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MESSENGER OF MATHEMATICS.

NOTE ON CONTINUED FRACTIONS.*

By *Henry J. Stephen Smith*, Savilian Professor of Geometry in the University of Oxford.

1. LET $\frac{p}{q}=\mu_1+\frac{1}{\mu_2+}\,\frac{1}{\mu_3+}\dots\frac{1}{\mu_s}$, p and q being two numbers relatively prime, of which p is the greater. Writing, for convenience $P=\frac{1}{p}$ and $Q=\frac{1}{q}$, we divide a line 01 of unit length (measured from left to right) into p equal parts at the points $1P$, $2P$, $3P$, ...$(p-1)P$; and also into q equal parts at the points $1Q$, $2Q$, $3Q$, ...$(q-1)Q$. We do not reckon 0 either as a point P or as a point Q, but we reckon 1 both as a point P and as a point Q, so that we have in all p points P, and q points Q, of which none are coincident, excepting the two extreme points, which coincide at 1.

2. It is the purpose of this note to show that the arrangement of the points P and Q upon the line 01, or, which is the same thing, the arrangement in order of magnitude of the proper fractions $\frac{x}{p}$ and $\frac{y}{q}$, may be inferred from the development of $\frac{p}{q}$ in a continued fraction; and that, *vicè versâ*, the development of $\frac{p}{q}$ may be inferred from an

* The substance of this note was communicated to the Mathematical Section of the British Association, at the Bristol Meeting in 1875.

inspection of the arrangement of the points. An example will serve to explain the nature of the relation which we have to establish.

3. Let $p=39$, $q=17$, so that we have the development $\frac{39}{17}=2+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{2}$; the arrangement of the points P and Q is indicated in the following scheme, in which transverse lines are placed at the close of each of the *sequences** to be presently defined.

$$P, 2P, Q \mid 3P, 4P, 2Q \mid 4P, 5P, 3Q \mid 7P, \| 8P, 9P, 4Q \mid,$$
$$10P, 11P, 5Q \mid 12P, 13P, 6Q \mid 14P \| 15P, 16P, 7Q \,|||,$$
$$17P, 18P, 8Q \mid 19P, 20P, 9Q. \mid 21P, 22P, 10Q \mid 23P \|$$
$$24P, 25P, 11Q \mid 26P, 27P, 12Q \mid 28P, 29P, 13Q \mid 30P, \|$$
$$31P, 32P, 14Q \,|||\, 33P, 34P, 15Q \mid 35P, 36P, 16Q \mid$$
$$37P, 38P, 17Q \mid 39P \,||||.$$

In this scheme, because $\mu_1=2$, we have two points P before we come to a point Q; the sequence PPQ, which consists of μ_1 points P followed by a point Q, we term a sequence of order 1; this sequence is repeated three times, because $\mu_2=3$, and is then followed by a single point P (which is a sequence of order zero); a sequence, such as $PPQ \mid PPQ \mid PPQ \mid P$, consisting of μ_2 sequences of order 1, followed by a sequence of order zero, we term a sequence

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* These sequences have been already noticed by M. Christoffel, in an interesting paper entitled "Observatio Arithmetica," (*Annali di Matematica*, 2nd series, vol. VI., p. 148), with which I unfortunately did not become acquainted until my own investigation was completed. M. Christoffel considers the least positive remainders of the series of numbers q, $2q$, $3q$... for the modulus p, and designates any remainder by the symbol c or d, according as it is less or greater than the remainder immediately following. It is easily seen that the sequences of the symbols c and d coincide with the sequences of the points P and Q. For if the remainder of sq is greater than the remainder of $(s+1)\,q$, we shall have, for some integral value of h, the inequalities

$$(h-1)\,p < sq < hp < (s+1)\,q < (h+1)\,p,$$

whence

$$\frac{s}{p} < \frac{h}{q} < \frac{s+1}{p},$$

or the point hQ lies between the points sP and $(s+1)\,P$. And, again, if the remainder of sq is less than the remainder of $(s+1)\,q$, the inequalities

$$(h-1)\,p < sq < hp,\quad (h-1)\,p < (s+1)\,q < hp,$$

which give immediately

$$\frac{h-1}{q} < \frac{s}{p},\ \frac{h}{q} > \frac{s+1}{p},$$

prove that no point Q can lie between the points sP and $(s+1)\,P$.

