Chapter 17. Fourier series

We have already met the simple periodic functions, of the form $\cos(\omega t - \theta)$. In this chapter we shall look at periodic functions of more complicated nature.

1. The basic results

A function $f(t)$ is said to be **periodic of period** T if $f(t+T) = f(t)$ for all values of t—that is to say, if its graph repeats every interval of T units. If $f(t)$ is periodic of period T then it is also periodic of period nT for every integer n. Sometimes the period of f is defined to be smallest possible T for which periodicity holds.

The constant functions have period T for any possible T. The functions $\cos(2\pi nt/T)$ and $\sin(2\pi nt/T)$ are all periodic of period T/n, hence of period T itself. Likewise the complex exponential function $e^{2\pi int/T}$.

The basic result in the theory of Fourier series asserts that any reasonable function with period T can be expressed as a possibly infinite sum of simple periodic functions with a period dividing T. There are several useful versions of this. First of all, we know that a simply periodic function of the kind we are looking for is of the form $C\cos(2\pi nt/T - \theta)$ with $n \neq 0$ an integer, or possibly just a constant function. So the basic claim is this:

- *If $f(t)$ is any reasonable function with $f(t+T) = f(t)$ for all t, then f can be expressed as an infinite linear combination*

$$f(t) = C_0 + \sum_{n>0} C_n \cos\big((2\pi nt/T) - \theta_n\big) .$$

The problem now is to find a way of calculating the coefficients C_n. To understand how this works, we first rewrite the expression for $f(t)$. We know that $\cos\big((2\pi nt/T) - \theta_n\big)$ can be written as a linear combination of $\cos(2\pi nt/T)$ and $\sin(2\pi nt/T)$, and equally well of $e^{2\pi int/T}$ and $e^{-2\pi int/T}$. So the first part of the following claims are more or less equivalent to the original one.

- *If $f(t)$ is any reasonable function with $f(t+T) = f(t)$ for all t, then f can be expressed as an infinite linear combination*

$$f(t) = \cdots + c_{-n}e^{-2\pi int/T} + \cdots + c_{-1}e^{-2\pi it/T} + c_0 + c_1 e^{2\pi it/T} + \cdots + c_n e^{2\pi int/T} + \cdots .$$

  For the (complex) coefficients c_n we have the formula

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}\,dt \\ &= \int_0^1 f(sT)e^{-2\pi ins}\,ds \quad (\textit{ setting } s = t/T) . \end{aligned}$$

  If $f(t)$ takes only real values then c_{-n} is the conjugate of c_n.

- *If $f(t)$ is any reasonable function with $f(t+T) = f(t)$ for all t, then f can be expressed as an infinite linear combination*

$$f(t) = a_0 + \sum_{n>0} a_n \cos(2\pi nt/T) + \sum_{n>0} b_n \sin(2\pi nt/T) .$$

  For the coefficients we have the formulas

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\,dt \\ a_n &= \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)\,dt \quad (n > 0) \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)\,dt \quad (n > 0) . \end{aligned}$$

We shall see later some justification for these formulas, and the relationship between them. The first version is the one which is conceptually the most important. The second gives the **complex Fourier series** of $f(t)$, the third gives the **real Fourier series** of $f(t)$. It is at any rate an easy step from the second to the first. If

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}\,dt$$

then

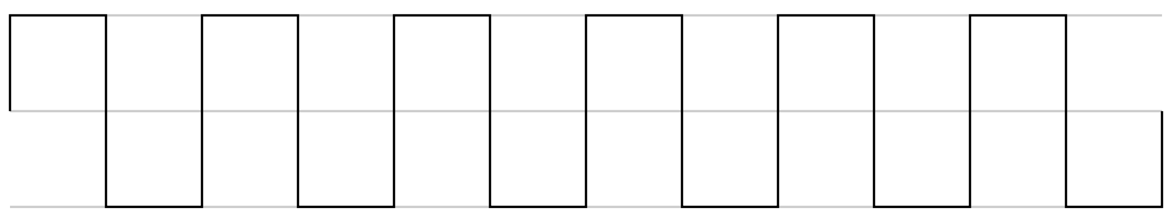
$$C_n = 2|c_n|, \quad \theta_n = -\arg c_n \ .$$

We shall not even attempt to prove these results, but we shall see in a while, as already mentioned, some kind of justification for them. Nor shall we consider carefully what 'reasonable' means here. In practice, the functions we shall apply this theory to are those which have simple formulas in a finite number of pieces of the interval $[0, T]$.

