# Chapter 11. Introduction to linear systems

Physical structures one wants to analyze in the real world are usually extremely complicated. They will be made up of a large number of components, and the way in which one component behaves will affect, and be affected by, the behaviour of many others. For example, if you shake a long pole—or if something larger than you shakes an airplane wing, or a tall building—the initial motion will propagate to one end of the structure and reflect back again, sometimes enhancing and sometimes cancelling out what motion already exists. *The motion of no part of the structure is independent of that of the rest.* In another example, the exact path and magnitude of electric currents in one part of a large circuit depend on all the components in the circuit. In these circumstances, what is required to describe the evolution of the system is a **system** of differential equations involving several state variables simultaneously.

## 1. Heated chambers

Suppose we go back again to Newton's Law of Cooling, but now considering two cooling objects that affect each other. Suppose $A$ and $B$ are two rooms of a house, cooling off to the outside but also exchanging heat between themselves. Complicate matters further by assuming that each has inside it a heat source. Each heat exchange will have associated to it a relaxation time: $\tau_A$ for $A$ interacting with the outside $\tau_B$ for $B$ interacting with the outside, $\tau_{AB}$ for the interaction between the two rooms. We shall specify the heat sources in terms of their effect on the rooms: room $A$ has a heat source which would raise the temperature of $A$ by $h_A$ degrees per unit of time if $A$ were insulated from all other places, and similarly for $B$. For each of the rooms, there are three ways to interchange heat: with the outside, with the other room, and with the heat source. Thus Newton's Law of Cooling becomes the pair of equations

$$\begin{aligned}
\theta'_A &= -\frac{1}{\tau_A}(\theta_A - \theta_{\text{outside}}) - \frac{1}{\tau_{AB}}(\theta_A - \theta_B) + h_A \\
\theta'_B &= -\frac{1}{\tau_B}(\theta_B - \theta_{\text{outside}}) - \frac{1}{\tau_{AB}}(\theta_B - \theta_A) + h_B
\end{aligned}$$

Each of these differential equations involves both $\theta_A$ and $\theta_B$, which makes it non-trivial to find formulas for $\theta_A$ and $\theta_B$. There is one feature of these equations that is worth noting: the single constant $\tau_B$ occurs in both. This means that, after all, a heat gain by $A$ from $B$ is equivalent to an equivalent to a loss from $B$ to $A$.

There is one special case, worth looking at. Suppose $A$ and $B$ are completely insulated from each other. Then $\tau_{AB} = \infty$, and the equations become

$$\begin{aligned}
\theta'_A &= -\frac{1}{\tau_A}(\theta_A - \theta_{\text{outside}}) + h_A \\
\theta'_B &= -\frac{1}{\tau_B}(\theta_B - \theta_{\text{outside}}) + h_B
\end{aligned}$$

Now each equation involves only one temperature, and can be solvede explicitly.

**Exercise 1.1.** *In the case when $A$ and $B$ are insulated from each other, find formulas for $\theta_A$ and $\theta_B$, assuming initial temperatures $\theta_{A,0}$ and $\theta_{B,0}$.*

It turns out to be convenient and useful to think of the two temperatures together being considered as the **components** of the temperature of the system as a whole. This means we should define a **temperature vector**

$$\theta = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} .$$

We can then write the system as a single vector equation

$$\begin{aligned}
\theta' &= \begin{bmatrix} \theta'_A \\ \theta'_B \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{\tau_A}(\theta_A - \theta_{\text{outside}}) - \frac{1}{\tau_{AB}}(\theta_A - \theta_B) + h_A \\ -\frac{1}{\tau_B}(\theta_B - \theta_{\text{outside}}) - \frac{1}{\tau_{AB}}(\theta_B - \theta_A) + h_B \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{\tau_A} - \frac{1}{\tau_{AB}} & \frac{1}{\tau_{AB}} \\ \frac{1}{\tau_{AB}} & -\frac{1}{\tau_B} - \frac{1}{\tau_{AB}} \end{bmatrix} \theta + \begin{bmatrix} \frac{\theta_{\text{outside}}}{\tau_A} + h_A \\ \frac{\theta_{\text{outside}}}{\tau_B} + h_B \end{bmatrix} \\
&= X\theta + Y
\end{aligned}$$

where

$$X = \begin{bmatrix} -\frac{1}{\tau_A} - \frac{1}{\tau_{AB}} & \frac{1}{\tau_{AB}} \\ \frac{1}{\tau_{AB}} & -\frac{1}{\tau_B} - \frac{1}{\tau_{AB}} \end{bmatrix}, \quad Y = \begin{bmatrix} \frac{\theta_{\text{outside}}}{\tau_A} + h_A \\ \frac{\theta_{\text{outside}}}{\tau_B} + h_B \end{bmatrix} .$$

