Chapter 19. Heat flow

So far we have used Newton's Law of cooling to analyze temperature changes in physical systems. In this chapter we shall introduce a more complicated and realistic model. The same mathematics we shall use here plays a role in a wide range of environments, including chemical and physical diffusion, and is also related to certain transmission problems over long wires.

1. The physics of heat flow

We have seen earlier the two basic rules governing how heat flow and temperature variation interact to produce varying temperatures in a physical system. They are (1) heat flows from hot to cold and (2) as heat flows into a point, its temperature rises. We must now make these intuitive principles into mathematically precise ones.

Ultimately, what we really want to answer are complicated questions like

'*How long does a turkey take to cook?*'

'*If I double the size of the roast beef, how much longer do I leave it in the oven?*'

'*After I turn off a light, when can I touch it without burning myself?*'

'*After the furnace is turned off, how long does a house take to cool off to the outside temperature?*'

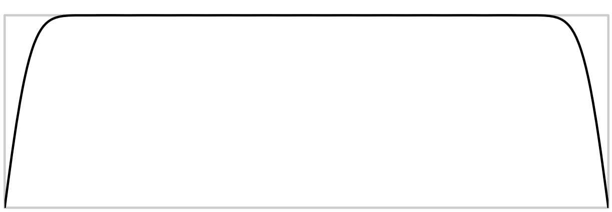
'*If the body was found with an internal temperature of* $16°$*, when did the person die?*'

'*What power can the board dissipate without melting?*'

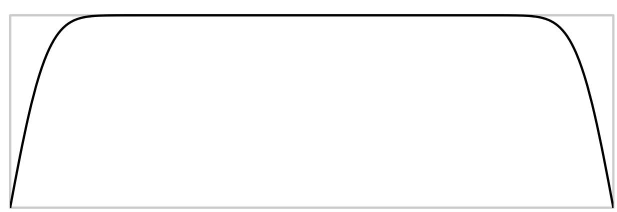
but of course they are too difficult for us. We shall instead restrict ourselves to some much simpler problems, mainly one dimensional in nature. More like this one:

- *A bar of copper* 1 *m. long is insulated along its sides, so that heat transfer takes place only at its ends. At the ends the transfer is extremely efficient. It is heated uniformly to* $100°$*, then allowed to cool in a room at* $0°$*. Describe the temperature distribution for times* $t > 0$*.*

Here is a sequence of graphs of the temperature in the bar as time proceeds. We must somehow account for what they show.