Example. Let $f(t)$ be a **square wave** function of period T:

$$f(t) = \begin{cases} 1/2 & 0 \le t < T/2 \\ -1/2 & T/2 \le t < T \end{cases}$$

Here is the graph of $f(t)$ with $T = 1$ (with vertical lines added to improve visibility):



In this case we have

$$\begin{aligned} c_n &= \int_0^1 f(sT)e^{-2\pi ins}\,ds \\ &= \frac{1}{2}\int_0^{1/2} e^{-2\pi ins}\,ds - \frac{1}{2}\int_{1/2}^1 e^{-2\pi ins}\,ds \\ &= 0 \qquad (n=0) \\ &= \frac{1}{2}\left[\frac{e^{-2\pi ins}}{-2\pi in}\right]_0^{1/2} - \frac{1}{2}\left[\frac{e^{-2\pi ins}}{-2\pi in}\right]_{1/2}^1 \qquad (n \ne 0) \\ &= \left(\frac{1-e^{-\pi in}}{4\pi in}\right) + \left(\frac{1-e^{-\pi in}}{4\pi in}\right) \\ &= \ 0 \qquad (n \text{ even}) \\ &= \frac{1}{\pi in} \qquad (n \text{ odd}) \end{aligned}$$

Note that if we sum the terms for n and $-n$ (n odd) we get

$$\left(\frac{1}{-\pi in}\right) e^{-2\pi int/T} + \left(\frac{1}{\pi in}\right) e^{2\pi int/T} = \frac{2}{2\pi in}\left(e^{2\pi int/T} - e^{-2\pi int/T}\right) = \left(\frac{2}{\pi n}\right) \sin(2\pi nt/T)$$