of order 2; it contains $\mu_1\mu_2+1$ points P, μ_2 points Q; this sequence of order 2 is, in the scheme before us, repeated twice, because $\mu_3=2$, and is then followed by a sequence of order 1; the sequence thus obtained, consisting of μ_3 sequences of order 2, followed by a sequence of order 1, we term a sequence of order 3. This sequence, containing $\mu_1\mu_2\mu_3+\mu_1+\mu_3=16$ points P, and $\mu_2\mu_3+1=7$ points Q, is in the scheme repeated twice, because $\mu_4=2$, and is followed by a sequence of order 2. We thus obtain a sequence of order 4, consisting of μ_4 sequences of order 3, followed by a sequence of order 2, and containing

$$\mu_1\mu_2\mu_3\mu_4+\mu_1\mu_2+\mu_1\mu_4+\mu_3\mu_4+1=39 \text{ points } P,$$

and $$\mu_2\mu_3\mu_4+\mu_2+\mu_4=17 \text{ points } Q.$$

This sequence, in the instance which we are considering, exhausts the whole system of points. We observe that all sequences begin with P, and that sequences of an uneven order end with PQ, sequences of an even order with QP.

4. In general, when the continued fraction is given, and it is required to obtain the arrangement of the points P and Q, we denote a sequence of order i by S_i, and we then find successively $S_1=P^{\mu_1}Q$, $S_2=S_1^{\mu_2}P$, $S_3=S_2^{\mu_3}S_1$, ..., the final sequence (which exhausts the whole series of points) being $S_s=S_{s-1}^{\mu_s}S_{s-2}$.

Vice versâ, when the arrangement of the points is given, and it is required to infer from it the development in a continued fraction, we count the points P till we come to the first point Q; if there are μ_1 of them, μ_1 is the first quotient, and $S_1=P^{\mu_1}Q$. If we can repeat this sequence μ_2 times, without departing from the given arrangement, the second quotient is μ_2, and the sequence of order 2 is $S_2=S_1^{\mu_2}P$. This sequence we now repeat as often as we can do so without departing from the given arrangement, observing, however, that the last repetition of S_2 is to be followed by a sequence S_1. If, subject to this condition, we can repeat S_2 μ_3 times, the third quotient is μ_3, and the sequence of order 3 is $S_2^{\mu_3}S_1$. The subsequent quotients and sequences are to be determined in the same manner; and, if $\frac{p_i}{q_i}$ is the convergent $\mu_1+\frac{1}{\mu_2+\ldots}\frac{1}{\mu_i}$, p_i and q_i are respectively the numbers of points P and points Q in the sequence S_i.

5. Since $\mu_1 < \frac{p}{q} < \mu_1 + 1$, or $\mu_1 P < Q < (\mu_1 + 1) P$, it is evident that the arrangement of the first $\mu_1 + 1$ points of the series is represented correctly by the sequence $S_1 = P^{\mu_1} Q$. We therefore proceed to show that the arrangement of the first $\mu_1 \mu_2 + 1$ points P, and the first μ_2 points Q is correctly represented by the sequence $S_2 = S_1^{\mu_2} P$. Since

$$\mu_1 + \frac{1}{\mu_2} > \frac{p}{q} > \mu_1 + \frac{1}{\mu_2 + 1},$$

we have $(\mu_1 k + 1) P > kQ$, for all values of $k \leqq \mu_2$, but $(\mu_1 k + 1) P < kQ$, if $k > \mu_2$.

If we write down the sequence $P^{\mu_1} Q$, $1 + \mu_2$ times over, so as to obtain the series

$$\begin{array}{lll}
1P,\ 2P, & \ldots & \mu_1 P,\ Q, \\
(1+\mu_1) P,\ (2+\mu_1) P, & \ldots & 2\mu_1 P,\ 2Q, \\
(1+2\mu_1) P,\ (2+2\mu_1) P, & \ldots & 3\mu_1 P,\ 3Q, \\
\ldots\ldots\ldots\ldots & & \\
(1+[\mu_2-1]\mu_1) P,\ (2+[\mu_2-1]\mu_1) P, & \ldots & \mu_2\mu_1 P,\ \mu_2 Q, \\
(1+\mu_2\mu_1) P,\ (2+\mu_2\mu_1) P, & \ldots & (1+\mu_2)\mu_1 P,\ (1+\mu_2) Q, \\
(1+[\mu_2+1]\mu_1) P, & & \ldots\ldots\ldots\ldots,
\end{array}$$

