

Chapter 18. Remarks on partial differential equations

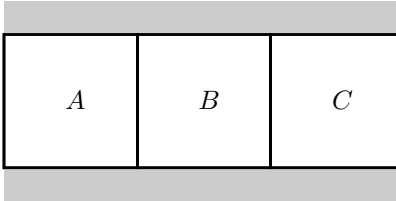
If we try to analyze heat flow or vibration in a continuous system such as a building or an airplane, we arrive at a kind of infinite system of ordinary differential equations. In fact, nearly all physical systems one usually wants to understand are essentially continuous in nature, resembling systems with an infinite number of components—for example, an electric field in empty space, or large physical structures like an aircraft fuselage or a large bridge. In these cases one can often approximate them by systems of finite if large dimension, but the underlying mathematics is more accurately expressed in terms of **partial** rather than ordinary differential equations. In this course we shall concentrate on demonstrating with relatively simple models what basic phenomena one expects to meet, rather than try to explain how realistic systems are dealt with. There are three basic types traditionally of interest to engineers. The first is related to how heat flows in complicated physical bodies; the second is related to equilibrium temperature distributions and equilibrium configurations of physical systems in general; the third is related to how waves travel in physical media. Real physical systems are complicated enough that one has to use a computer to understand what happens, but the models we construct will show us essentials.

The two main tools we shall use are Fourier series and variants, and a technique of solving partial differential equations called **separation of variables**.

1. Three cooling rooms

Before we look at a new type of physical system, we shall look more closely at a type only slightly more complicated than those we have looked at before. Some of the phenomena we want to understand appear here in a simple way.

Suppose we are considering three rooms A , B , and C , with A and C exposed to an outside environment of 0° but B in between them only exposed to A and C .



Suppose also that all relaxation times τ_A , τ_C , τ_{AB} , and τ_{BC} are equal to 1. Then the evolution in time of temperature is controlled by the system of differential equations

$$\begin{aligned}\theta'_A &= -\frac{1}{\tau_A} \theta_A - \frac{1}{\tau_{AB}} (\theta_A - \theta_B) \\ \theta'_B &= -\frac{1}{\tau_{AB}} (\theta_B - \theta_C) - \frac{1}{\tau_{BC}} (\tau_B - \tau_C) \\ \theta'_C &= -\frac{1}{\tau_C} \theta_C - \frac{1}{\tau_{BC}} (\theta_C - \theta_B)\end{aligned}$$

or

$$\theta' = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \theta, \quad \theta = \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{bmatrix}.$$

Since we have fixed the outside temperature at 0° , this is a homogeneous system. In order to solve the system, we must find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Its characteristic polynomial is

$$\lambda^3 + 6\lambda^2 + 10\lambda + 4 = 0$$

Trying out various small integers, we find that $\lambda = -2$ is a root. This polynomial factors as

$$(\lambda + 2)(\lambda^2 + 4\lambda + 2)$$

and therefore the other two roots are

$$-2 \pm \sqrt{2}.$$

The corresponding eigenvectors are (in order of the magnitude of the eigenvalues)

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} (\lambda = -2 + \sqrt{2}), \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} (\lambda = -2), \quad \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} (\lambda = -2 - \sqrt{2}).$$

The general solution of the system is hence

$$c_1 e^{(-2+\sqrt{2})t} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{(-2-\sqrt{2})t} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Let's now suppose that the initial temperatures are all equal to 20° :

