n \end{bmatrix}_q$;
- *the trace polynomial* τ_k^n *of* $S^n(\sigma_k)$ *is* $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$.

The proof of the first is by induction, applying Pascal's recursion. The proof of the second exhibits a weight-compatible bijection of bases of $\bigwedge^m(\sigma_{n-1})$ **and** $S^m(\sigma_{n-m})$**.**

I recall: if

$$S^m(\sigma_k) = \sum c_i \cdot \sigma_{km-2i} \cdot \det^i .$$

the decomposition polynomial is

$$\delta(q) = \delta_k^m(q) = \sum_{i=0}^{\lfloor km/2 \rfloor} c_i \, q^i$$

The trace polynomial of $S^m(\sigma_k)$ is then

$$\sum_{0}^{\lfloor km/2 \rfloor} c_i \cdot q^i \cdot \frac{q^{km-2i+1} - 1}{q - 1} = \frac{\sum_{i=0}^{\lfloor km/2 \rfloor} c_i \cdot q^i - \sum_{i=0}^{\lfloor km/2 \rfloor} c_i \cdot q^{km-i+1}}{1-q}$$

$$= \frac{\delta(q) - q^{km - \lfloor km/2 \rfloor + 1} \delta^\vee(q)}{1-q} .$$

Here $\delta^\vee = q^{\lfloor km/2 \rfloor} \delta(q^{-1})$ is the Poincaré dual of δ.

As I have said before:

Proposition. *The decomposition polynomial of $S^m(\sigma_k)$ is obtained from the trace polynomial $(1-q)\tau_k^m(q)$ by truncating all terms of degree larger than $\lfloor km/2 \rfloor$.*

3. A more explicit formulation

Let's see how the classical formula agrees with what we know and what we conjecture. What does it say about $S^m(\sigma_2)$?

$$\frac{(1-q^{m+1})(1-q^{m+2})}{1-q^2} = \begin{cases} (1+q^2+\cdots+q^{2n})(1-q^{m+1}) & \textbf{if } m=2n \\ (1+q^2+\cdots+q^{2n})(1-q^{m+2}) & \textbf{if } m=2n+1 \end{cases},$$

which matches exactly with what we saw before.

One thing that you can see from this example is that although we know that

$$\begin{bmatrix} m \\ k \end{bmatrix}_q$$

is a polynomial, evaluating it explicitly as a polynomial in q will depend on certain congruence conditions.

Since δ_3^m is a truncation of

$$(1-q)\tau_3^m = \frac{(1-q^{m+1})(1-q^{m+2})(1-q^{m+3})}{(1-q^2)(1-q^3)} \, .$$

we can already understand part of our conjecture. *The decomposition polynomial agrees with the Taylor series of $1/(1-q^2)(1-q^3)$ up through terms of degree m.*

