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Essays on representations of p-adic groups

Smooth representations

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In this chapter I'll define admissible representations and prove their basic properties. For technical reasons, I shall initially take the coefficient right to be an arbitrary Noetherian ring.

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1. Definitions

Let $\mathcal{R}$ be a commutative Noetherian ring containing the field $\mathbb{Q}$. A **smooth** G module over $\mathcal{R}$ is a representation (π, V) of G on a module V over $\mathcal{R}$ such that each v in V is fixed by an open subgroup of G. The smooth representation (π, V) is said to be **admissible** if for each open subgroup K in G the subspace V^K of vectors fixed by elements of K is finitely generated over $\mathcal{R}$. Usually R will be a field (necessarily of characteristic 0) in which case this means just that V^K has finite dimension.

Suppose (π, V) to be a smooth representation of G over $\mathcal{R}$. If D is a smooth distribution of compact support with values in $\mathcal{R}$, there is a canonical operator $\pi(D)$ on V associated to it. Fix for the moment a right-invariant Haar measure dx on G. Recall that given the choice of dx, smooth R-valued functions may be identified with smooth distributions:

$$\varphi \longmapsto D_\varphi = \varphi(x)\,dx$$

Suppose φ to be the smooth function of compact support on G such that $D = D_\varphi$. Then for v in V we define

$$\pi(D)v = \int_G \varphi(x)\pi(x)\,dx\ .$$

If K is a compact open subgroup then the distribution μ_K amounts to integration over K, scaled so as to be idempotent. The operator $\pi(\mu_K)$ is projection from V onto its subspace V^K of vectors fixed by K.

Suppose that D_1 and D_2 are two smooth distributions of compact support, corresponding to smooth function φ_1 and φ_2. Then

$$\begin{aligned}
\pi(D_1)\pi(D_2)v &= \int_G \varphi_1(x)\pi(x)\,dx \int_G \varphi_2(y)\pi(y)v\,dy \\
&= \int_{G\times G} \varphi_1(x)\varphi(y)\pi(xy)v\,dx\,dy \\
&= \int_G \varphi(z)\pi(z)v\,dz \\
&= \pi(D_\varphi)v \\
\text{where}\quad \varphi(z) &= \int_G \varphi_1(zy^{-1})\varphi_2(y)\,dy
\end{aligned}$$

The distribution D_φ, also smooth and of compact support, is called the **convolution** $D_1 * D_2$ of the two operators D_1 and D_2.

The point of allowing a smooth representation to have a rather arbitrary coefficient ring is essentially a matter of book-keeping, so as to keep track of the kind of formulas that arise. If K is a compact open subgroup of G, with V^K is free over the ring R (which will often be the case) and D is right- and left-invariant under K, then $\pi(D)$ will be represented by a matrix whose coefficients are in R.

The space of smooth R-valued distributions of compact support, with convolution as product, is called the **Hecke algebra** $\mathcal{H}_R(G)$ of the group with coefficients in R. It does not have a multiplicative unit. The subalgebra $\mathcal{H}_R(G/\!/K)$ of distributions right- and left- invariant with respect to a compact open subgroup K has the multiplicative unit μ_K.

For every closed subgroup H of G, define $V(H)$ to be the subspace of V generated by the $\pi(h)v - v$ for h in H.

[projection] **Proposition 1.1.** *For any compact open subgroup K and smooth representation V, we have an equality*

$$V(K) = \{v \in V \mid \pi(\mu_K)v = 0\}$$

and a direct sum decomposition

$$V = V(K) \oplus V^K \ .$$

Proof. If v is fixed by K_* then

$$\frac{1}{[K\colon K_*]} \sum_{K/K_*} \pi(k)v = \pi(\mu_K)v$$

and of course trivially

$$\frac{1}{[K\colon K_*]} \sum_{K/K_*} v = v \ .$$

If we subtract the second from the first, we get

$$v - \pi(\mu_K)v = \frac{-1}{[K\colon K_*]} \sum_{K/K_*} \big(\pi(k)v - v\big)$$ ▯

[vkexactness] **Corollary 1.2.** *The functor $V \rightsquigarrow V^K$ is exact for every compact open subgroup K of G.*

[abelian] **Corollary 1.3.** *If*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence of smooth representations, V is admissible if and only if both U and W are.