$$\theta_A = \theta_B = \theta_C = 20.$$

Then we must solve

$$c_1 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

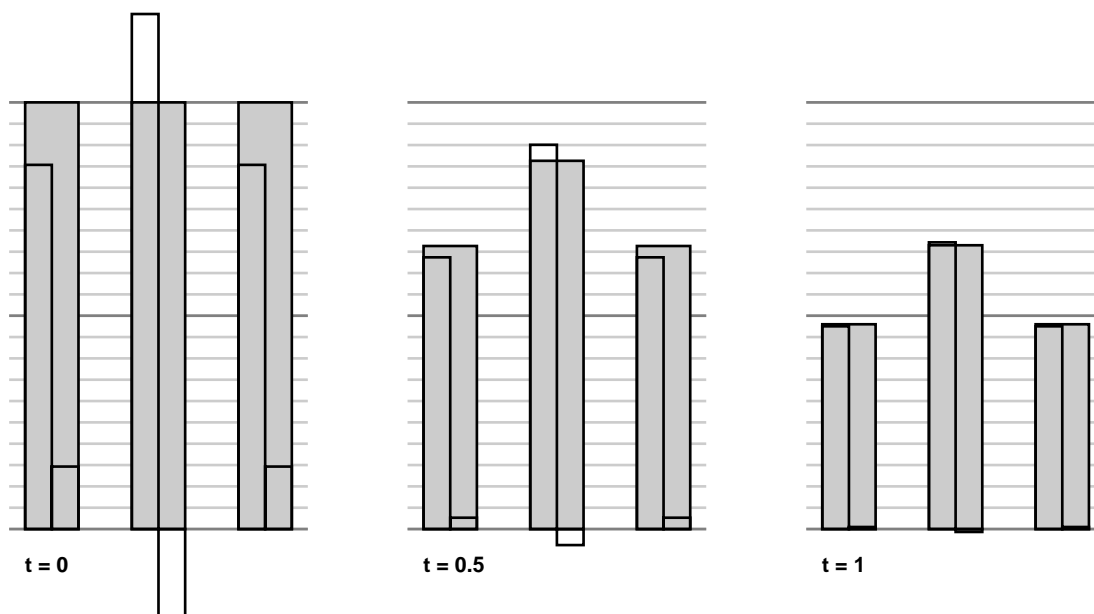
or

$$\begin{array}{rcccc} c_1 & + & c_2 & + & c_3 & = & 20 \\ c_1\sqrt{2} & & & - & c_3\sqrt{2} & = & 20 \\ c_1 & - & c_2 & + & c_3 & = & 20 \\ 2c_1 & & & + & 2c_3 & = & 40 \\ & & & & c_2 & = & 0 \\ & & & & c_1 & = & 10 + 5\sqrt{2} \\ & & & & c_3 & = & 10 - 5\sqrt{2} \end{array}$$

leading to the solution

$$\begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{bmatrix} = (10 + 5\sqrt{2})e^{(-2+\sqrt{2})t} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} + (10 - 5\sqrt{2})e^{(-2-\sqrt{2})t} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

In order to understand this, I use bar graphs below to illustrate how the temperature of the three rooms changes as time goes on. Eventually, of course, all the rooms will cool off towards the outside temperature. In each case, the temperature is a sum of two terms, one for each of the contributing eigenvalues, and these terms are also indicated.



The important thing to realize is that although both terms decrease exponentially, one decreases much more rapidly than the other. This is because $e^{(-2-\sqrt{2})t} \sim e^{-3.4t}$ will be much much smaller than $e^{(-2+\sqrt{2})t} \sim e^{-0.6t}$ for even moderate values of t . Not only do both terms get small with time, but the difference between the two terms gets small even more rapidly. In effect, after a short time the temperature of each room decays at the same essentially exponential rate. In other words, Newton's Law of Cooling for the system as a whole should be considered as an **asymptotic approximation** to a more complicated situation for moderate and large values of t .

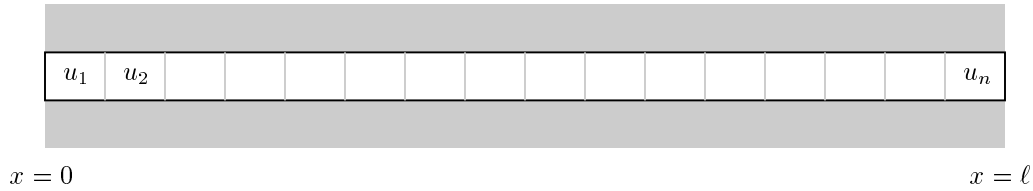
Another important thing to see is that each of the terms in the expression for θ corresponds to a separate **mode of decay**, each mode decaying at its own exponential rate. *The rate in a mode is determined by an eigenvalue, and the temperature distribution in a mode is determined by the corresponding eigenvector.* In other words, the evolution of the system in time brings out the latent modes in the initial temperature distribution among the rooms.

Exercise 1.1. Suppose rooms A and B start off at 20° while room C starts off at 0° . Find a formula for the temperatures of all three rooms as a function of t .

2. The transition from large finite systems to infinite ones

The partial differential equations we shall see are just analogues of systems of ordinary equations we have already seen. In this section we shall see how this works in one example.