But we can actually prove the conjecture by induction. Define

$$\mu_3^m = \sum \mu_{3,i}^m \cdot q^i \, .$$

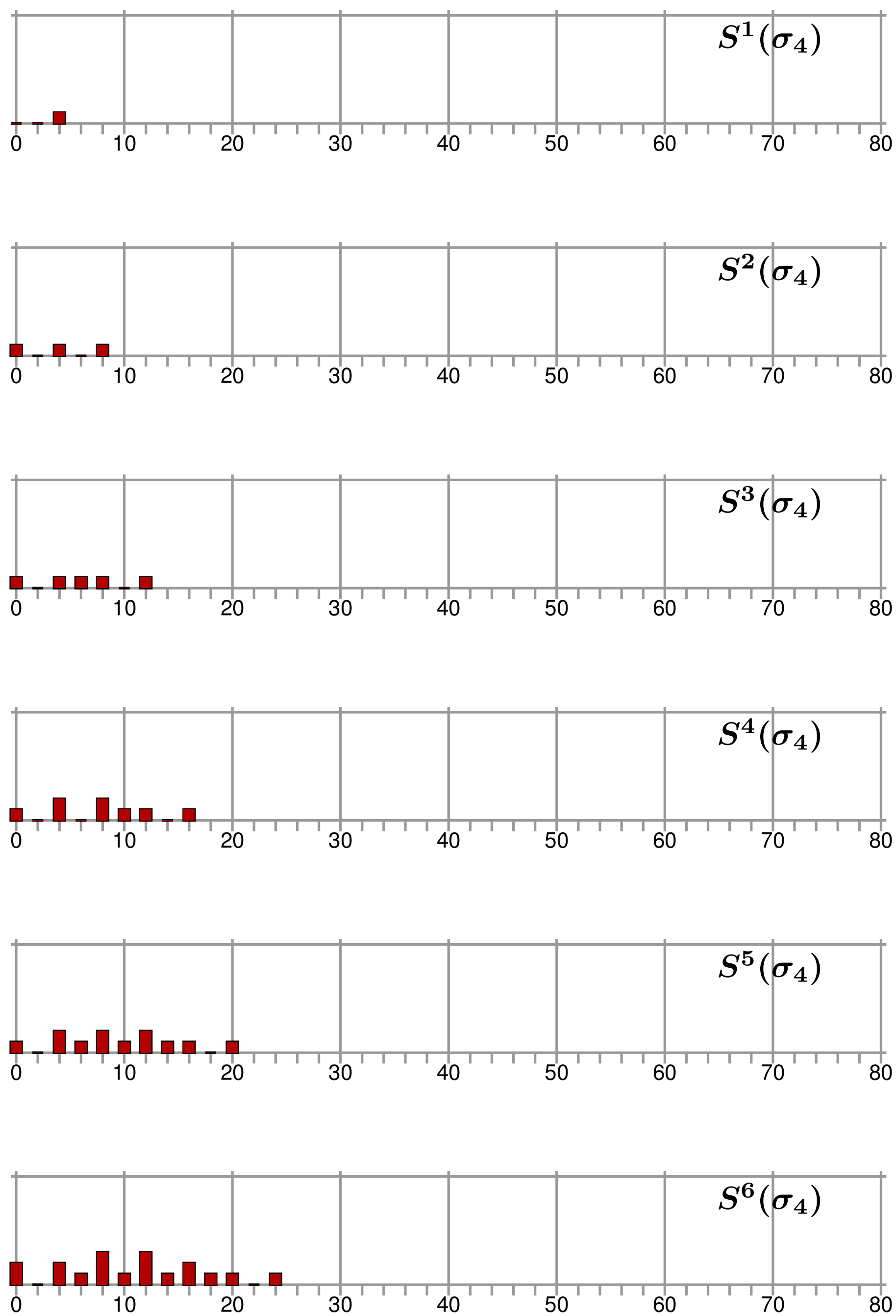