the inequalities $k\mu_1 P < kQ < (k\mu_1 + 1) P$, which hold as long as $k \leqq \mu_2$, show that all these points, with the exception of the last of them $(1 + \mu_2) Q$, succeed one another in the proper order. But the last is in error; for, putting $k = 1 + \mu_2$, $(1 + \mu_1 + \mu_1\mu_2) P < (1 + \mu_2) Q$, and consequently $(1 + \mu_2) Q$ does not follow immediately after $(1+\mu_2)\mu_1 P$. We conclude, therefore, that we can repeat the sequence $P^{\mu_1} Q$ μ_2 times, but that we cannot repeat it $1 + \mu_2$ times. And, since two points Q cannot come together, the series $(P^{\mu_1} Q)^{\mu_2}$ is necessarily followed by a point P, so that the sequence $S_2 = S_1^{\mu_2} P$ correctly represents, as far as it goes, the arrangement of the points.

6. We have thus shown that the relation between the continued fraction and the sequences S_1, S_2, $S_3 \ldots$ holds as far as S_2. Assuming, therefore, that it holds as far as S_i, where $i \gtreqless 2$, we have to prove that it holds as far as S_{i+1}.

The proof depends on an elementary theorem relating to continued fractions, which was first established by Lagrange.

"If $\frac{p_{i-1}}{q_{i-1}}$, $\frac{p_i}{q_i}$ are consecutive convergents to the same rational or irrational quantity θ, $p_{i-1} - q_{i-1}\theta$ is less in absolute magnitude than any quantity of the form $y - x\theta$, where x and y are positive integers, of which x is less than q_i."

Supposing, for brevity, that i is uneven, we infer from this principle that the least segment in the sequence S_i is its last segment $q_iQ - p_iP$, and that the next least segment in S_i is the last segment of S_{i-1}, viz. $p_{i-1}P - q_{i-1}Q$. We have to add that $p_{i-1}P - q_{i-1}Q$ is also less than the segment $P(1 + p_i) - q_iQ$ which immediately follows S_i. For if

$$q_{i+1} > 1 + \mu_{i+1},$$

a condition which is certainly satisfied when $i > 1$, we have $\frac{p_i - p_{i-1} + 1}{q_i - q_{i-1}} > \frac{p_{i+1}}{q_{i+1}}$, *i.e.* $> \frac{p}{q}$, because $i + 1$ is even. Let us write down the sequence S_i $1 + k$ times over, and let $yQ - xP$ be any segment of S_i contained between two consecutive points P and Q, of which Q is to the right of P; the corresponding segment in $(1 + k)$ S_i will be

$$(kq_i + y)\,Q - (kp_i + x)\,P = k\,(q_iQ - p_iP) + (yQ - xP);$$

i.e. Q will be still further to the right of P, and the distance between P and Q be increased. Next, let $xP - yQ$ be a segment of S_i, contained between two consecutive points P and Q, of which P lies to the right of Q; or, again, let $xP - yQ$ represent the segment $(1 + p_i)\,P - q_iQ$, which immediately follows S_i. The corresponding segment in $(k+1)$ S_i, or immediately following $(k + 1)$ S_i, will be

$$(kp_i + x)\,P - (kq_i + y)\,Q = xP - yQ - k\,(q_iQ - p_iP);$$

so that, if k be not too great, the two new points P and Q will lie in the same relative position with regard to one another as the two points originally considered, the distance between them being diminished; but, for values of k which surpass a certain limit, the point Q will be shifted to the right of P, and the segment QP will be replaced by a segment PQ. As long as this interchange of places between two consecutive points Q and P does not occur, so long the successive repetitions of S_i will represent with accuracy the arrangement of the points P and Q. Now the least of the segments $xP - yQ$ is $p_{i-1}P - q_{i-1}Q$, and

$$p_{i-1}P - q_{i-1}Q - \mu_{i+1}\,(q_iQ - p_iP)$$

is still positive; therefore we may repeat S_i $1+\mu_{i+1}$ times, but we cannot repeat it $2+\mu_{i+1}$ times, for