**Exercise 1.2.** *We suspect that the temperatures of both rooms will in time settle down very close to steady state temperatures. This gives the state of the system in which all heat interchange cancels out, and temperatures do not change. Find a formula for the steady state temperature of each room.*

**Exercise 1.3.** *Write down the equations for the case when $A$ and $B$ interact with each other, but are as a pair completely insulated from the outside. Find the steady state temperatures in this case. Add the two differential equations together to get a single simple differential equation for the sum $\theta_A + \theta_B$. Find a formula for the sum. Use this to find a forula for each temperature separately.*

## 2. Electric circuits

Large electric circuits are perfect examples of large physical systems where the components interact in a complicated way.

Their behaviour is governed essentially by some relatively simple rules called **Kirchhoff's Laws**:

- **(Kirchhoff's current law)** *At any point of the circuit the total amount of current flowing into that point must balance that flowing out.*
- **(Kirchhoff's voltage law)** *The voltage drop around any closed loop in the circuit must be $0$.*

Using these, every circuit made up of the four electric components we are using (resistors, inductors, capacitors, voltage sources) gives rise to a system of differential equations describing the flow in all of its parts.

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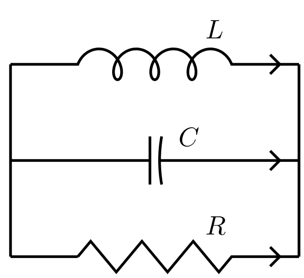
**Example.** Consider the following electric circuit:



In other words, we have three circuit elements in parallel. Let $L$ be the inductance, $C$ the capacitance, $R$ the resistance of the various elements of the circuit.

We ask the question: *suppose we start up a current in the circuit and then let it decay. How does the circuit behave?*

From Kirchhoff's Laws we deduce

$$I_R + I_C + I_L = 0$$

and

$$V_R = V_L = V_C$$

Combining all these we get

$$\begin{aligned} I' &= \frac{V}{L} \\ V' &= -\frac{I}{C} - \frac{V}{RC} \end{aligned}$$

where

$$I = I_L, \quad V = V_C \ .$$

We can write the system in compressed form using matrices:

$$\begin{bmatrix} I' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} \ .$$

Or we can think of the pair $(I, V)$ as making up a vector $y$ whose coordinates are functions of $t$ and write this as

$$y' = Ay$$

where

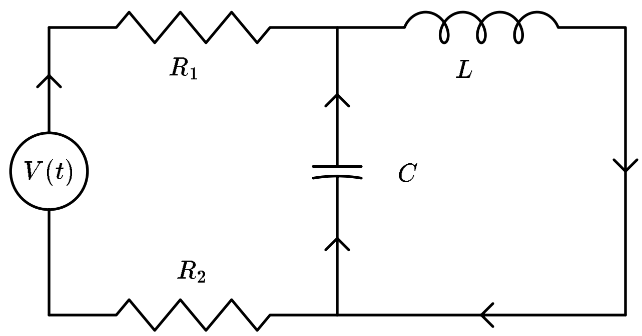
$$A = \begin{bmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix}$$

and

$$y = \begin{bmatrix} I \\ V \end{bmatrix} \ .$$

It is easy enough to follow the derivation of this system, but perhaps difficult to see what reasoning to follow when looking at a new circuit. There are in fact different methods possible. One method that works except in unusual circumstances is to choose as the basic variables (1) the currents through inductors and (2) the voltage drops across capacitors. This choice is determined by the fact that the potential energy of the circuit is stored in its capacitors and inductors. The evolution of the system is completely determined by its energy distribution.

**Exercise 2.1.** *Write down the differential equations for this circuit:*




### 3. The basic principles of solution

For nearly all of this part of the course, we shall look only at linear systems of equations

$$y' = Ay + b(t)$$

where $y = y(t)$ is a vector

$$y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \ldots \\ y_n(t) \end{bmatrix}$$

whose components are functions of $t$, $b(t)$ is a similar vector, and $A$ is an $n \times n$ matrix whose entries are constants.