We need only verify initial conditions and the recursion

$$\mu_3^m = \mu_3^{m-1} + q^n \mu_2^{m-1} \, ,$$

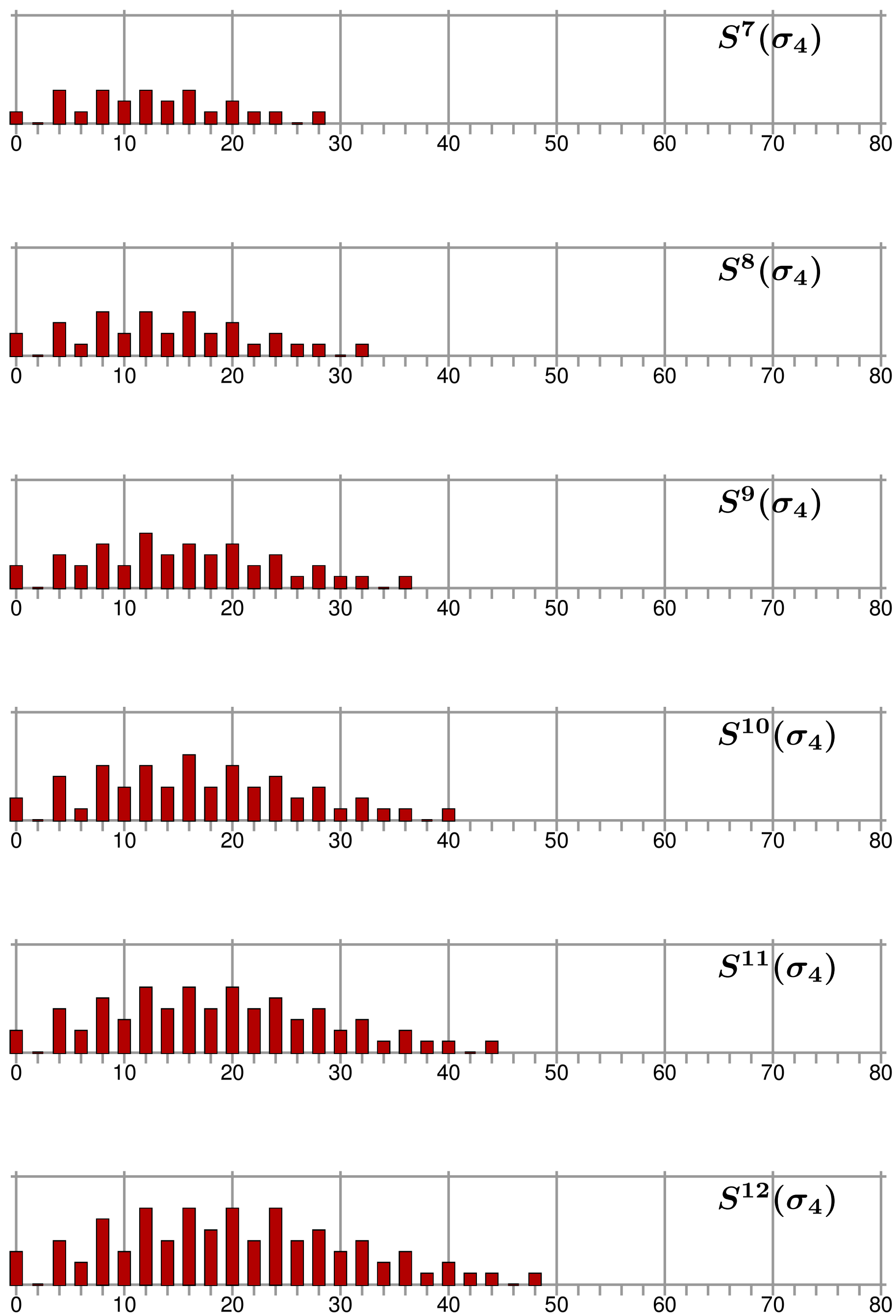
which is straightforward if a bit tedious.