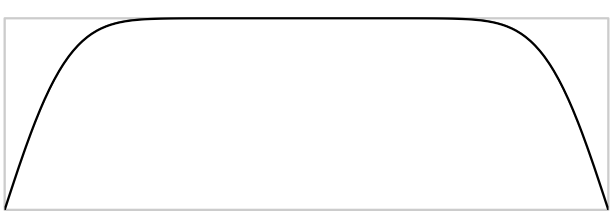


t = 5

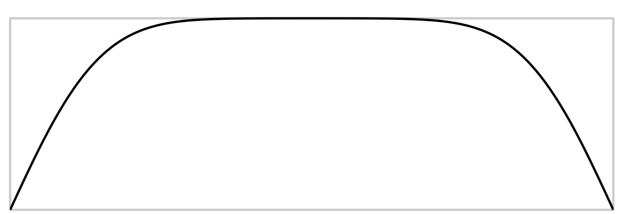


t = 10

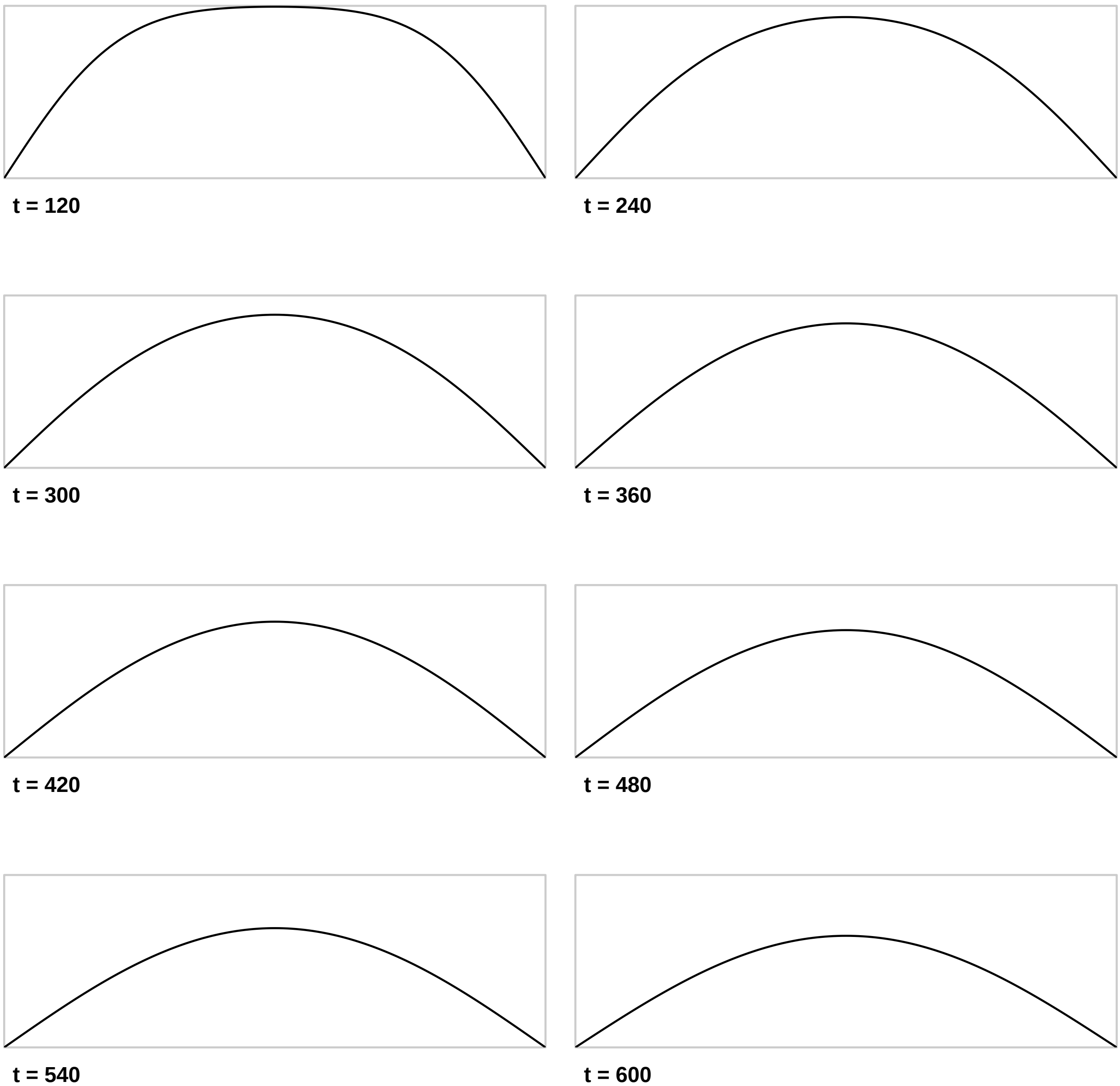
One important thing to notice is that at a mere instant after the the cooling starts, the temperature at the ends is equal to $0°$. This is what 'extremely efficient' means.



t = 30



t = 60



Notice that in the last few figures the basic shape of the graph isn't changing; it seems to be scaling vertically. We shall understand this later.

2. The heat equation

We begin our analysis by finding a single **partial differential equation** underlying the theory. It is called the **heat equation**. Let $u(t, x)$ be the temperature inside some object at time t and position x. For the moment we shall not even assume the object to be one dimensional, so x could actually be a pair or a triple of coordinates. There are two physical constants associated to the object that we have to know about. The first is its **conductivity** κ. The higher κ is, the more easily heat flows. Silver and copper are good heat conductors with high values of κ. At first we shall allow for the possibility that κ varies with location, so $\kappa = \kappa(x)$.

