CHAPTER V

THEOREM 1: PROOF FOR ABELIAN EXTENSIONS

Let $G = G(K : k)$ be the Galois group of finite abelian extension $K/k$. $G$ is a direct product of cyclic groups.

$$G = G_1 \times \cdots \times G_r$$

Put $G_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_r$. Let $K_i$ be the fixed field of $G_i$. Then $G(K_i : k) = G/G_i$ is cyclic. The fixed field of $K_1K_2, \ldots K_j$ is $G_1 \cap \cdots \cap G_j$, and

$$G_1 \cap \cdots \cap G_j = \{(\sigma_1, \ldots, \sigma_r) \in G_1 \times \cdots \times G_r \mid \sigma_1 = \cdots = \sigma_j = 1\}$$

The fixed field of $K_1 \ldots K_j \cap K_{j+1}$ is $(G_1 \cap \cdots \cap G_j)G_{j+1} = G$. Therefore

$$K_1 \ldots K_j \cap K_{j+1} = k$$

so

$$G(K_1 \ldots K_jK_{j+1}) \simeq G(K_1 \ldots K_j : k) \times G(K_{j+1} : k)$$

Arguing by induction, we arrive at

$$G(K_1 \ldots K_r : k) \simeq G(K_1 : k) \times \cdots \times G(K_r : k)$$

This isomorphism maps $\sigma$ to $(\sigma_1, \ldots, \sigma_r)$ where $\sigma_i$ is the restriction of $\sigma$ to $K_i$.

Let $E$ be the set of primes of $k$ containing all infinite primes and all primes which are ramified in $K$. For $p \notin E$ the Artin symbols are defined and we have

$$\left(\frac{K : k}{p}\right) = \left(\left(\frac{K_1 : k}{p}\right), \ldots, \left(\frac{K_r : k}{p}\right)\right),$$

and for $i \in I_k \{E\}$ we have

$$\phi_{K/k}(i) = \prod_{p \notin E} \left(\frac{K : k}{p}\right)^{u_p} \quad \text{where } |i|_p = Np^{-u_p}$$

$$= \left(\left(\frac{K_1 : k}{p}\right)^{u_p}, \ldots, \left(\frac{K_r : k}{p}\right)^{u_p}\right),$$
or

\[ \phi_{K/k}(i) = (\phi_{K_1/k}(i), \ldots, \phi_{K_r/k}(i)). \]

The right side of (5.1) agrees with (2.1) on \( I_k \{ E \} \), is defined for all \( i \) in \( I_k \), and the kernel contains \( k^* \). Define \( \phi_{K/k}(i) \) on \( I_k \) by (5.1). Except for the proofs of the first and second fundamental inequalities, this completes the proof of theorem 1 for finite abelian extensions.