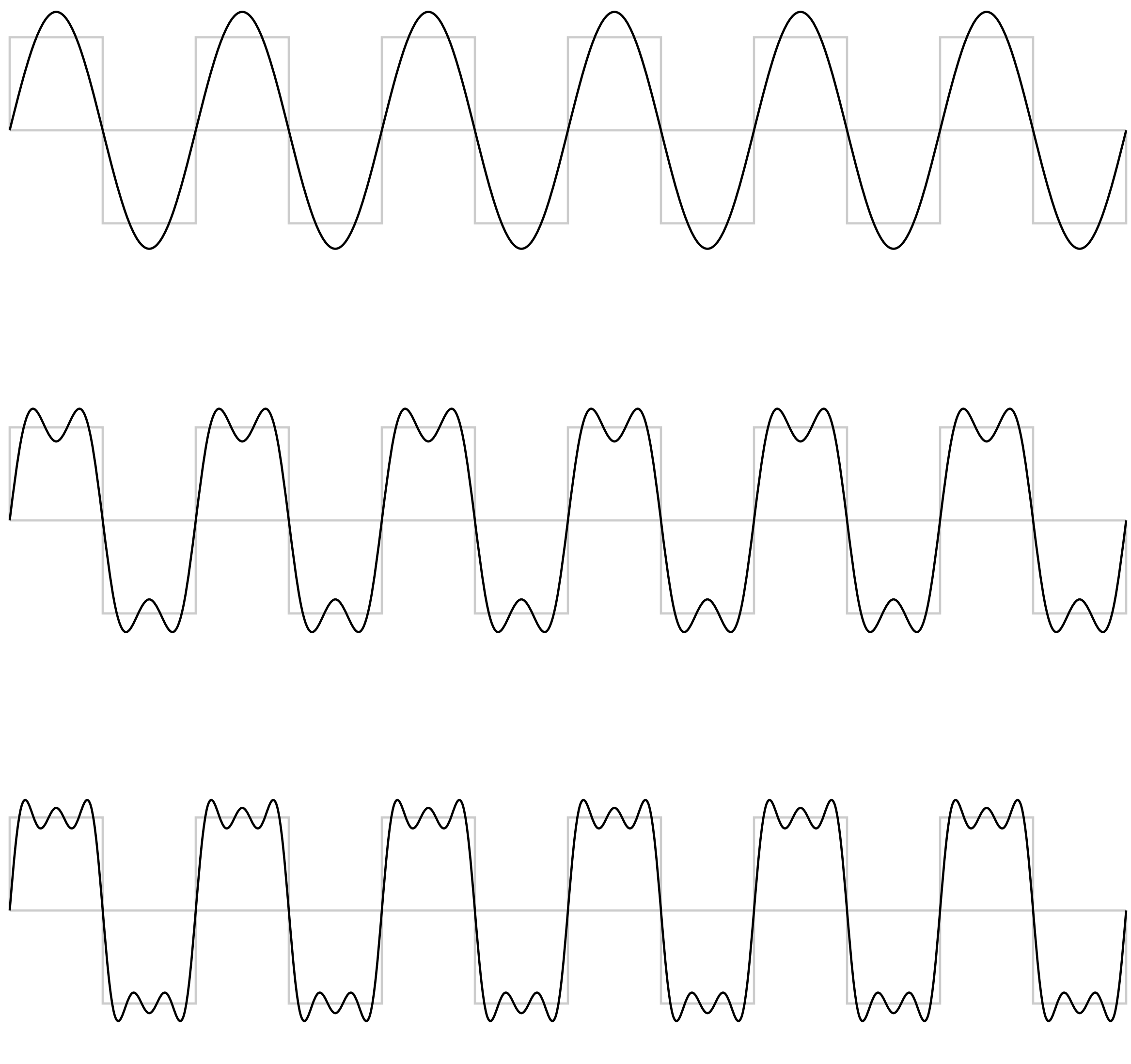
since

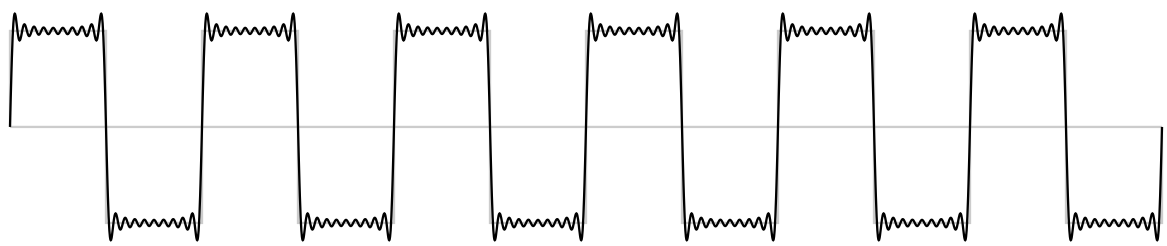
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i},$$

and finally

$$f(t) = \sum_{n \mathrm{odd}, n>0} \left(\frac{2}{\pi n}\right) \sin(2\pi n t/T)\ .$$

In the figure below we show successive sums of terms from this series.





The series seem to converge reasonably well, if not very rapidly, at most points—but there are some problems at those sharp edges. These problems persist even if a very large number of terms are used. (In the bottom graph 10 non-zero terms were used.) The 'ears' remain for all these finite series, and their height remains essentially constant. This sort of behaviour is not unusual for Fourier series.

Exercise 1.1. *Let $f(t)$ be the function which is equal to t in the range $[0,1)$, and extended periodically outside this range. Draw the graph of f. Find the complex and real Fourier series of f.*

Exercise 1.2. *Let a be a constant between 0 and 1, and let*

$$f(t) = \begin{cases} 1 & 0 \le t < a \\ 0 & a \le t < 1 \end{cases}$$

With $a = 1/4$, graph $f(t)$. With an arbitrary a, find its real Fourier series.

Exercise 1.3. *Let a be a constant between 0 and 1, and let*

$$f(t) = \begin{cases} t/a & 0 \le t < a \\ (1-t)/(1-a) & a \le t < 1 \end{cases}$$

With $a = 1/2$ and $1/4$, graph $f(t)$. With an arbitrary a, find its real Fourier series.