$$p_{i-1}P-q_{i-1}Q-(1+\mu_{i+1})(q_iQ-p_iP)$$

is negative. The sequence $S_i^{\mu_{i+1}}S_{i-1}$ will therefore truly represent, as far as it goes, the arrangement of the points P and Q; but the sequence $S_i^{1+\mu_{i+1}}S_{i-1}$ would fail to do so. We should in fact come to an error in the last two points of S_{i-1}, which, according to the law of that sequence, we should have to write down as QP, whereas the true arrangement of these points is PQ. This suffices to establish the general theorem of Art. 4; but it is of interest to add, that the error which we have just shown must occur in the last two points of the sequence $S_i^{1+\mu_{i+1}}S_{i-1}$, is the only error that can occur in that sequence. And this is certain; for, in the first place, we have seen that there is no error in $S_i^{1+\mu_{i+1}}$; and, in the second place, if $xP-yQ$ be any segment of S_{i-1} of the same positive sign as $p_{i-1}P-q_{i-1}Q$, $xP-yQ-(1+\mu_{i+1})(q_iQ-p_iP)$ is necessarily positive; for, by the theorem of Lagrange, $xP-yQ>q_{i-2}Q-p_{i-2}P$; and

$$xP-yQ+p_iP-q_iQ>(p_i-p_{i-2})P-(q_i-q_{i-2})Q>p_{i-1}P-q_{i-1}Q$$

by the same theorem; whence

$$xP-yQ-(1+\mu_{i+1})(q_iQ-p_iP)$$

is positive, because

$$p_{i-1}P-q_{i-1}Q-\mu_{i+1}(q_iQ-p_iP)$$

is positive.

7. It will be noticed that the sequence S_1 can only be repeated μ_2 times, whereas any subsequent sequence S_i can be repeated $1+\mu_{i+1}$ times. The exception in the case of S_1 is apparent rather than real, and arises from the fact, that the period S_0 consists of only one term. If we were to attempt to repeat the sequence S_1 $2+\mu_2$ times, the sequence S_0, which commences the last repetition of S_1, ought, according to the general theory, to be in error; viz. its last point must be interchanged with the preceding point; and, as S_0 contains but one point, this interchange vitiates the sequence S_1 immediately preceding.

8. Any finite continued fraction may be written either with an even or with an uneven number of quotients, because

the last quotient may be made either equal to unity or greater than unity. If the number of quotients be even, the two extreme points P and Q, which coincide with 1, must be written in the order QP; if the number of quotients be uneven these points must be written in the order PQ.

9. If we omit these two last coincident points, the remaining $p-1$ points P and $q-1$ points Q evidently form a symmetric series, being similarly distributed on either side of the middle point of the line. And, similarly, if we remove from any sequence whatever its two final points, we obtain a symmetrical series, because the sequence S_i corresponds to the division of a line into p_i equal parts and also into q_i equal parts.

10. If we wish, from the arrangement of the points P and Q, to infer the arrangement corresponding to the fraction $\mu_{i+1}+\frac{1}{\mu_{i+2}}+\dots$, obtained from the fraction $\frac{p}{q}$ by omitting its first i quotients, we have only to replace the sequences S_i and S_{i-1} by single points. Thus, in the example of Art. 3, if we put $S_1=A$, $S_0=B$, we find

$$A,\ 2A,\ 3A,\ B \mid 4A,\ 5A,\ 6A,\ 2B \mid 7A \parallel 8A,\ 9A,\ 10A,\ 3B \mid$$
$$11A,\ 12A,\ 13A,\ 4B \mid 14A \parallel 15A,\ 16A,\ 17A,\ 5B \,|||,$$

corresponding to $\frac{17}{5}=3+\frac{1}{2+}\,\frac{1}{2}$. And, again, if we wish to obtain the arrangement corresponding to the fraction $\mu_i+\frac{1}{\mu_{i+1}+\dots}\,\frac{1}{\mu_{i+j}}$, where $i+j<s$, we first replace S_i, S_{i-1} by single points, and then consider in the resulting arrangement the sequence of order j. Thus the arrangement

$$A,\ 2A,\ 3A,\ B \mid 4A,\ 5A,\ 6A,\ 2B \mid 7A \parallel$$

corresponds to the fraction $3+\frac{1}{2}$.

Addition to the preceding note.

11. The theorem of Lagrange, on which the demonstration in the preceding note depends, will be found in the second paragraph of his Additions to Euler's Algebra. But as this theorem is no longer included in elementary treatises, we shall here place Lagrange's demonstration of it.

