Chapter 8. Inhomogeneous equations

We shall now look at equations

$$y'' + a(t)y' + b(t)y = c(t)$$

where $c(t)$ is allowed not to vanish. At the beginning, we shall not even have to assume that $a(t)$ and $b(t)$ are constants. But you might as well.

1. A formula for solving second order inhomogeneous equations

We shall start with a kind of recipe for solving the inhomogeneous equation & initial conditions

$$y'' + a(t)y' + b(t)y = c(x), \quad y(t_0) = y_0,\ y'(t_0) = v_0\ .$$

- Find the fundamental solutions y_{0,t_0}, y_{1,t_0} and a formula for all $y_{1,s}$.
- The formula we want is

$$y(x) = y_0 \cdot y_{0,t_0} + v_0 \cdot y_{1,t_0} + \int_{t_0}^{t} y_{1,s}(t)c(s)\,ds\ .$$

Example. If we want to solve

$$y'' + y = c(t), \quad y(0) = 2,\ y'(0) = -1$$

we know that $y_{0,0} = \cos t$, $y_{1,0} = \sin t$. Then

$$\begin{aligned}
y_0 &= 2 \\
v_0 &= -1 \\
y_{1,s} &= \sin(t-s) \\
y &= 2\cos t - \sin t + \int_0^t \sin(t-s)c(s)\,ds \\
&= 2\cos t - \sin t + \int_0^t (\sin t \cos s - \cos t \sin s)c(s)\,dt \\
&= 2\cos t - \sin t + \sin t \int_0^t c(s)\ \cos s\,ds - \cos t \int_0^t c(s)\ \sin s\,ds
\end{aligned}$$

If you put the formula in this section together with an explicit formula for $y_{1,s}$, you will arrive at a version of this formula usually called the **variation of parameters** formula.

2. Analogies with the formula for first order linear equations

The formula for solving the first order linear equation

$$y' + a(t)y = b(t), \quad y(t_0) = y_0$$

is

$$y = y_0 \cdot e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} e^{A(s)} b(s)\,ds$$

where

$$A(t) = \int_{t_0}^{t} a(s)\,ds\ .$$

The function $e^{-A(t)}$ is the solution of the homogeneous equation taking the value 1 at t_0. For any other point s, $y(t) = e^{A(s)}e^{-A(t)}$ will be the solution of the homogeneous which takes the value 1 at s. In other words

$$y_{0,s} = e^{A(s)-A(t)}$$

and we can write the first order formula as

$$\begin{aligned} y &= y_0 \cdot y_{0,t_0} + \int_{t_0}^{t} e^{-A(t)+A(s)} b(s)\, ds \\ &= y_0 \cdot y_{0,t_0} + \int_{t_0}^{t} y_{0,s}(t) b(s)\, ds \end{aligned}$$

which is analogous to the formula for second order equations.

- *The essential difference between first and second order equations is that for first order equations we can always find solutions of the associated homogeneous equation, whereas for second order this may not be possible.*

3. Forgetting the initial conditions

If we are only interested in the general solution, we can use this formula for a solution:

$$y(x) = c_1 z_1 + c_2 z_2 + \int^{t} y_{1,s}(x) c(s)\, ds$$

Here z_1 and z_2 are any two essentially distinct solutions of the homogeneous equation. We can get by with the indefinite integral because for any fixed t_0 and t_1 the integral

$$\int_{t_0}^{t_1} y_{1,s}(x) c(s)\, ds$$

is a solution to the homogeneous equation. This is because

$$y_{1,s}(t) = c_1(s) z_1(t) + c_2(s) z_2(t)$$

for some coefficients $c_1(s)$ and $c_2(s)$ and the integral is

$$C_1 z_1(t) + C_2 z_2(t) = z_1(t) \int_{t_0}^{t_1} c_1(s) c(s)\, ds + z_2(t) \int_{t_0}^{t_1} c_2(s) c(s)\, ds \ .$$

Exercise 3.1. *Find the general solution of* $y'' + 3y' + 2y = e^{ct}$.