2. Justification

If we have an expression

$$f(t) = \cdots + c_{-1}e^{-2\pi it/T} + c_0 + c_1 e^{2\pi it/T} + \cdots + c_n e^{2\pi int/T} + \cdots \ .$$

Then we can multiply both sides of this equation by $e^{-2\pi imt/T}$ to get

$$e^{-2\pi imt/T} f(t) = \cdots + c_{-1}e^{-2\pi imt/T}e^{-2\pi it/T} + c_0 e^{-2\pi imt/T} + c_1 e^{-2\pi int/T}e^{2\pi it/T} + \cdots + c_n e^{-2\pi imt/T}e^{2\pi int/T} + \cdots$$

or

$$f(t)e^{-2\pi imt/T} = \sum_n c_n e^{2\pi i(n-m)t/T} \ .$$

If we integrate both sides over $[0,T]$ we get

$$\int_0^T f(t)e^{-2\pi imt/T}\,dt = \int_0^T \sum_n c_n e^{2\pi i(n-m)t/T}\,dt \ .$$

Now if $k = 0$ we have

$$\int_0^T e^{2\pi ikt/T} dt = \int_0^T dt = 1$$

while if $k \neq 0$ we have

$$\int_0^T e^{2\pi ikt/T} dt = \left[\frac{e^{2\pi ikt/T}}{2\pi ik} \right]_0^T = 0 \ .$$

Therefore all but one of the integrals in the sum on the right vanish and we have

$$\int_0^T f(t) e^{-2\pi imt/T} \, dt = c_m T, \qquad c_m = \frac{1}{T} \int_0^T f(t) e^{-2\pi imt/T} \, dt \ .$$

Exercise 2.1. *From the formulas for* $\cos(x+y)$, $\sin(x+y)$ *add and subtract to get formulas for* $\cos(x)\cos(y)$, $\cos(x)\sin(y)$, $\sin(x)\sin(y)$ *in terms of* $\cos(x+y)$, $\cos(x-y)$, $\sin(x+y)$, $\sin(x-y)$.

Exercise 2.2. *Calculate explicitly the following integrals, using the previous exercise, to show*

$$\begin{aligned}
\int_0^T \cos(2\pi nt/T) \sin(2\pi mt/T) \, dt &= 0 \qquad (\text{ all } m, n) \\
\int_0^T \cos(2\pi nt/T) \cos(2\pi mt/T) \, dt &= T/2 \quad (m = n > 0) \\
&= 0 \qquad (m \neq n) \\
\int_0^T \sin(2\pi nt/T) \sin(2\pi mt/T) \, dt &= T/2 \quad (m = n) \\
&= 0 \qquad (m \neq n)
\end{aligned}$$

Exercise 2.3. *Use the previous result to show that if*

$$f(t) = a_0 + \sum_{n>0} a_n \cos(2\pi nt/T) + \sum_{n>0} b_n \sin(2\pi nt/T) \ .$$

then for the coefficients we have the formulas

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_0^T f(t) \, dt \\
a_n &= \frac{2}{T} \int_0^T f(t) \cos(2\pi nt/T) \, dt \quad (n > 0) \\
b_n &= \frac{2}{T} \int_0^T f(t) \sin(2\pi nt/T) \, dt \quad (n > 0) \ .
\end{aligned}$$

Exercise 2.4. *Another way to derive the formulas is to set* $c_{\pm n} = A_n \pm iB_n$ *and then combine* $c_{-n} e^{-2\pi int/T} + c_n e^{-2\pi int/T}$ *using Euler's formula. Do this.*

3. Even and odd functions

There are special circumstances in which the calculation of Fourier series becomes a bit simpler than usual. Recall that the function $f(t)$ is called **even** if $f(-t) = f(t)$, and **odd** if $f(-t) = -f(t)$. All the functions $\cos 2\pi nt/T$ are even, for example, and all the functions $\sin 2\pi nt/T$ are odd. Because of this:

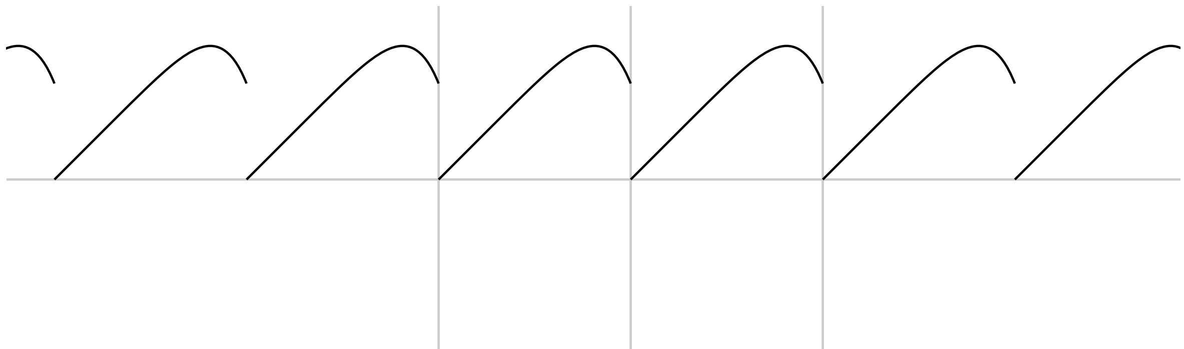
- *If $f(t)$ is even its Fourier series can only have cosine terms, and if $f(t)$ is odd it can only have sine terms.*