If ϕ_i is the complete quotient of order i in the development of θ, we have

$$\theta = \frac{\phi_i p_{i-1} + p_{i-2}}{\phi_i q_{i-1} + q_{i-2}}, \text{ or } \phi_i = -\frac{p_{i-2} - q_{i-2}\theta}{p_{i-1} - q_{i-2}\theta}.$$

But ϕ_i is positive and greater than unity; hence, $p_{i-2} - q_{i-2}\theta$, and $p_{i-1} - q_{i-1}\theta$ are of opposite signs, and $p_{i-1} - q_{i-1}\theta$ is less in absolute magnitude than $p_{i-2} - q_{i-2}\theta$.

Again, since $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$, we can always find, whatever the given integral numbers x and y may be, two integral numbers λ and μ satisfying the equations

$$x = \lambda q_{i-1} + \mu q_i, \; y = \lambda p_{i-1} + \mu p_i,$$

whence we obtain

$$y - x\theta = \lambda (p_{i-1} - q_{i-1}\theta) + \mu (p_i - q_i\theta).$$

As $p_{i-1} - q_{i-1}\theta$ and $p_i - q_i\theta$ are of opposite signs, if $y - x\theta$ is less than $p_{i-1} - q_{i-1}\theta$, λ and μ must be of the same sign; that is to say, x and y are either respectively equal to q_i and p_i, or else they are respectively greater than q_i and p_i.

12. In the same place Lagrange has also established the converse theorem, that if $b - a\theta$ is a minimum difference, *i.e.* if $b - a\theta$ is less in absolute magnitude than any difference $y - x\theta$, in which x is less than a, $\frac{b}{a}$ is a convergent to θ. Writing p_{i-1} for b, and q_{i-1} for a, we first determine the positive numbers p_{i-2} and q_{i-2}, respectively less than p_{i-1} and q_{i-1}, which satisfy the equation $p_{i-1}q_{i-2} - p_{i-2}q_{i-1} = \varepsilon$, ε denoting an unit of the same sign as $p_{i-1} - q_{i-1}\theta$. If we write $p_{i-1} - q_{i-1}\theta = u_{i-1}$, $p_{i-2} - q_{i-2}\theta = u_{i-1}$, we find, on eliminating θ,

$$\frac{\varepsilon}{q_{i-1}q_{i-2}} = \frac{u_{i-1}}{q_{i-1}} - \frac{u_{i-2}}{q_{i-2}}.$$

In this equation u_{i-2} is greater, by hypothesis, than u_{i-1}, because $q_{i-2} < q_{i-1}$; *à fortiori* $\frac{u_{i-2}}{q_{i-2}}$ is greater than $\frac{u_{i-1}}{q_{i-1}}$. But u_{i-1} is of the same sign as ε; therefore, u_{i-1} and u_{i-2} must have contrary signs. Consequently the quotient

$$\phi_i = -\frac{u_{i-2}}{u_{i-1}} = -\frac{p_{i-2} - q_{i-2}\theta}{p_{i-1} - q_{i-1}\theta}$$

is positive, and greater than unity; and if $\frac{p_{i-1}}{q_{i-1}}$ be developed in a continued fraction having $\frac{p_{i-2}}{q_{i-2}}$ for its last convergent (which is always possible) we obtain

$$\theta = \frac{p_{i-2} + \phi_i p_{i-1}}{q_{i-2} + \phi^i q_{i-1}} = \mu_1 + \frac{1}{\mu_2 +}\, \frac{1}{\mu_3 + \ldots}\, \frac{1}{\mu_{i-1} +}\, \frac{1}{\phi_i};$$

i.e. $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_{i-1}}{q_{i-1}} = \frac{b}{a}$ are successive convergents to θ.

13. Combining the two theorems of Lagrange, we see that if we have ascertained, by observation, that $p_{i-1} - q_{i-1}\theta$ is less than any difference $y - x\theta$ in which x is less than q_{i-1}, we can at once infer that $p_{i-1} - q_{i-1}\theta$ is also less than any difference $y - x\theta$ in which $q_{i-1} < x < q_i$.