Suppose we want to analyze heat flow in a long metallic bar. Suppose that the bar is of length ℓ , and that it is insulated completely along its sides, so that the only way in which heat can flow in or out of it is through its ends. We further assume that at the ends of the bar heat conduction is extremely efficient, so that in effect the ends of the bar have exactly the same temperature as their environment. A typical question might be this: *Suppose that the bar is heated to 100° and then placed in a room at 0° . What is the temperature at the centre at time t ?* In these circumstances, temperature inside the bar will be constant in a cross-section, hence a function of the single variable x which varies from 0 at one end to ℓ at the other. In the next chapter we shall analyze this situation in more detail, but here we want to try to understand what is going on through some kind of approximation. We divide the bar into a large number of chambers, and apply Newton's law of cooling to this collection, as though each chamber behaved as a simple object. If there are n chambers in all, then the length of each chamber will be $h = \ell/n$, and we have to look at temperatures u_1, u_2, \dots, u_n where u_i is the temperature in the i -th chamber.



We apply Newton's law of cooling. The chambers at the ends lose heat to the environment, but all other heat transfer takes place between two neighbouring chambers. We shall assume that the bar is uniform, which means here that different chambers have exactly the same cooling properties. We are assuming that heat transfer at the ends is completely efficient, which means here that $u_1 = 0$, and $u_n = 0$. Otherwise, let τ_h be the relaxation time of cooling between neighbouring chambers. The subscript h is to remind us that smaller objects cool faster than large ones, so that τ will depend on the size of the chambers. Then Newton's law gives the equations

$$\begin{aligned} \frac{du_k}{dt} &= -\frac{1}{\tau_h} (u_k - u_{k-1}) - \frac{1}{\tau_h} (u_k - u_{k+1}) \\ &= \frac{1}{\tau_h} (u_{k-1} - 2u_k + u_{k+1}) \end{aligned}$$

for $k = 2, \dots, n-1$, where we take $u_1 = u_n = 0$. These give us a system of $n-2$ ordinary differential equations

$$u' = Au$$

with

$$A = \left(\frac{1}{\tau_h} \right) \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & 0 & 0 & 1 & -2 \end{bmatrix}$$

If our model is to be a reasonably close approximation to reality, we must take n to be very large. In order to understand what happens for large values of n , we have to rewrite our differential equations slightly. If n is large then the length of each chamber will be very small, and the expression

$$u_k - u_{k-1}$$

will be small, also. But the quotient

$$\frac{u_k - u_{k-1}}{h}$$

will be a close approximation to the *derivative* of the true temperature $u(x_k)$ where $x_k = kh$. Furthermore, the quotient difference

$$\left(\frac{1}{h} \right) \left(\frac{u_{k+1} - u_k}{h} - \frac{u_{k+1} - u_k}{h} \right)$$

will be close to the *second derivative* of u at x_h . We therefore rewrite our equations as

$$\frac{du_k}{dt} = -\frac{h^2}{\tau_h} \frac{1}{h^2} (u_{k-1} - 2u_k + u_{k+1}) = -\frac{1}{\tau_h/h^2} \frac{1}{h^2} (u_{k-1} - 2u_k + u_{k+1}).$$

We know that the second part of this equation has as its limiting value $u''(x_k)$, while the left hand side has as limit du/dt evaluated at x_k . This suggests that the expression

$$\tau_h/h^2$$

also has a limiting value as $h \rightarrow 0$. This is certainly very plausible. We know intuitively that the length of time it takes for an object to cool off is much less for a small object than for a large one, and by asserting that τ_h/h^2 has a limit as $h \rightarrow 0$ we are claiming that the dependency of relaxation time on linear size is that it is essentially proportional to the square of its size. The limit will be a positive number, say α^2 for some constant α which will depend on the material of which the bar is constructed. Thus as $n \rightarrow \infty$ the system of differential equations becomes the partial differential equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

if $u(t, x)$ is the temperature at time t and position x . This is called the **heat equation** in one dimension. Imposed also are the conditions at the end points

$$u(t, 0) = 0, \quad u(t, \ell) = 0$$

which are called the **boundary conditions**.

We shall derive the heat equation again from more direct physical principles in a later Chapter. The point to be understood here is that partial differential equations can arise as the limit of a system of ordinary differential equations as the system gets very large. This ought to give you more intuition about how solutions to partial differential equations behave, and suggest a possible method of solution.