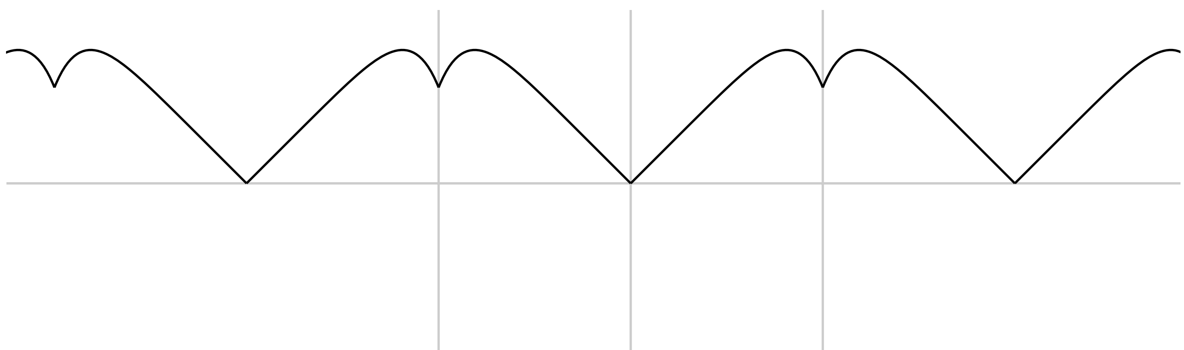
This explains why the Fourier series of the shifted square wave function has the expansion

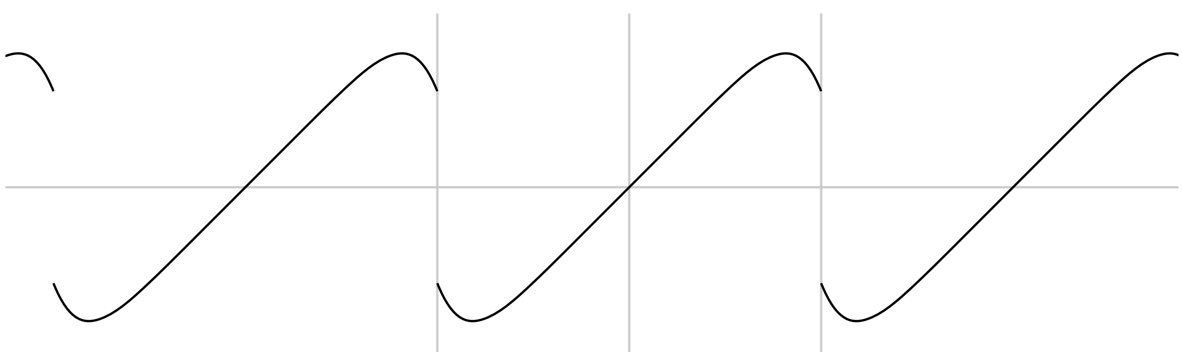
$$f(t) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{\pi(2n+1)} \sin \pi(2n+1)t$$

since $f(t) - 1/2$ is odd.

There are other related consequences we shall use later on in the course. Suppose $f(t)$ to be any function defined in the interval $[0, \ell]$. We can extend it to be defined for all numbers in any of several ways. (1) We can extend it to a function of period ℓ by just repeating it over again in every interval $[n\ell, (n+1)\ell]$. (2) We extend it to be an even function on $[-\ell, \ell]$ and then extend it everywhere as a function of period 2ℓ. (3) We extend it to be an odd function on $[-\ell, \ell]$ and then extend it everywhere as a function of period 2ℓ.