4. The structure of the formula

The formula as I have stated it gives the solution for the equation

$$y'' + a(t) y' + b(t) y = f(t)$$

with initial conditions

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= v_0 \end{aligned}$$

It says that

$$y = y_0 \cdot y_{0,t_0}(t) + v_0 \cdot y_{1,t_0}(t) + \int_{t_0}^{t} y_{1,s}(t) f(s)\, ds$$

The first component of the formula is

$$y_0 \cdot y_{0,t_0}(t) + v_0 \cdot y_{1,t_0}(t)$$

is a linear combination of the fundamental solutions $y_{0,t_0}(t)$ and $y_{1,t_0}(t)$ of the homogeneous equation, and is consequently itself a solution to the homogeneous equation. Of course its derivative is

$$y_0 \cdot y'_{0,t_0}(t) + v_0 \cdot y'_{1,t_0}(t)$$

If you set $t = t_0$ you will see, by definition of the fundamental solutions, that its value at t_0 is y_0 and the value of its derivative is v_0.

- *The first term in the basic formula can be characterized as the unique solution y of the homogeneous equation with $y(t_0) = y_0$ and $y'(t_0) = v_0$.*

The formula

$$y = y_0 \cdot y_{0,t_0}(t) + v_0 \cdot y_{1,t_0}(t)$$

for it is certainly correct, but you should be aware that there may be more efficient ways to find it.

The second component of the basic formula is

$$\int_{t_0}^{t} y_{1,s}(t) f(s)\, ds\ .$$

This is the most important and interesting of the two components. It depends linearly on $f(t)$, and in particular vanishes if the equation is homogeneous.

- *The second component of the basic formula is the solution y of the inhomogeneous equation with $y(t_0) = 0$, $y'(t_0) = 0$.*

I remind you that, in case the coefficients $a(t)$ and $b(t)$ are constants, in order to find $y_{1,s}$ we use the translation principle to write

$$y_{1,s}(t) = y_{1,0}(t - s)$$

If we want to know the general solution of the inhomogeneous equation, without initial conditions specified, we may replace the first component by the general solution of the homogeneous equation and the second component by an indefinite integral

$$\int^{t} y_{1,s}(t) f(s)\, ds\ .$$

The point here is that

- *We can find the general solution to the inhomogeneous equation by adding the general solution of the homogeneous equation to any one solution of the inhomogeneous we can find.*

Sometimes there is a solution of the inhomogeneous equation much simpler than the rest.

5. When the right hand side is a simple exponential function

The basic formula for solving inhomogeneous linear equations in terms of the solutions of the associated homogeneous equation has the virtue of working under just about all circumstances, but it is usually more complicated to use than necessary.

In most circumstances you will meet in practice there is a way to solve an inhomogeneous equation which is simpler than using the basic formula. I will explain this by an example.

Suppose we want to find the general solution of the equation

$$y'' + 3y' + 2y = e^{ct}$$

where c is an unspecified constant.

The general solution to the homogeneous equation is

$$y = Ae^{-t} + Be^{-2t}$$

with

$$y' = -Ae^{-t} - 2Be^{-2t}$$

To find $y_{1,0}$ we solve

$$\begin{aligned} A + B &= 0 \\ -A - 2B &= 1 \end{aligned}$$

to find

$$A = 1, \quad B = -1$$

and

$$\begin{aligned} y_{1,0}(t) &= e^{-t} - e^{-2t} \\ y_{1,s}(t) &= e^{-(t-s)} - e^{-2(t-s)} \\ &= e^{-t}e^{s} - e^{-2t}e^{2s} \end{aligned}$$