14. The two theorems of Lagrange serve to define the successive minima of the expression $y - x\theta$. The theory of the successive minima of the expression $\frac{y}{x} - \theta$ is perhaps less complete. Thus we have the elementary theorem, that the difference $\frac{p_{i-1}}{q_{i-1}} - \theta$ is less than any difference $\frac{y}{x} - \theta$ in which x does not surpass q_{i-1}, and is also less than any difference of the same sign with itself, in which x does not surpass q_i; but there may be differences of a contrary sign to $\frac{p_{i-1}}{q_{i-1}} - \theta$, in which x does not surpass q_i, and which are less in absolute magnitude than $\frac{p_{i-1}}{q_{i-1}} - \theta$. And again, if $\frac{b}{a} - \theta$ be a minimum difference (*i.e.* if $\frac{b}{a} - \theta$ be less in absolute magnitude than any difference $\frac{y}{x} - \theta$, in which x is less than a), we cannot in general infer that $\frac{b}{a}$ is a convergent to θ. We shall attempt, in what follows, to define accurately the successive minima of the expression $\frac{y}{x} - \theta$, and thus to give a greater amount of precision to this part of the theory of continued fractions.

15. We still consider a rational or irrational quantity θ, of which the development is

$$\theta = \frac{\phi_i p_{i-1} + p_{i-2}}{\phi_i q_{i-1} + q_{i-2}} = \mu_1 + \frac{1}{\mu_2 + \ldots}\, \frac{1}{\mu_{i-1} +}\, \frac{1}{\phi_i};$$

and, adopting the designation of Lagrange, we term the fractions $\frac{P_k}{Q_k} = \frac{kp_{i-1} + p_{i-2}}{kq_{i-1} + q_{i-2}}$, where $0 > k < \mu_i$, *intermediate* fractions. These fractions are evidently intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$; hence θ lies between any one of them and $\frac{p_{i-1}}{q_{i-1}}$. If $\frac{b}{a} - \theta$ is a minimum difference, we can, by reasoning as in Art. 12, arrive at an equation of the form

$$\theta = \mu_1 + \frac{1}{\mu_2 +} \frac{1}{\mu_3 + \ldots} \frac{1}{\mu_{j-1} +} \frac{1}{\lambda +} \frac{1}{\psi},$$

where
$$\frac{b}{a} = \mu_1 + \frac{1}{\mu_2 + \ldots} \frac{1}{\mu_{j-1} +} \frac{1}{\lambda};$$

and we can prove that in this equation ψ is positive. But we cannot prove that ψ is greater than unity; *i.e.* instead of the equation $\lambda = \mu_j$, we have the inequalities $0 < \lambda \leq \mu_j$; and thus from the hypothesis that $\frac{b}{a} - \theta$ is a minimum difference, we cannot infer that $\frac{b}{a}$ is a convergent to θ, but only that $\frac{b}{a}$ is either a convergent or an intermediate fraction. But not every intermediate fraction can give a minimum difference; for in order that $\frac{P_k}{Q_k} - \theta$ should be a minimum difference, $\frac{P_k}{Q_k} - \theta$ must (at any rate) be less than $\frac{p_{i-1}}{q_{i-1}} - \theta$, because $q_{i-1} < Q_k$. The absolute value of

$$\frac{p_{i-1}}{q_{i-1}} - \theta \text{ is } \frac{1}{q_{i-1}(\phi_i q_{i-1} + q_{i-2})},$$

and the absolute value of

$$\frac{P_k}{Q_k} - \theta \text{ is } \frac{\phi_i - k}{(kq_{i-1} + q_{i-2})(\phi_i q_{i-1} + q_{i-2})};$$

whence, if $\frac{P_k}{Q_k} - \theta$ is a minimum difference, we must have

$$\left.\begin{array}{l} \phi_i < 2k + \dfrac{q_{i-2}}{q_{i-1}}, \\ \text{or} \quad \mu_i + \dfrac{1}{\mu_{i+1}} + \ldots < 2k + \dfrac{1}{\mu_{i-1} +} \dfrac{1}{\mu_{i-2} + \ldots} \end{array}\right\} \ldots\ldots (A).$$

And this necessary condition is also sufficient. For, since θ lies between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$, and since (if the condition A be satisfied) θ also lies nearer to $\frac{P_k}{Q_k}$ than to $\frac{p_{i-1}}{q_{i-1}}$, any fraction which is nearer to θ than $\frac{P_k}{Q_k}$ must lie between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q^i_{-1}}$, and must therefore have a denominator greater than Q_k, because $\frac{P_k}{Q_k} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_{i-1} Q_k}$.