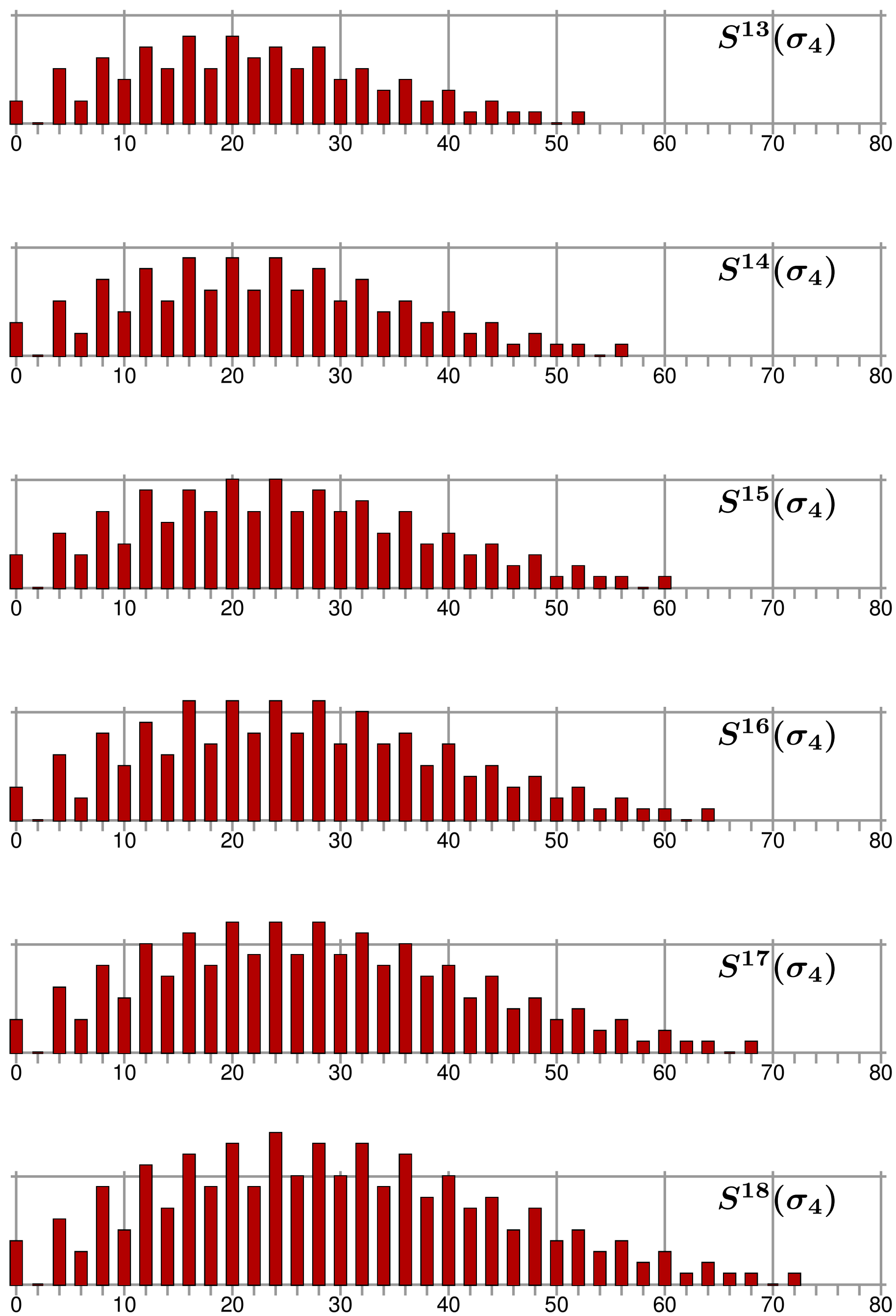
4. Other examples

First σ_4.



$S^1(\sigma_4)$
$S^2(\sigma_4)$
$S^3(\sigma_4)$
$S^4(\sigma_4)$
$S^5(\sigma_4)$
$S^6(\sigma_4)$
0 10 20 30 40 50 60 70 80


$S^7(\sigma_4)$
$S^8(\sigma_4)$
$S^9(\sigma_4)$
$S^{10}(\sigma_4)$
$S^{11}(\sigma_4)$
$S^{12}(\sigma_4)$
0 10 20 30 40 50 60 70 80



$S^{13}(\sigma_4)$
$S^{14}(\sigma_4)$
$S^{15}(\sigma_4)$
$S^{16}(\sigma_4)$
$S^{17}(\sigma_4)$
$S^{18}(\sigma_4)$
0 10 20 30 40 50 60 70 80

A1_4.pdf

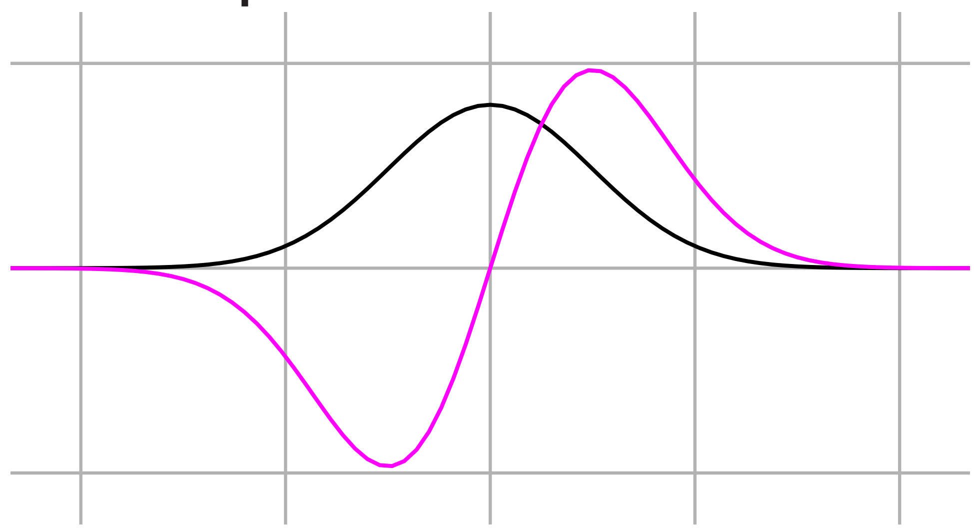
A1_6.pdf

There is one observation that should make things clearer. Multiplying by $1-q$ replaces the polynomial $\sum c_i q^i$ by $\sum (c_i - c_{i-1}) q^i$. Taking our scaling into account, it becomes a derivative. This suggests that it might be a fruitful idea to graph the weight polynomial τ_k^m rather than the decomposition polynomial δ_k^m.

a1_wts_3.pdf

a1_wts_3a.pdf

I.e. weights versus decomposition:



a1_wts_16.pdf

It looks very much as though the trace polynomials become closer and closer to the distribution of a normal curve as $k \to \infty$, and the decomposition polynomial would then have as limit the negative of its derivative. So we are presumably dealing with some new version of the central limit theorem!

This is remarkable. There is a known limit formula (due to Gerrit Heckman) for the weights of the irreducible representation $\pi^{n\lambda}$ (any reductive group) as $n \to \infty$, but although quite elegant, it is also much more complicated. It $appears$ that the symmetric powers are simpler. But I have very little evidence for this for groups of higher rank.

One obstacle is that I do not have a very fast way to compute the weights of symmetric powers, which grow in size very rapidly.

We know that

$$\tau_k^m = \frac{(1-q^{m+1})\dots(1-q^{m+k})}{(1-q)\dots(1-q^k)}\,.$$

Up through terms of degree m this agrees with the Taylor series of

$$\frac{1}{(1-q)\dots(1-q^k)}\,.$$

What is that series?