Therefore the second component of the basic formula (with indefinite integrals) is

$$\begin{aligned} \int^t y_{1,s}(t)f(t)\,ds &= \int^t \left[e^{-t}e^{s} - e^{-2t}e^{2s}\right]e^{cs}\,ds \\ &= e^{-t}\int^t e^{(c+1)s}\,ds - e^{-2t}\int^t e^{(c+2)s}\,ds \\ &= e^{-t}\left[\frac{e^{(c+1)s}}{(c+1)}\right] - e^{-2t}\left[\frac{e^{(c+2)t}}{(c+2)}\right] \\ &= \frac{e^{ct}}{(c+1)} - \frac{e^{ct}}{(c+2)} \\ &= \frac{e^{ct}}{(c^2+3c+2)} \end{aligned}$$

This procedure doesn't work if $c + 1 = 0$ or $c + 2 = 0$.

These calculations suggest what happens in general.

- *If c is not a root of the characteristic equation*

  $$c^2 + ac + b = 0$$

  then for the inhomogeneous linear differential equation with constant coefficients

  $$y'' + ay' + y = e^{ct}$$

  the function

  $$y = \frac{e^{ct}}{c^2 + ac + b}$$

  is a solution.

This works in circumstances more varied than you might think.

Example. To solve

$$y'' + 3y' + 2y = \cos t$$

we recall that $\cos t$ is the real part of e^{it}. Therefore we first solve

$$y'' + 3y' + 2y = e^{it}$$

We get

$$\begin{aligned} y &= \frac{e^{it}}{i^2 + 3i + 2} \\ &= \frac{e^{it}}{1 + 3i} \\ &= \frac{1 - 3i}{1 - 3i}\,\frac{e^{it}}{1 + 3i} \\ &= \frac{(1 - 3i)(\cos t + i \sin t)}{10} \\ &= \frac{(\cos t + 3\sin t) + i(-3\cos t + \sin t)}{10} \end{aligned}$$

whose real part, the solution we are looking for, is

$$\frac{\cos t + 3 \sin t}{10} \ .$$

We now know how to solve any equation

$$y'' + ay' + by = e^{ct}$$

when c is not a root of the characteristic polynomial. There are various rules for what happens when c is a root of the characteristic equation, but probably the most straightforward thing to do in this exceptional circumstance is go back to the basic formula.

There is one case which is worth while handling explicitly, however. Suppose we are solving

$$y'' + \omega_0^2 y = \cos \omega t \ .$$

The solutions to the associated homogeneous equation are

$$e^{\pm i\omega_0 t}$$

or

$$\cos \omega_0 t, \quad \sin \omega_0 t \ .$$

If $\omega \neq \omega_0$ we apply the technique explained earlier in this section with right hand side $e^{i\omega t}$ to get the general solution

$$c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{\cos \omega t}{-\omega^2 + \omega_0^2} \ .$$

When $\omega = \omega_0$ we have to apply the basic formula. The second fundamental solution to the homogeneous equation is

$$\frac{\sin \omega_0 t}{\omega_0} \ .$$

We therefore get

$$
\begin{aligned}
y &= \frac{1}{\omega}\int^t \sin\omega(t-s)\cos\omega s\,ds \\
&= \frac{\sin\omega t}{\omega}\int^t \cos^2\omega s\,ds - \frac{\cos\omega t}{\omega}\int^t \sin\omega s\,\cos\omega s\,ds \\
&= \frac{\sin\omega t}{\omega}\int^t \frac{1+\cos 2s}{2}\,ds - \frac{\cos\omega t}{\omega}\int^t \frac{\sin 2s}{2}\,ds \\
&= \frac{t\sin\omega t}{2\omega} + \frac{\sin\omega t\sin 2\omega t}{4\omega^2} - \frac{cos\omega t\cos 2\omega t}{4\omega^2} \\
&= \frac{t\sin\omega t}{2\omega} - \frac{\cos(2t-t)\omega}{4\omega^2} \\
&= \frac{t\sin\omega t}{2\omega} - \frac{\cos\omega t}{4\omega^2}\,.
\end{aligned}
$$

