Chapter 2. Complex numbers

If we want to solve a differential equation

$$y' = ay + \cos bt$$

we obtain the formula

$$y = C^{at} + e^{at} \int^t e^{-as} \cos bs \, ds \ .$$

Integrals like

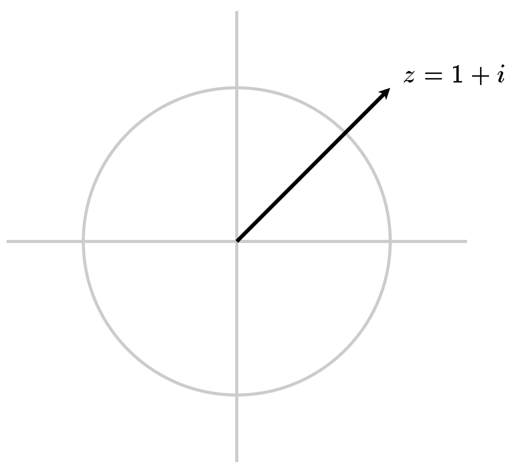
$$\int^t e^{-as} \cos bs \, ds$$

are a bit tricky, as I shall recall a bit later on. There is one technique to apply to integrals like this, and even more complicated ones, which uses complex numbers. Since complex numbers will turn out to be extremely important in this course, we shall look at them now.

1. Complex numbers and geometry

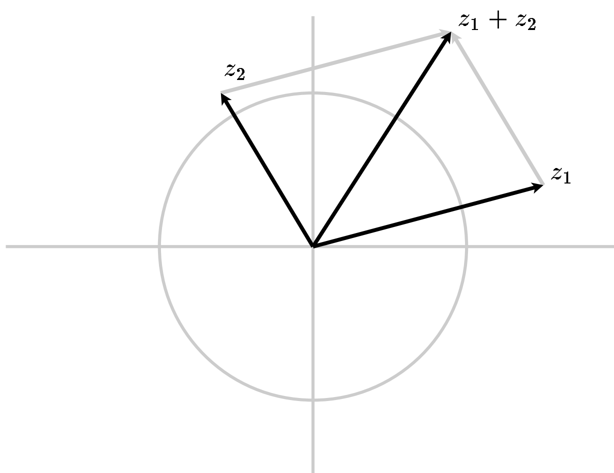
A **complex number** is one of the form $x + iy$ where $i = \sqrt{-1}$. They are required in order to solve all quadratic equations.

We can picture complex numbers $z = x + iy$ by plotting them in the (x, y) plane. Essentially, then, the complex number z becomes a $2D$ vector from the origin to (x, y).



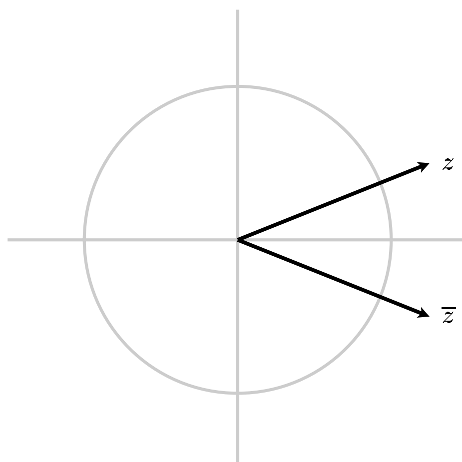
The real numbers lie along the x-axis. The number i and all pure imaginary numbers lie along the y-axis. If $z = x + iy$ then x is called its **real component** and y is called its **imaginary component**.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. In other words, the addition of complex numbers is just vector addition.



The **conjugate** $\overline{z}$ of a complex number z is what we get when we reflect it in the x axis.

$$\overline{x + iy} = x - iy \ .$$



If $z = x + iy$ and $\overline{z} = x - iy$ then

$$\begin{aligned} z + \overline{z} &= 2x \\ x &= \frac{z + \overline{z}}{2} \\ &= \frac{1}{2}\, z + \frac{1}{2}\, \overline{z} \\ z - \overline{z} &= 2iy \\ y &= \frac{z - \overline{z}}{2i} \ . \\ &= \frac{1}{2i}\, z - \frac{1}{2i}\, \overline{z} \ . \end{aligned}$$

The product of z and $\overline{z}$ has geometrical significance:

$$z\overline{z} = \|z\|^2 = (x + iy)(x - iy) = x^2 + y^2 = R^2$$

if R is the length of the $2D$ vector z. The radius $R = \sqrt{x^2+y^2}$ is called the **magnitude** or **amplitude** or **absolute value** of z. Sometimes (in other texts) it is written as $|z|$.

If $w = a + ib$, $z = c + id$ then

$$wz = (a+ib)(c+id) = (ac-bd) + i(ad+bc) \ .$$

If $z = x + iy$ then

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x-iy}\,\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \ .$$

For example, we see that

$$\frac{a+ib}{c+id} = \frac{(c-id)(a+ib)}{c^2+d^2} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

which shows how to divide complex numbers and retain the $x + iy$ form. Here is a numerical example:

$$\frac{i}{1+2i} = \frac{1-2i}{1-2i}\,\frac{i}{1+2i} = \frac{2+i}{5} \ .$$

Exercise 1.1. *Find* $1/(1+i)$, $1/(3+2i)$.

Exercise 1.2. *Write down* i^n *for* $n = -4$ *to* $n = 8$. *For* $n = 101$.