We are thus led to divide the fractions, intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$ into two sets, according as they do or do not satisfy the condition (A). We may call those fractions which do not satisfy that condition the *inferior*, and those which do satisfy it the *superior* intermediate fractions. We then have the theorem,—

"The complete series of successive minima of the expression $\frac{y}{x} - \theta$ is obtained by taking in succession for $\frac{y}{x}$ the convergents, and the superior intermediate fractions in their natural order."

16. If $k > \frac{1}{2}\mu_i$, the condition (A) is satisfied; if $k = \frac{1}{2}\mu_i$, the condition is satisfied if $\mu_{i-1} < \mu_{i+1}$; if $k = \frac{1}{2}\mu_i$, $\mu_{i-1} = \mu_{i+1}$, the condition is satisfied if $\mu_{i-2} > \mu_{i+2}$, and so on continually. If the continued fraction be finite, symmetrical, and of an uneven number of quotients, $\mu_i = 2k$ being the middle quotient, we have a singular case in which the errors of $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{P_k}{Q_k}$ are exactly equal; we may in this case regard $\frac{P_k}{Q_k}$ as an inferior fraction. It will be seen that, as nearly as possible, one-half of the fractions intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$ are superior. Thus, if $\mu_i = 2h + 1$ is uneven, there are h inferior and h superior intermediate fractions; if $\mu_i = 2h$ is even, there are certainly $h - 1$ inferior, and $h - 1$ superior intermediate fractions; but whether $\frac{P_h}{Q_h}$ is inferior or superior, can only be decided (as we have just seen) by comparing the quotients which precede μ^i with those which follow it.

17. The difference $\frac{p_{i-1}}{q_{i-1}} - \theta$ is of the sign $(-1)^{i-1}$; the differences $\frac{P}{Q} - \theta$, which, in forming the complete series of minima of $\frac{y}{x} - \theta$, we have to intercalate between $\frac{p_{i-1}}{q_{i-1}} - \theta$, and $\frac{p_i}{q_i} - \theta$ are of the same sign as the latter of these differences, *i.e.* they are of the sign $(-1)^i$. Thus, after every convergent there is a change of sign in the series of minimum differences, and the minimum differences formed with convergents are distinguished by this criterion from the minimum differences formed with superior intermediate fractions.

18. Again, if $\frac{b}{a} - \theta$ be any minimum difference, and if $q_{i-1} \leqq a < q_i$, the only differences $\frac{y}{x} - \theta$, which are less than $\frac{b}{a} - \theta$, and which have denominators x less than q_i, are the minimum differences which lie between $\frac{b}{a} - \theta$ and $\frac{p_i}{q_i} - \theta$. It is sufficient to prove this for the case in which $a = q_{i-1}$, $b = p_{i-1}$. Let $\frac{P_{\lambda_{-1}}}{Q_{\lambda_{-1}}}$ be the last of the inferior fractions, intermediate between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$; then θ, which lies between $\frac{P_{\lambda_{-1}}}{Q_{\lambda_{-1}}}$ and $\frac{p_{i-1}}{q_{i-1}}$, is nearer to the latter than to the former of those fractions. If then $\frac{y}{x}$ be nearer to θ than $\frac{p_{i-1}}{q_{i-1}}$ is, $\frac{y}{x}$ must itself lie between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{P_{\lambda_{-1}}}{Q_{\lambda_{-1}}}$. But, if $x < q_i$, $\frac{y}{x}$ cannot lie between $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$; hence, $\frac{y}{x}$ must lie between $\frac{p_i}{q_i}$ and $\frac{P_{\lambda_{-1}}}{Q_{\lambda_{-1}}}$. But the only fractions between these limits, which have denominators less than q_i, and lie nearer to θ than $\frac{p_{i-1}}{q_{i-1}}$, are the superior fractions intermediate between $\frac{p_{i-2}}{q_{i-2}}$ and $\frac{p_i}{q_i}$. For all such fractions are of the type $\frac{\sigma p_{i-1} + \tau p_{i-2}}{\sigma q_{i-1} + \tau q_{i-2}}$, the relatively prime numbers σ and τ satisfying the inequalities