Recall here that

$$
\begin{aligned}
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\cos 2x &= 2\cos^2 x - 1 \\
\sin 2x &= 2\sin x\cos x\,.
\end{aligned}
$$

Also note that the second term $\cos\omega t/4\omega^2$ in the expression above is a solution to the homogeneous linear equation, and can be absorbed into the other component. We discover in the end that the solution to the equation

$$y'' + \omega^2 y = \cos\omega t$$

(the special case when $c = i\omega$ is a characteristic root) is equal to

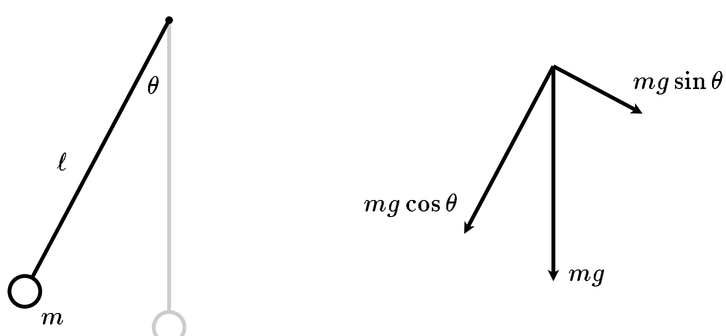
$$c_1\cos\omega t + c_2\sin\omega t + \frac{t\sin\omega t}{2\omega}$$

Exercise 5.1. *Suppose $\lambda^2 + a\lambda + b = (\lambda - c)(\lambda - d)$ where $d \neq c$. Find the general solution of $y'' + ay' + by = e^{ct}$.*

Exercise 5.2. *Suppose $\lambda^2 + a\lambda + by = (\lambda - c)^2$. Find the general solution of $y'' + ay' + by = e^{ct}$.*

6. Impulse response and inhomogeneous equations

Suppose the pendulum sits at rest until $t = 0$, when it is hit sharply with a hammer, say in the direction of positive displacement (from the left, in the picture below).

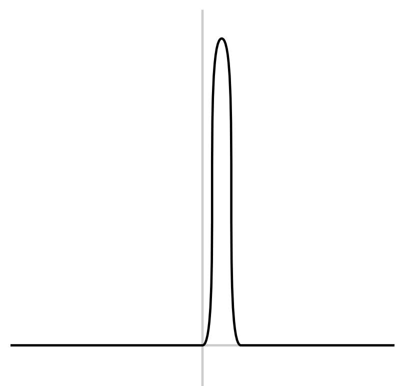


What determines how the pendulum moves? The details are messy, of course. When the hammer head hits the bob, the left hand side of the bob squishes in a bit, reacting elastically against the head of the hammer but also transmitting the effect of the blow throughout the bob, eventually to the rod of the pendulum, and then up the rod. But in a short time things stabilize and the pendulum begins to swing back and forth in a simple fashion. After this happens, the force $F(t)$ is equal to 0 and the system obeys the homogeneous equation

$$my'' + cy' + ky = 0 \ .$$

The period when interesting things are going on will last a fraction of a second. The subsequent motion will depend only on the state of the pendulum at the end of that brief period. So we must answer the question: *At the end of that short period, where is the pendulum, and how fast is it moving?* To determine the motion from that point on, we just solve the homogeneous equation with appropriate initial conditions.