It is

$$\sum_0^\infty N_n q^n$$

with

$$N_n = \left|\{(m_i)_1^k \mid \sum m_i \cdot i = n\}\right|.$$

This is the same as the number of integral points $(m_i)_{i\geq 2}$ such that $\sum m_i \cdot i \leq n$, which is asymptotically of the form Cn^{k-1}. An exact formula will depend on n modulo $k!$.

I believe I can show that asymptotically τ_k^m is equal to a polynomial of degree $k-1$ on each of several intervals that can be specified simply. I seem to have an algorithm for finding these polynomials for any given m.

There is a way to see intuitively what's going on. As m grows large by k remains fixed, the weight polynomial looks more and more like

$$\frac{(1-q^{m+1})^k}{(1-q^k)} = (1+\cdots+q^m)^k \,,$$

which does indeed, upon scaling, record the distribution of k uniformly distributed random variables. So at the moment I conjecture that the *limit distribution of the weight polynomial is such a distribution*.

5. Basic functions

How and why did Bob arrive at the problem of symmetric power decomposition? There is a formula attributed in the literature to Molien that tells us that if T is any linear transformation on a finite dimensional vector space

$$\frac{1}{\det(I - Tx)} = \sum_{m \geq 0} x^m \,\mathrm{trace}\;\, S^m(T)$$

This can be applied to the case where $T = \sigma(\mathfrak{F}_\pi)$, with $\mathfrak{F}_\pi$ equal to what Bob calls the Frobenius-Hecke element of an L-group ${}^LG$.

If $x = q^{-s}$, the left-hand side becomes $L(s, \pi, \sigma)$. Each term in the infinite sum defines a conjugation-invariant affine function on the L-group, and is therefore (according to one of Bob's original observations) in the image of the Satake transform.

In some circumstances, the L-group has a center isomorphic to $\mathbb{C}^\times$, the unramified group G possesses an analogue of the determinant map, and

$$\sigma(\pi)\cdot q^{-s} = \sigma(\pi)\cdot|\det|^s\,.$$

In these circumstances, let f_σ be the sum of inverse Satake transforms. Each term will have support on $|\det| = q^{-ms}$, and the sum will be locally finite. It has been suggested that it will make some kind of sense to use f_σ in the trace formula, even though it does not have compact support.

What is the asymptotic behaviour of f_σ as $|\det| \to 0$?

In general, finding an answer to this question depends on two things—the inverse Satake transform and the decomposition of symmetric powers. For GL_2, the Satake transform is simple enough that the answer depends only on the symmetric power decomposition.

Proposition. *Suppose λ a dominant weight for GL_2, and that*

$$S^m(\sigma_\lambda) = \sum c_i\,\sigma_{m\lambda - i\alpha}\,.$$

Then the basic function evaluated at $m\lambda - i\alpha$ is

$$\Phi_{m\lambda - i\alpha} = \sum_{0\le\ell\le i} c_\ell q^\ell\,.$$

Proof. **Suppose that**

$$S^m(\sigma_\lambda) = \sum c_i\sigma_{m\lambda - i\alpha}\,.$$

As is well known,

$$\mathfrak{S}^{-1}\tau_{m\lambda - i\alpha} = \sum q^{-j} f_{m\lambda - i\alpha - j\alpha}$$

which leads to

$$S^m(\sigma_\lambda) = \sum_{i,j} c_i f_{m\lambda - i\alpha - j\alpha} = \sum_\ell f_{m\lambda - \ell\alpha} \sum_i q^{-(\ell - i)} c_i\,.$$

in which the first sum is over all ℓ for which $m\lambda - \ell\alpha \ge 0$, the second over $0 \le i \le \ell$. Dualize. **QED**

For groups other than GL_2, such as GL_3 and Sp_4, I have some intriguing evidence that an eventual answer is not out of reach. For GSp_4 I have an extremely simple conjecture, based on extensive computation, for the basic function associated to the standard representation. As a result of joint work with Tom Hales on a completely different matter, I have a method to work with arbitrary unramified groups. For the p-adic group SU_3 this reduces to computations for GL_2, and in general the Langlands L-function is always related to an L-function for a split group. This is because of the curious form of the 'twisted Weyl character formula'.

There is some reason to think that for groups of higher rank it is the basic function for which one expects a relatively simple formula. The geometry of the Vinberg monoids suggests this.

But, as Bob has said, at this point the technology of symmetric power decomposition, although in a poor state, is in advance of possible applications to the trace formula.

I wish to thank Ali Altug for encouragement.

6. References

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