$$\lambda - 1 < \frac{\sigma}{\tau} < \mu_i \text{ ....(1)},$$

$$\sigma q_{i-1} + \tau q_{i-1} < q_i, \text{ whence } \sigma < \mu_i \ldots\ldots\ldots\ldots (2),$$

$$\phi_i < 2\frac{\sigma}{\tau} + \frac{q_{i-2}}{q_{i-1}} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now if $\mu_i = 2h+1$, we have $\lambda - 1 = h$, and the inequalities $h < \frac{\sigma}{\tau}$, and $\sigma < 2h+1$ (from which equality is excluded), show that unity is the only admissible value for τ. Again, if $\mu_i = 2h$ and $\frac{P_h}{Q_h}$ is an inferior fraction, we have $h < \frac{\sigma}{\tau}$, $\sigma < 2h$, and unity is the only admissible value for τ. In both these cases, therefore, the only fractions having denominators less than q_i, which lie between $\frac{P_{\lambda-1}}{Q_{\lambda-1}}$ and $\frac{p_i}{q_i}$, are the superior intermediate fractions. If, however, $\mu_i = 2h$, and $\frac{P_h}{Q_h}$ is a superior fraction, the inequalities (1) and (2) are satisfied by the values $\sigma = 2h-1$, $\tau = 2$, so that, besides the superior intermediate fractions, the fraction $\frac{(2h-1)\,p_{i-1} + 2p_{i-2}}{(2h-1)\,q_{i-1} + 2q_{i-2}}$ lies between the limits $\frac{P_{h-1}}{Q_{h-1}}$ and $\frac{p_i}{q_i}$. But this fraction is more remote from θ than $\frac{p_{i-1}}{q_{i-1}}$ is, because the equation $\frac{\sigma}{\tau} = \frac{2h-1}{2}$ is inconsistent with the inequality (3).

19. The inferior intermediate fractions $\frac{P_k}{Q_k}$, $k \leqq \lambda - 1$, do not give minimum differences, because $q_{i-1} < Q_k$, and $\frac{p_{i-1}}{q_{i-1}} - \theta$ is less in absolute magnitude than $\frac{P_k}{Q_k} - \theta$. But, with the single exception of $\frac{p_{i-1}}{q_{i-1}} - \theta$, all other differences $\frac{y}{x} - \theta$, in which x is less than Q_k, are greater than $\frac{P_k}{Q_k} - \theta$. For, if $\frac{y}{x}$ lie between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$, x must be greater than Q_k; if $\frac{P_k}{Q_k}$ lie between $\frac{y}{x}$ and $\frac{p_{i-1}}{q_{i-1}}$, the difference $\frac{y}{x} - \theta$ is certainly greater than $\frac{P_k}{Q_k} - \theta$, because θ lies between $\frac{P_k}{Q_k}$ and $\frac{p_{i-1}}{q_{i-1}}$; lastly, if $\frac{p_{i-1}}{q_{i-1}}$ lies between $\frac{y}{x}$ and $\frac{P_k}{Q_k}$, we find (taking the case in which i is uneven)

$$\frac{y}{x}-\theta > \frac{y}{x}-\frac{p_{i-1}}{q_{i-1}} > \frac{1}{xq_{i-1}} > \frac{1}{Q_k q_{i-1}} = \frac{p_{i-1}}{q_{i-1}} - \frac{P_k}{Q_k} > \theta - \frac{P_k}{Q_k}.$$

20. The theorem of Lagrange admits of an important geometrical interpretation. If with a pair of rectangular axes in a plane we construct a system of unit points (*i. e.* a system of points of which the coordinates are integral numbers), and draw the line $y=\theta x$, we learn from that theorem that if (x, y) be an unit point lying nearer to that line than any other unit point having a less abscissa (or, which comes to the same thing, lying at a less distance from the origin), $\frac{y}{x}$ is a convergent to θ; and, *vice versâ*, if $\frac{y}{x}$ is a convergent, (x, y) is one of the "nearest points." Thus the "nearest points" lie alternately on opposite sides of the line, and the double area of the triangle, formed by the origin and any two consecutive "nearest points," is unity.

In particular, if $\theta=\frac{p}{q}$, p and q being relatively prime integers, the coordinates of the two "nearest points" above and below the finite line joining the origin to the unit point (q, p) satisfy respectively the equations $px-qy=1$, and $px-qy=-1$. We thus obtain a simple geometrical method of finding the least solution in integral numbers of either of those indeterminate equations.