Let's answer the second question first: *how fast is the bob moving when the hammer stops contact?* Of course this depends on how the force acts. Here, for example, is a plausible graph of $F(t)$:



What we shall see is that the effect of the force depends in a very simple way on this graph, at least if the time interval Δt is small. Of course the effect of the force is mainly to determine the position and velocity at the end of the initial period. • The claim is certainly true as far as position is concerned. The effect on position is a second-order effect, which is to say it is proportional to the square of the length of the time interval in which things are going on. This is just what happens when you let an object drop from your hand—the distance travelled from your hand is proportional to t^2. The force causes acceleration, which causes the velocity to build up noticeably, but it takes a relatively long time for this increase in velocity to affect position. If the time interval is small then the position of the bob won't have changed appreciably. • As far as velocity is concerned, we can use Newton's law $F = ma$, which can be reformulated approximately as

$$F = m\,\Delta v/\Delta t, \quad F\Delta t = m\,\Delta v$$

for small intervals Δt. Several smaller intervals will add their effects together, which means that roughly speaking *the change in momentum mv is the integral of $F(t)\,dt$, which is the area under the graph of $F(t)$.* We are assuming that the effects of the force of gravity are small over the small interval, compared to the force F. This is not badly inaccurate.

So the effect of the hammer blow on position is practically negligible and that on velocity depends primarily on the area under its graph. It is in fact proportional to this area, and the constant of proportionality is the mass m. The integral of Fdt is called **impulse** or **action**. It measures the transfer of momentum. An **ideal** hammer would transfer the momentum instantaneously, and we can measure even such ideal hammers according to the amount of momentum transferred. The ideal hammer of unit size can be thought of as the limit of ones acting over a very small but still finite interval, each with unit area under the graph of $F(t)$. We can now ask a simpler equation than the original one: *What is the effect of a unit ideal hammer?* It changes the momentum of the pendulum from 0 to 1 in an instant, and does not in that instant affect position. After that instant, the pendulum motion follows the homogeneous differential equation. So up to $t = 0$ the graph of position is dead horizontal, and at $t = 0$ it starts to rise suddenly with a slope equal to $1/m$. In other words, for $t > 0$ the position is given by the solution $y(t)$ of the homogeneous equation with $y(0) = 0$, $y'(0) = 1/m$. *This is what I have called, in the case where $m = 1$, the second fundamental solution $y_{1,0}(t)$ of the homogeneous equation.* So we see that *there is a simple interpretation of this solution in terms of a transfer of unit momentum to a physical system at $t = 0$.*

Here is the graph of the solution of the equation

$$y'' + 0.4y' + y = F(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$F(t) = \begin{cases} 5 & \text{if } 0 \le t \le 1/5 \\ 0 & \text{otherwise .} \end{cases}$$

I have chosen the friction coefficient low enough to allow some oscillation, and chosen $\Delta t = 1/5$ large enough to show what happens in that small interval. The light gray line is the actual function $y_{1,0}(t)$.

If force acts on the linear system over a long period of time, we can think of it as being broken up into a succession of forces of short duration, displaced in time. What happens during the interval $[s, s + \Delta s]$ is added to the accumulated effect. This new addition is equal to $F(s)y_{1,s}(t)\Delta s$ for $t \ge s$. If the system is at rest and the force does not act until $t = 0$, we get an approximation

$$y(t) \doteq \frac{1}{m} \sum_{0 \le s \le t} y_{1,0}(t - s)F(s)\,\Delta s$$

and in the limit

$$y(t) = \frac{1}{m} \int_0^t y_{1,0}(t - s)F(s)\,ds$$

for the solution of the inhomogeneous system with vanishing initial conditions. This tells us the physical significance of this formula.

Exercise 6.1. *Find an explicit formula for the solution of*

$$y'' + 2y' + 2y = F(t), \quad y(0) = 0,\ y'(0) = 1$$

for $t \ge 0$, where

$$F(t) = \begin{cases} h & \text{for } 0 \le t < 1/h \\ 0 & \text{otherwise} \end{cases}$$

Exercise 6.2. *Find an explicit formula for the solution of the equation*

$$y'' + 0.4y' + y = F(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$F(t) = \begin{cases} 5 & \text{if } 0 \le t \le 1/5 \\ 0 & \text{otherwise .} \end{cases}$$