3. More about approximating systems

If we are given a system

$$u' = Au$$

then its solutions are linear combinations of solutions

$$e^{\lambda t}u$$

where λ is an eigenvalue of A and u is a corresponding eigenvector. In our case

$$A = \left(\frac{1}{\tau_h} \right) \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & 0 & 0 & 1 & -2 \end{bmatrix}$$

is symmetric, so the eigenvalues are real, and the eigenvectors may be taken orthogonal to each other. If the size of A is large, then one's first impression is that the problem of finding the eigenvalues and eigenvectors of A is nearly impossible. It turns out, however, that because A has a regular structure its eigenvalue and eigenvectors can be found with relative ease.

Suppose that A has size $n \times n$. An eigenvector $u = (u_1, u_2, \dots, u_n)$ for A with eigenvalue λ satisfies the equations

$$\begin{aligned} -2u_1 + u_2 &= \lambda u_1 \\ u_1 - 2u_2 + u_3 &= \lambda u_2 \\ u_2 - 2u_3 + u_4 &= \lambda u_3 \\ &\dots \\ u_{n-1} - 2u_n &= \lambda u_n \end{aligned}$$

or, in other words, the set of equations

$$u_{k+1} = (2 + \lambda)u_k - u_{k-1}$$

where implicitly $u_0 = u_{n+1} = 0$. An equation of the form

$$v_{k+1} = av_k + bv_{k-1}$$

where a and b are constants is called a **difference equation**. Solving it is very similar to solving an ordinary differential equation of second order. First of all, any solution is determined by any two successive values, since if we know v_0 and v_1 we can determine v_2 by looking forward and v_{-1} by looking backwards. Then also v_3, v_4 , etc. The equation is also linear, since the linear combination of solutions will be a solution. And since a and b are constants, we have **shift invariance**, taking the sequence v_k into the sequence $w_k = v_{k+1}$. Therefore shifting indices is a linear transformation on a two-dimensional vector space, hence has an eigenvalue we call γ . If v_k is an eigenvector of the shift operation then $v_1 = \gamma v_0, v_2 = \gamma v_1 = \gamma^2 v_0$, etc. so $v_k = \gamma^k v_0$. If we substitute this into the difference equation, we see that γ must be root of the **characteristic equation**

$$\gamma^2 = a\gamma + b$$

and that the general solution to the difference equation is a linear combination

$$v_k = c_1 \gamma_1^k + c_2 \gamma_2^k$$

if γ_i are its roots. In our explicit example the characteristic equation is

$$\gamma^2 = (2 + \lambda)\gamma - 1$$

If γ_1 and γ_2 are its roots then

$$\gamma_1 + \gamma_2 = 2 + \lambda, \quad \gamma_1 \gamma_2 = 1$$

and they must be inverses of each other. In other words, if $\gamma_1 = \gamma$ is one of them, the other is $\gamma_2 = \gamma^{-1}$. Since $u_0 = u_{n+1} = 0$

$$c_1 + c_2 = 0, \quad c_2 = -c_1 = \text{say } c$$

and then

$$c\gamma^{n+1} - c\gamma^{-(n+1)} = 0$$

or

$$\gamma^{2(n+1)} = 1$$

which means that γ must be one of the complex $2(n+1)$ -th roots of unity, say

$$\gamma = e^{2\pi i \ell / 2(n+1)}$$

for some ℓ . If $\gamma = \pm 1$ then we just get the zero solution, so $\ell = 0$ and $\ell = n+1$ are not allowed, so we have $2n$ legitimate possibilities in all. Then

$$\lambda = 2 - (\gamma + \gamma^{-1}) = 2 - 2\cos \pi \ell / (n+1).$$

Finally, the components of the eigenvector are

$$u_k = (\gamma^k - \gamma^{-k}) = 2i \sin \pi k \ell / (n+1)$$

which means that $u_k = \sin \pi k \ell / (n+1)$ are also the components of an eigenvector.

The eigenvalues are all in the range $[-2, 0]$, and in particular they are all negative numbers as they should be in a cooling problem.

The general solution is then

$$u_k = \sum_{\ell=1}^{\ell=n} c_k \sin \pi k \ell / (n+1)$$

for arbitrary coefficients c_k determined by initial conditions.

Exercise 3.1. Find by these formulas the eigenvectors and eigenvalues of the matrices

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$