The principle we consider first is that *heat flows from hot places to cold ones*. To be precise, it flows in the direction in which temperature decreases most rapidly, or in the direction opposite to that in which it increases most rapidly. The direction in which this occurs is the direction in which the **temperature gradient** points. In fact, heat flow H is a vector field. Its direction is opposite to the gradient of temperature and its magnitude is the number of calories crossing a unit area perpendicular to the direction of flow per unit of time. To a good approximation it is proportional to the temperature gradient. The constant of proportionality is the conductivity κ, specific to the substance at hand. In a precise formula

$$H = -\kappa \, \mathrm{grad}_x u \, .$$

Both sides of this equation are vector fields which vary in space and time. Recall that the components of the gradient of a function f are

$$\begin{aligned} \mathrm{grad}\, f &= \partial f/\partial x \\ \mathrm{grad}\, f &= (\partial f/\partial x, \partial f/\partial y) \\ \mathrm{grad}\, f &= (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) \end{aligned}$$

in dimensions one, two, and three.

Now we consider the principle that *temperature rises at a point when heat flows into it*. Her again we need a physical constant, namely the **specific heat** s of a substance. This measures how much heat it takes to raise a unit mass of the substance by 1°. Water has a high specific heat, silver a low one. This constant is on the whole independent of κ. Now the way in which heat flow changes temperature is somewhat subtle—it is not heat flow which causes temperature change, but only **differential heat flow**. In other words, if the heat flow into one end of a bar is the same as the heat flow out at the other end, then the temperature will not rise at all. It will rise in a region only if the heat flow into that region is greater than the heat flow out of it. The mathematical version of this in one dimension, where the heat flow is measured by a single number H, is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho s} \frac{\partial H}{\partial x} \, .$$

In two dimensions it is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho s} \left(\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} \right)$$

since there are more directions by which heat can flow in and out, and in three

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho s} \left(\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} + \frac{\partial H_3}{\partial z} \right)$$

If we combine the equations we get a partial differential equation involving temperature alone:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\frac{1}{\rho s} \right) \frac{\partial H}{\partial x} \\ &= \left(\frac{\kappa}{\rho s} \right) \frac{\partial^2 u}{\partial x^2} \, . \end{aligned}$$

In all dimensions we get get

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u$$

where

$$\alpha^2 = \frac{\kappa}{\rho s}$$

and in various dimensions

$$\Delta u = \frac{\partial^2 u}{\partial x^2}$$
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

The constant α^2 is called the **thermal diffusivity** and the operator Δ is called the **Laplacian**. Here are rough values of α^2 for a few substances:

Material	α^2 (in cm^2 / sec)
Silver	1.71
Copper	1.14
Cast iron	0.12
Water	0.00144

3. A formula for the temperature

Let's repeat the first problem we are going to look at, with a slight variation:

- *A rod of copper of length ℓ is circular in cross-section, 1 cm. in radius. It is insulated along its sides, so that heat transfer takes place only at its ends. It is heated uniformly to $100°$, then allowed to cool in a room at $0°$. Describe the temperature distribution for times $t > 0$.*

The condition that the bar starts at a flat $100°$ is called the **initial condition**. The room temperature forces the bar to have temperature $0°$ at its ends for time $t > 0$. The two conditions, one for each end of the bar, are called the **boundary conditions**. Note that the right hand side of the partial differential equation is **linear** in the function u. The partial differential equation is somewhat analogous to a system of differential equations with a very large number of components—intuitively, one component for every small interval $(x, x + \Delta x)$ along the bar. The initial condition is very much like an initial condition for a first order homogeneous linear system.

The principles we shall follow now are the same basic principles we have met before—**time invariance** and **linearity**, following the analogy of our problem with that of solving a homogeneous liner system. In that case, time invariance and linearity led us to these two steps, at least most of the time:

(1) We first looked for solutions of the system of the form $e^{\lambda t} v$ where v is a constant vector. It turned out that λ was an eigenvalue of the matrix of the system, and v an eigenvector. All solutions were linear combinations of the ones we found in this way.