As with second order differential equations, the system will be called **homogeneous** if $b$ vanishes. Solutions of a homogeneous linear system obey the linearity principle:

- *If $y(t)$ is a solution, so is $cy(t)$ where $c$ is a constant.*
- *If $y_1(t)$ and $y_2(t)$ are solutions, so is the sum $y_1(t) + y_2(t)$.*

Furthermore, in analogy with first order equations, a solution of any $n \times n$ system is completely determined by a single initial vector value:

- *If $y_0$ is an $n$-dimensional vector and $t_0$ any value of $t$, then there exists a unique solution of the system*

  $$y' = Ay + b(t)$$

  *with $y(t_0) = y_0$.*

The set of all solutions to a homogeneous linear system

$$y' = Ay$$

therefore forms a vector space of dimension $n$. If the coefficient matrix $A$ is constant, solutions also satisfy **time invariance**:

- *If $y(t)$ is a solution of the homogeneous system $y' = Ay$ with constant coefficient matrix $A$, then so is $y(t-h)$ for any time shift $h$.*

If $y(t)$ is a solution, so is

$$\frac{y(t+h) - y(t)}{h}$$

by applying linearity and time invariance together. If we take $h$ very small, and let $h \to 0$:

- *If $y(t)$ is a solution of the homogeneous system $y' = Ay$ with constant coefficient matrix $A$, then so is $y'(t)$.*

Differentiation $y \mapsto y'$ is therefore a linear operator on an $n$-dimensional vector space. It must have an eigenvalue, which means that there exists at least one non-trivial solution $y(t)$ of the system $y' = Ay$ such that $y' = \lambda y$. This equation amounts to a simultaneous set of equations $y_i' = \lambda y_i$, so that there exist constants $C_i$ with $y_i = C_i e^{\lambda t}$ for all $i$. If we substitute this into the original differential equation we get

$$\lambda e^{\lambda t} C = A C e^{\lambda t}, \qquad AC = \lambda C$$

if

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \dots \\ C_n \end{bmatrix} .$$

This means that $\lambda$ is an eigenvalue of $A$ and $C$ is a corresponding eigenvector.

- *To find solutions of a homogeneous linear system*

$$y' = Ay$$

*where $A$ is an $n \times n$ matrix with constant coefficients, we must find the eigenvalues and eigenvectors of $A$.*

Let's look at an example, a pair of rooms cooling off in the open. We shall assume that there are no heat sources, and that the outside temperature is $0$. Then the system of equations becomes

$$\theta' = X\theta$$

with

$$X = \begin{bmatrix} -\dfrac{1}{\tau_A} - \dfrac{1}{\tau_{AB}} & \dfrac{1}{\tau_{AB}} \\ \dfrac{1}{\tau_{AB}} & -\dfrac{1}{\tau_B} - \dfrac{1}{\tau_{AB}} \end{bmatrix},$$

Let's assume that $B$ is somewhat less well insulated than $A$, and that the rooms interchange heat easily. More precisely set $\tau_A = 1$, $\tau_B = 1/2$, $\tau_{AB} = 1/4$. Then we have

$$X = \begin{bmatrix} -5 & 4 \\ 4 & -6 \end{bmatrix} .$$

As we shall see later, the eigenvalues of $X$ are

$$-11/2 \pm \sqrt{65}/2$$

or approximately $-1.5$ and $-9.5$. Forget about the explicit eigenvectors are for the moment, just let them be $(x_1, y_1)$ and $(x_2, y_2)$. Then the general solution to the system is approximately

$$c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} e^{-1.5t} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} e^{-9.5t} .$$

What is the significance of this formula?

The rooms both cool off towards $0$, as we would expect. Since $e^{-9.5t}$ will be far smaller than $e^{-1.5t}$ for even moderate values of $t$, for most practical purposes we need only consider the first term. So to a good approximation for values of $t$ even a bit larger than $0$, the temperature vector will be essentially a scalar multiple of

$$\begin{bmatrix} x_1 e^{-1.5t} \\ y_1 e^{-1.5t} \end{bmatrix} .$$

In other words, to a good approximation the two rooms together will cool off as though they were isolated in the outside, but with a relaxation time somehow dependent on their interaction, different from the relaxation times of either room by itself. This is only reasonable, since we know that larger objects cool off more slowly than small ones, and the two rooms together might be considered in a way as a single object. What is especially curious is that in this approximation, the ratio of the temperatures of the two rooms depends somehow on eigenvectors on the matrix $X$. Another way of understanding the exact formula involving both $e^{-1.5t}$ and $e^{-9.5t}$ is that the pair of rooms together act almost as if they were made up of two objects—which you might call **virtual rooms**—each cooling off independently of the other. The temperature of the system is a linear combination of two **modes** of temperature decay, each with its own relaxation time.

This will all, I hope, become somewhat clearer in time. What is important to keep in mind now is that

- *The motion in time of many physical structures can be expressed as a linear combination of motions in certain simpler* **modes of motion***, each one corresponding to an eigenvalue of an associated matrix.*

**Exercise 3.1.** *Assume there are no heat sources and that $\theta_{\mathrm{outside}} = 0$. Suppose that that $\tau_A = \tau_B$. Add the two differential equations together to get a simple differential equation for the sum of the temperatures. Then use this to find a formula for each of the two separate temperatures.*