[restriction-to-K] **Proposition 1.4.** *Suppose K to be a fixed compact open subgroup of G. A smooth representation is admissible if and only if its restriction to K is the direct sum of irreducible smooth representations of K, each with finite multiplicity.*

Proof. Choose a sequence of compact open subgroups K_n normal in K and with $\{1\}$ as limit. Then $V = V(K_n) \oplus V^{K_n}$. The representation of K/K_n decomposes into a finite sum of irreducible representations of K. ▯

2. The contragredient

If (π, V) is an admissible representation of G, the smooth vectors in its linear dual $\mathrm{Hom}_{\mathcal{R}}(V, \mathcal{R})$ define its **contragredient** representation $(\widetilde{\pi}, \widetilde{V})$. If K is a compact open subgroup of G then because $V = V^K + V(K)$ the subspace of K-fixed vectors in $\widetilde{V}$ is equal to

$$\mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R}) .$$

From the exact sequence of R-modules

$$\mathcal{R}^n \longrightarrow V^K \longrightarrow 0$$

we deduce

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(V^K, \mathcal{R}) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^n, \mathcal{R}) \cong \mathcal{R}^n .$$

Therefore $\widetilde{V}^K$ is finitely generated over $\mathcal{R}$, and $\widetilde{\pi}$ is again admissible. If $\mathcal{R}$ is a field, which is often the only case in which contragredients are significant, the assignment of $\widetilde{\pi}$ to π is exact, and the canonical map from V into the contragredient of its contragredient will be an isomorphism.

If (π, V) is an admissible representation of G then each space V^K is stable under the centre Z_G of G. Assume for the moment that $\mathcal{R}$ is an algebraically closed field. The subgroup $Z_G \cap K$ acts trivially on it, and the quotient $Z_G/Z_G \cap K$ is finitely generated, so it elementary to see that V^K decomposes into a direct sum of primary components V^K_ω parametrized by homomorphisms

$$\omega\colon\ Z_G \longrightarrow \mathcal{R}^\times .$$

For each ω occurring there exists an integer n such that $\big(\pi(z) - \omega(z)\big)^n = 0$ on V^K_ω. If π is irreducible there is just one component and the centre must act by scalars. In general, I call an admissible representation **centrally simple** if this occurs, and in this case the character $\zeta_\pi\colon Z_G \to \mathcal{R}^\times$ by which Z_G acts is called the **central character** of π. If Z_G acts through the character ω then π is called an ω-representation. For any central character ω with values in $R^\times$ the Hecke algebra $\mathcal{H}_{R,\omega}$ is that of uniformly smooth functions on G compactly supported modulo Z_G such that

$$f(zg) = \omega(z)^{-1} f(g) .$$

If π is centrally simple with central character ω it becomes a module over this Hecke algebra:

$$\pi(f)v = \int_{G/Z_G} f(x)\pi(x)v \, dx ,$$

which is well defined since $f(zx)\pi(zx) = f(x)\pi(x)$.

3. Admissible representations of parabolic subgroups

Let $P = M_P N_P = MN$ be a parabolic subgroup of G, $A = A_P$ the split centre of M_P. There exists a basis of neighbourhoods of P of the form $U_M U_N$ where U_M is a compact open subgroup of M, U_N is one of N, and U_M conjugates U_N to itself.

♣ [induced-admissible] Since $P\backslash G$ is compact, Proposition 4.1 implies immediately:

[parabolic-induced-admissible] **Proposition 3.1.** *If (σ, U) is an admissible representation of P then $\mathrm{Ind}(\sigma \,|\, P, G)$ is one of G.*

The group M may be identified with a quotient of P, and therefore the admissible representations of M may be identified with those of P trivial on N. It happens that there are no others:

[parabolic-admissible] **Proposition 3.2.** *Every admissible representation of P is trivial on N.*

Proof. We begin with a preliminary result.

[noetherian] **Lemma 3.3.** *Suppose B to be a finitely generated module over the Noetherian ring R. If $f\colon B \to B$ is an R-injection with the property that for each maximal ideal $\mathfrak{m}$ of R the induced map $f_\mathfrak{m}\colon B/\mathfrak{m}B \to B/\mathfrak{m}B$ is also injective, then f is itself an isomorphism.*

Proof. Let C be the quotient $B/f(B)$. The exact sequence

$$0 \longrightarrow B \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

induces for each $\mathfrak{m}$ an exact sequence

$$0 \longrightarrow B/\mathfrak{m}B \xrightarrow{f_\mathfrak{m}} B/\mathfrak{m}B \longrightarrow C/\mathfrak{m}C \longrightarrow 0\,.$$

It is by assumption that the left hand map is injective. Since $F = R/\mathfrak{m}$ is a field and B is finitely generated, the space $B/\mathfrak{m}B$ is a finite-dimensional vector space over F, and therefore $f_\mathfrak{m}$ an isomorphism. Hence $C/\mathfrak{m}C = 0$ for all $\mathfrak{m}$. The module C is Noetherian, which means that if $C \neq 0$, it possesses at least one maximal proper submodule D. The quotient C/D must be isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. But then $C/\mathfrak{m}C \neq 0$, a contradiction. Therefore $C = 0$ and f an isomorphism. ▯

Let (π, V) be an admissible representation of P, and suppose v in V. We want to show that $\pi(n)v = v$ for all n in N.