(2) Next, to solve for a given initial condition at $t = 0$ we expressed the initial vector v as a linear combination of eigenfunctions

$$v = \sum c_i v_i \ .$$

The solution to the original problem was then

$$\sum c_i e^{\lambda_i t} v_i \ .$$

Here, we shall interpret a 'vector' to mean a function $f(x)$ defined on the interval $[0, \ell]$ and satisfying the boundary conditions $f(0) = 0$, $f(\ell) = 0$. So we try to find a solution of the heat equation of the form

$$e^{\lambda t}\varphi(x), \qquad \varphi(0) = \varphi(\ell) = 0 \ .$$

If we substitute into the heat equation we get

$$\lambda e^{\lambda t}\varphi(x) = \alpha^2 e^{\lambda t}\varphi''(x) \ .$$

this leads to the ordinary differential equation

$$\varphi'' = (\lambda/\alpha^2)\varphi$$

with φ assumed to satisfy the conditions $\varphi(0) = \varphi(\ell) = 0$. We know from the discussion of even and odd functions in the theory of Fourier series that

$$\varphi_n(x) = \sin(\pi n x/\ell), \qquad \lambda = -\pi^2\alpha^2 n^2/\ell^2$$

for some positive integer n. This finishes the analogue of step (1) above. We have the 'eigenvectors' and eigenvalues.

For step (2), suppose $f(x)$ is the initial temperature distribution along the bar. We would like to write

$$f(x) = \sum c_n \varphi_n(x) = \sum c_n \sin(\pi n x/\ell) \ .$$

But again from the discussion on Fourier series we know that we can do this with the formulas

$$c_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(\pi n x/\ell)\, dx \ .$$

This gives the final solution to the problem as

$$u(t,x) = \sum c_n e^{-\pi^2\alpha^2 n^2 t/\ell^2} \sin(\pi n x/\ell) \ .$$

Note that

$$e^{-\pi^2\alpha^2 n^2 t/\ell^2} = q^{n^2 t}$$

if we set

$$q = e^{-\pi^2\alpha^2/\ell^2}$$

so that we can also write the solution as

$$\sum c_n q^{n^2 t} \sin(\pi n x/\ell) \ .$$

In the explicit case we are looking at, $f(x)$ is the square wave, so

$$c_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{\pi n} & n \text{ odd.} \end{cases}$$

The final solution is then

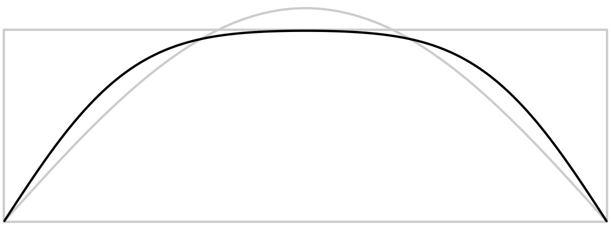
$$\frac{400}{\pi}\left(q\sin(\pi x/\ell) + \frac{q^{9t}\sin(3\pi x/\ell)}{3} + \frac{q^{25t}\sin(5\pi x/\ell)}{5} + \cdots \ \right) \ .$$

For large values of t, q^t will be rather small and the terms $q^{n^2 t}$ will be much smaller when $n > 0$. Therefore the first term in the solution will be by far the largest. The first term in the general case is

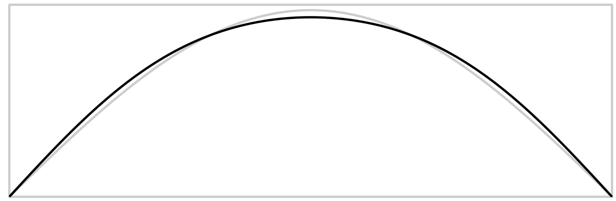
$$\frac{400}{\pi} q^t \sin(\pi x/\ell)$$

so that for large values of t the temperature distribution will decrease at an exponential rate throughout the bar. In this way we recover Newton's Law of cooling as an approximation to a more detailed theory.

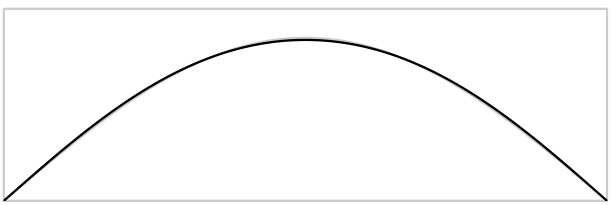
We can see in the following figures how the first term does indeed dominate the cooling for large t. If we look carefully we can even see how the difference is approximately proportional to $\sin(3\pi x/\ell)$.