In order to understand well the multiplication of complex numbers we have to look at polar coordinates. If the polar coordinates of $z = x + iy$ are (R, θ) then

$$\begin{aligned}
x &= R\cos\theta \\
y &= R\sin\theta \\
z &= x + iy \\
&= R\cos\theta + iR\sin\theta \\
&= R(\cos\theta + i\sin\theta)
\end{aligned}$$

The angle θ is called the **argument** or **phase** of z. The numbers z with $R = \|z\| = 1$, for example, are those of the form

$$z = \cos\theta + i\sin\theta \ .$$

If

$$z_1 = R_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = R_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$\begin{aligned}
z_1 z_2 &= R_1R_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= R_1R_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \\
&= R_1R_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)
\end{aligned}$$

because of the rules for calculating the trigonometrical functions for sums of angles.

- *The magnitude of the product of two complex numbers is the product of their magnitudes. The argument of the product of two complex numbers is the sum of their arguments.*

Multiplication of a complex number by a real positive number R just scales it by the factor R. As a consequence of the rule above, multiplication by a number $\cos\theta + i\sin\theta$ amounts to rotation by θ. In particular, multiplication by i means rotation by $90°$.

If $w = z^{-1}$ then $wz = 1$. This means that the magnitude of $1/z$ is $1/R$, while the argument of $1/z$ is the negative of the argument of z.

Exercise 1.3. *Find the amplitude and arguments of* $3 + 2i$, $1/(3+2i)$.

Exercise 1.4. *By writing $(\cos\theta + i\sin\theta)^3$ in two ways, find a formula for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$. (Hint: First expand $(a+b)^3$.)*

Exercise 1.5. *(a) Find and plot in the (x, y) plane all the roots of $z^3 = 1$. (b) Of $z^4 = 1$. (c) Of $z^8 = 1$. (d) Of $z^4 = 2$.*

Exercise 1.6. *Plot roughly the path traversed by*

$$z^2 - 3z + 2$$

as z moves around the circle $\|z\| = 1$.

Exercise 1.7. *Plot roughly the path traversed by the complex numbers*

$$\frac{1}{1 + z - z^2}$$

as z goes from $-i\infty$ to $i\infty$ along the imaginary axis. (Hint: do 0 to $i\infty$ first.)

2. The complex exponential function

The exponential function e^x and the trigonometric functions $\cos x$ and $\sin x$ are related to each other. The relationship involves complex numbers. It simplifies many of the calculations involved in solving differential equations.

The exact relationship is called **Euler's equation**:

$$e^{ix} = \cos x + i\sin x\ .$$

In other words, $\cos x$ *is the real part of the complex exponential function* e^{ix}, *and* $\sin x$ *is its imaginary part.* The complex exponential obeys all the usual rules that the real one does.

Euler's equation can be proven by using Taylor's series. If we recall that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and we set $z = ix$ then we get since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \cdots \\ &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] \\ &= \cos x + i\sin x\ . \end{aligned}$$

From this we see that

- *If $c = a + ib$ is any complex number then*

$$e^{cx} = e^{ax+ibx} = e^{ax}e^{ibx} = e^{ax}\cos bx + ie^{ax}\sin bx,$$

or equivalently

- *The function $e^{ax}\cos bx$ is the real part of e^{ax+ibx} and $e^{ax}\sin bx$ is its imaginary part.*

Euler's equation 'explains' the trigonometrical sum formulas, since they turn out just be a different way of expressing the identity $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\,e^{i\theta_2}$.

Exercise 2.1. *Find* $e^{\pi i}$; $e^{-\pi i}$; $e^{\pi i/2}$.

3. Integrals

Let's look again at exponentials of the form

$$\int e^{ax}\cos bx\,dx, \quad \int e^{ax}\cos bx\,dx, \ .$$

First recall what you may have learned earlier. For example, let

$$I_1 = \int e^x \cos x\,dx, \quad I_2 = \int e^x \sin x\,dx$$

Then integration by parts with $u = e^x$, $v = \sin x$ gives us

$$I_1 = \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx = e^x \sin x - I_2$$

and with $u = e^x$, $v = -\cos x$

$$I_2 = \int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx = -e^x \cos x + I_1$$

from which we can solve the pair of linear equations

$$\begin{aligned} I_1 + I_2 &= e^x \sin x \\ I_1 - I_2 &= e^x \cos x \ . \end{aligned}$$

This is rather complicated and error-prone.

Now let's use complex exponentials to find the general formula more directly. Recall that for any constant c

$$\int e^{cx}\,dx = \frac{e^{cx}}{c} \ .$$

This works even if c is complex! So we write the integrals

$$\int e^{ax}\cos bx\,dx, \quad \int e^{ax}\sin bx\,dx$$

as the real and imaginary parts of the complex integral

$$\begin{aligned} \int e^{(a+ib)x}\,dx = \int e^{cx}\,dx \quad (c = a + ib) \\ &= \frac{e^{cx}}{c} \\ &= \frac{e^{(a+ib)x}}{a+ib} \\ &= \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} \\ &= \frac{e^{ax}}{a^2+b^2}(a-ib)(\cos bx + i\sin bx) \\ &= \frac{e^{ax}}{a^2+b^2}\big((a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)\big) \end{aligned}$$

Equating real and imaginary components, we get

$$\int e^{ax}\cos bx\,dx = e^{ax}\,\frac{a\cos bx + b\sin bx}{a^2+b^2}$$
$$\int e^{ax}\sin bx\,dx = e^{ax}\,\frac{a\sin bx - b\cos bx}{a^2+b^2}\ .$$

Exercise 3.1. *Find*

$$\int e^t\cos t\,dt$$

Exercise 3.2. *Find the integral*

$$\int te^{ct}\,dt$$

by parts. Find the integral

$$\int t\cos t\,dt$$

by applying this for a suitable complex number c*.*