Let $U = U_M U_N$ be a compact opne subgroup of P fixing v. We can find a in A such that $a^{-1}Ua \subseteq U$. For u in U we have

$$\pi(u)\pi(a)v = \pi(a)\pi(a^{-1}ua)v = \pi(a)v$$

♣ [noetherian] so that V^U is stable under $\pi(a)$. That $\pi(a)$ is surjective on V^U, hence bijective, follows from Lemma 3.3.

For any n in N there will exist some power b of a such that $b^{-1}nb$ lies in U. But then for v in V^U the vector $v_* = \pi(b)^{-1}v$ will also lie in V^U and

$$\begin{aligned}\pi(n)v &= \pi(n)\pi(b)\pi(b^{-1})v \\ &= \pi(n)\pi(b)v_* \\ &= \pi(b)\pi(b^{-1}nb)v_* \\ &= \pi(b)v_* \\ &= v\end{aligned}$$

which means that N fixes all vectors in V^U and in particular v. ▯

4. Induced representations

If H is a closed subgroup of G and (σ, U) is a smooth representation of H, the **unnormalized** smooth representation $\mathrm{ind}(\sigma \mid H, G)$ **induced** by σ is the right regular representation of G on the space of all uniformly smooth functions $f \colon G \to U$ such that

$$f(hg) = \sigma(h) f(g)$$

for all h in H, g in G. The **normalized** induced representation is

$$\mathrm{Ind}(\sigma \mid H, G) = \mathrm{ind}\big(\sigma \delta_H^{1/2} \delta_G^{-1/2} \mid H, G\big) \ .$$

Compactly supported induced representations ind_c and Ind_c are on spaces of functions of compact support on G modulo H.

[induced-admissible] **Proposition 4.1.** *If $H\backslash G$ is compact and (σ, U) admissible then $\mathrm{Ind}(\sigma \mid H, G)$ is an admissible representation of G.*

Proof. If $H\backslash G/K$ is the disjoint union of cosets HxK (for x in a finite set X), then the map

$$f \longmapsto \big(f(x)\big)$$

is a linear isomorphism

$$\mathrm{Ind}(\sigma \mid H, G)^K \cong \bigoplus_{x \in X} U^{H \cap xKx^{-1}}$$ ▯

[also-free] **Corollary 4.2.** *If U is free over $\mathcal{R}$ so are the induced representations.*

This follows from the proof.

Suppose (π, V) to be a smooth representation of G, (σ, U) one of H. The map

$$\Lambda \colon \mathrm{Ind}(\sigma \mid H, G) \to U$$

taking f to $f(1)$ is an H-morphism from $\mathrm{Ind}(\sigma)$ to $\sigma \delta_H^{1/2} \delta_G^{-1/2}$. If we are given a G-morphism from V to $\mathrm{Ind}(\sigma \mid H, G)$ then composition with Λ induces an H-morphism from V to $\sigma \delta_H^{1/2} \delta_G^{-1/2}$.

[frobenius] **Proposition 4.3.** *If π is a smooth representation of G and σ one of H then evaluation at 1 induces a canonical isomorphism*

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma \mid H, G)\big) \to \mathrm{Hom}_H\big(\pi, \sigma \delta_H^{1/2} \delta_G^{-1/2}\big) \ .$$

♣ [one-densities] For F in $\mathrm{Ind}(\widetilde{\sigma} \mid H, G)$ and f in $\mathrm{Ind}_c(\sigma \mid H, G)$ then according to •the function $\langle F(g), f(g) \rangle$ is a left-H-invariant one-density of compact support on $H\backslash G$. If we are given right invariant Haar measures dg on G and dh on H then we can define a canonical pairing between $\mathrm{Ind}(\widetilde{\sigma} \mid H, G)$ and $\mathrm{Ind}_c(\sigma \mid H, G)$ according to the formula

$$\langle F, f \rangle = \int_{H\backslash G} \langle F(x), f(x) \rangle \, dx$$

Thus there is an essenially canonical G-covariant map from $\mathrm{Ind}(\widetilde{\sigma} \mid H, G)$ to the smooth dual of $\mathrm{Ind}_c(\sigma \mid H, G)$. ▌ In particular, if $\mathcal{R} = \mathbb{C}$ and σ is unitary so is $\mathrm{Ind}(\sigma \mid H, G)$.