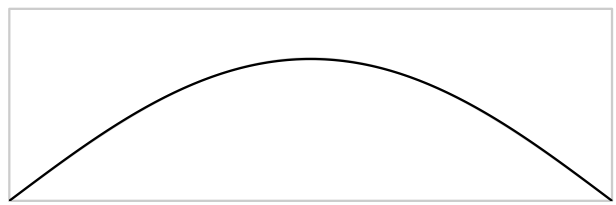
t = 120



t = 240



t = 360



t = 480

Even for t not so large, the powers of q^t decrease at better than a geometric rate, and just a few terms will be required for an accurate estimate.

Exercise 3.1. *Two bars, each* 50 *cm long and made of copper, one at* $100°$ *and the other at* 0 *are fused together at* $t = 0$ *and then allowed to cool in an environment at* $0°$*. Find a formula for the temperature at time* t*. Find approximate temperatures at the centre for* $t = 200$*,* $t = 400$*. Sketch the temperature distribution at* $t = 200$*,* $t = 400$.

Exercise 3.2. *We shall say that an object becomes cooked when its centre reaches a certain temperature, say* $120°$*. If we remove a copper bar* 20 *cm long from the refrigerator at* $0°$ *and put it in an oven at* $150°$*, how long does it take to cook it? How much longer for a bar* 40 *cm long?*

4. Other boundary conditions

The boundary conditions

$$u(t, 0) = u(t, \ell) = 0$$

amount to assuming that the room temperature penetrates immediately to just inside the bar as soon as it starts to cool. This is not usually realistic. At the other extreme is the condition that the ends are completely insulated. This means that there is no heat flow at the ends, and since we know that heat flow is proportional to the gradient of temperature this gives boundary conditions

$$u'(0) = u'(\ell) = 0$$

In this case the eigenvectors we get are the functions

$$\varphi(x) = \cos(\pi n x/\ell)$$

and the same eigenvalues

$$e^{-\pi^2\alpha^2 n^2 t/\ell^2}$$

except that now $n = 0$ is allowed. To solve for an initial temperature $f(x)$ we use the cosine series

$$f(x) = c_0 + \sum c_n \cos(\pi n x/\ell) \ .$$

The solution to the heat problem is then

$$u(t,x) = c_0 + \sum_{n>0} q^{n^2 t} \cos(\pi n x/\ell)$$

with q as before. Here again the first term is dominant for large values of t. The first term is the average temperature at $t = 0$, so that intuitively as time proceeds the in the bar averages out to a constant distribution.

In between the two extremes of instant conductivity at the ends and complete insulation we get conditions

$$u(0) - h_0 u'(0) = 0, \qquad u(\ell) + h_\ell u'(\ell) = 0$$

where the coefficients h measure effectiveness of conduction at the ends.

Exercise 4.1. *Two bars, each 50 cm long and made of copper, one at 100° and the other at 0 are fused together at $t = 0$, insulated at both ends and then left by itself. Find a formula for the temperature at time t. Find approximate values for the temperature at the centre for $t = 200$, $t = 400$. Sketch the temperature distribution at $t = 200$, $t = 400$.*

5. The inhomogeneous problem

Suppose now we consider this problem:

- *One end of the bar is maintained at temperature u_0, the other at temperature u_ℓ.*

This is an inhomogeneous problem, since scaling a solution will satisfy different boundary conditions. Intuitively, we expect the temperature distribution to pass to some steady state solution $u_\sigma(x)$. The steady state solution doesn't change with time, so it must satisfy the simple ordinary differential equation

$$u''_\sigma(x) = 0 \ .$$

In other words, u_σ must be a linear polynomial $a + bx$, and in fact it must be the unique linear polynomial with value u_0 at $x = 0$ and value u_ℓ at $x = \ell$. We can solve

$$\begin{aligned} a &= u_0 \\ a + b\ell &= u_\ell \end{aligned}$$

for a and b.

But now we are in good shape. If we subtract off the steady state solution we get a solution to the homogeneous problem, with the same boundary conditions as before. So the function $v(t,x) = u(t,x) - u_\sigma(x)$ must be a solution of the problem

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial^2 v}{\partial x^2}, \quad v(t,0) = 0 = v(t,\ell)$$

but it has a different initial condition

$$v(0,x) = u(0,x) - u_\sigma(x) = f(x) - u_\sigma(x) \ .$$

Exercise 5.1. *A copper bar 100 cm long and at constant temperaure 100° is put into an environment with one end at 100 and the other at 0. Find a formula for the temperature distribution at time t. Estimate the temperature at the centre at $t = 200$*