If $f(t)$ is even with period T then we only need to calculate the cosine coefficients. We can restrict the integration to half the period, since integration over the second half just repeats the integration on the first half. We get in this case

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(2\pi n t/T)\, dt \; .$$

If $f(t)$ is odd with period T then we only need to calculate the sine coefficients. We can restrict the integration to half the period, since again integration over the second half just repeats the integration on the first half. We get in this case

$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(2\pi n t/T)\, dt \; .$$

If we put the two features of this section together, we get:

- *If f is any function defined on the interval $[0,\ell]$, there exists a decomposition of f into components*

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(\pi n t/\ell)$$

  where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(t)\sin(\pi n t/\ell)\,dt\ .$$

  This is called the **sine series** *for f.*

- *If f is any function defined on the interval $[0,\ell]$, there exists a decomposition of f into components*

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(\pi n t/\ell)$$

  where

$$a_n = \frac{1}{\ell}\int_0^{\ell} f(t)\,dt$$
$$a_n = \frac{2}{\ell}\int_0^{\ell} f(t)\cos(\pi n t/\ell)\,dt\ .$$

  This is called the **cosine series** *for f.*

We get these decompositions by extending f to an odd and an even function of period 2ℓ, respectively.

Exercise 3.1. *Let*

$$f(t) = 1$$

in $[0,1]$. Extend $f(t)$ as an even function of period 2. Graph the extended function. Find the Fourier series for it.

Exercise 3.2. *Same $f(t)$. Extend as an odd function.*

Exercise 3.3. *Let*

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \end{cases}$$

Extend $f(t)$ as an odd function of period 4. Graph the extended function. Find the Fourier series for it.

Exercise 3.4. *Let*

$$f(t) = t$$

in $[0,1]$. Extend $f(t)$ as an even function of period 2. Graph the extended function. Find the Fourier series for it.

Exercise 3.5. *Same $f(t)$. Extend as an odd function.*

4. The rate of decrease of Fourier coefficients

Properties of the true convergence of Fourier series are very subtle and not usually of practical importance. The basic fact that is important and roughly valid always is that if the Fourier series converges at all, in any sense, then the coefficients decrease to 0.

$$f(t) = \sum c_n e^{2\pi i n t}$$

then

$$f'(t) = \sum c_n 2\pi n e^{2\pi i n t}\ .$$

However, we have to be careful. For example, if $f(t)$ is a square wave then

$$f(t) = \sum_{n \mathrm{odd}, n>0} \left(\frac{2}{\pi n}\right) \sin(2\pi nt/T) \ .$$

Taking the derivative on both sides we get the formal identity

$$f'(t) = \sum_{n \mathrm{odd}, n>0} \left(\frac{4}{T}\right) \cos(2\pi nt/T) \ .$$

This violates the rule I just asserted! What's going on? The point here is that the derivative of the square wave makes no sense—the function $f(t)$ has a **step discontinuity** at a couple of places in every period, so of course it has no well defined slope and no derivative there. Therefore its Fourier series does not in fact converge to it in the normal way, and the claim that the Fourier coefficients decrease is no longer valid.

But we do have the following consequence: if the series for $f'(t)$ also converges then the coefficients nc_n must also decrease to 0 as $|n| \to \infty$, so that in fact c_n must decrease to 0 faster than $1/|n|$. And so on, if $f(t)$ has higher derivatives.

- *Roughly speaking, the smoother a function $f(t)$ is the more rapidly its Fourier coefficients c_n decrease to 0 as $n \to \infty$.*

This is actually a matter of practical importance! Many compression schemes for electronic display apply Fourier analysis to small chunks of the image. For example, the common JPEG scheme looks at the image on squares of size 8×8 pixels. It applies a two dimensional version of Fourier analysis to the image, and in order to compress the amount of storage the image requires, it throws away the terms in the Fourier series of high frequency assuming that the high frequency coefficients will be smaller than those of low frequency. The first thing it must do is make a periodic function out of these images. One possibility would be to just extend it by the obvious tiling of squares. But if this is done, there will be sharp discontinuities at the boundaries of the squres, and this will cause the high frequency components not to be small, which wrecks the scheme. The trick is to extend the small image to one of size 16×16 through even reflection. The boundaries of these images will be continuous, and the high frequency components can indeed be expected to be small. Of course in doing this even extension you will be calculating a two-dimensional version of the **cosine transform**.