Computing with series

A series is an infinite sum

$$c_0 + c_1 + \cdots + c_n + \cdots$$

The **partial sums** of the series are

$$c_0, \quad c_0 + c_1, \quad c_0 + c_1 + c_2, \quad \ldots$$

and the series is said to **converge** if these partial sums have a limit.

It should be clear that a *necessary* condition for convergence is that the terms c_n thmeselves have 0 as limit. *This is not, however, a sufficient condition.*

Geometric series

Perhaps the simplest convergent series si

$$1 + 1/2 + 1/4 + \cdots + 1/2^n + \cdots$$

The partial sums are

$$1, \quad 1\,{}^1\!/\!_2, \quad 1\,{}^3\!/\!_4, \quad 1\,{}^7\!/\!_8, \quad 1\,{}^{15}\!/\!_{16}, \quad \ldots$$

and it is easy to show that they converge to 2.

Any series of the form

$$1 + q + q^2 + q^3 + \cdots$$

is called a **geometric series**. If $|q| \geq 1$ this cannot possibly converge, but if $|q| < 1$ then it does converge. In fact, we can see exactly what it converges to, by doing some simple algebra.

Lemma. *For any q, the finite geometric series*

$$1 + q + q^2 + q^3 + \cdots + q^n$$

is equal to

$$\frac{1 - q^{n+1}}{1 - q}\,.$$

The best way to convince yourself of this is to do a few simple cases. One case is well known:

$$1 + q = \frac{1 - q^2}{1 - q}\,.$$

The general case can be proven by multiplying

$$\begin{aligned}(1-q)(1 + q + q^2 + \cdots + q^n) &= 1 + q + q^2 + \cdots + q^n - q(+q + q^2 + \cdots + q^n) \\ &= 1 + q - q + q^2 - q^2 + \cdots - q^{n+1} \\ &= 1 - q^{n+1}\end{aligned}$$

From this it follows immediately that the partial sums of the geometric series converge to

$$\frac{1}{1 - q}$$

if $|q| < 1$.

We'll see that finding convergent series isn't difficult, but that finding convergent series that converge to something explicitly know is rather rare. The geometric series is the basis for understanding the way many series behave.

The exponential series

The Taylor series for a function $f(x)$ at $x = a$ is

$$f(a+h) = f(a) + f'(a)h + f''(a)\,\frac{h^2}{2!} + f'''(a)\,\frac{h^3}{3!} + \cdots$$

For $f(x) = e^x$ with $a = 0$ this becomes

$$e^x = 1 + x = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and in particular it gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

There is really no other way to define e except by this series, which does converge. This series not only converges, but it converges rapidly, and can be used to compute e to any desired accuracy. Here, for example, are the first several partial sums according to my computer:

1.0	2.0
0.5	2.5
0.16666666666666666	2.6666666666666665
0.041666666666666664	2.7083333333333330
0.008333333333333333	2.7166666666666663
0.001388888888888889	2.7180555555555554
$1.984126984126984E{-4}$	2.7182539682539684
$2.48015873015873E{-5}$	2.7182787698412700
$2.7557319223985893E{-6}$	2.7182815255731922
$2.7557319223985894E{-7}$	2.7182818011463845
$2.505210838544172E{-8}$	2.7182818261984930
$2.08767569878681E{-9}$	2.7182818282861687
$1.6059043836821616E{-10}$	2.7182818284467594
$1.1470745597729726E{-11}$	2.7182818284582300
$7.647163731819817E{-13}$	2.7182818284589950
$4.779477332387386E{-14}$	2.7182818284590430
$2.811457254345521E{-15}$	2.7182818284590455
$1.5619206968586228E{-16}$	2.7182818284590455
$8.220635246624331E{-18}$	2.7182818284590455

It seems pretty clear that the last sum is very close to the true value of e, and that adding more terms will not affect the calculation of the first 16 decimal digits. The reason we can be so sure of that is that beyond this table each term is less that the previous one divided by 10, so that in effect we are compare this series with a geometric series for $q = 10$.

This always happens for any value of x, although there are some tricky points to be dealt with. If x is large, the terms will start by growing, and only when n exceeds x will they start to shrink, and the series look like a converging geometric series.

To see hwo this comparison works, let's try to figure out how many terms of the series for e would nbe becessary to calculate it correct to 100 significant figures. Answering a question like this means that we have to find N such that the **tail** of the series

$$\frac{1}{N!} + \frac{1}{(N+1)!} + \cdots$$

is smaller than, say, 10^{-100}. In this case, we can factor the tail as

$$\frac{1}{N!}\left(1+\frac{1}{N+1}+\frac{1}{(N+1)(N+2)} makescpseries.psger + \ldots\right)$$

which although quite complicated, is bounded from above by the geometric series

$$\frac{1}{N!}\,\frac{1}{1-1/(N+1)}=\frac{N+1}{NN!}\ .$$

if N is large, this is just about the same as $1/N!$. So our task now becomes that of finding N such that with $N! > 10^{100}$. A little exploring with your calculator will show that $N = 70$ will do.

Comparison to integrals

I have said that in order to check whether or not a series converges or not, it is not sufficient to know that the terms themselves converge to 0. In other words, there are series in which the individual terms get arbitrarily small but the series does not converge. For example, the series

$$1 + 1/2 + 1/3 + 1/4 + \cdots$$

does not converge. That is to say, sooner or later we can find partial sums that are arbitrarily large. I'll show this in two ways.

For the first, group the terms like thsi:

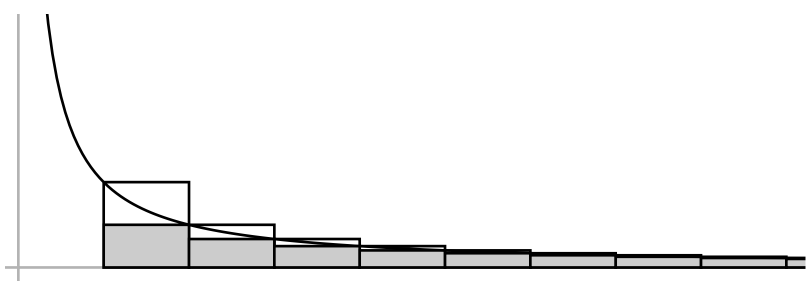
$$1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + \cdots + 1/16) + \cdots$$

Each group is larger than $1/2$, so sooner or later the partial sums must exceed any given bound. It takes a while, though, since the number of terms necessary to exced a bound N is roughly proportional to 2^N!

A second way to see that this series does not converge comes from a trick that is generally quite useful.

Proposition. *Suppose $f(x)$ to be a function defined and continuous in the range $(0, \infty)$, constantly decreasing as x grows. Then*

$$\int_2^{n+1} f(x)\,dx < f(2) + \cdots + f(n) < \int_1^n f(x)\,dx\ .$$



This bounds partial sums above and below by integrals, and gives extremely fine estimates for them. Thus

$$\int_2^{n+1} \frac{dx}{x} = \ln(n+1) - \ln(2) < 1/2 + 1/3 + \cdots + 1/n$$

which again shows that the series sums to arbitrarily large numbers. But since also

$$f(2) + f(3) + f(n) < \log n$$

if we want the sum to exceed 100 we have to choose $\log n > 99$, $n > e^{99}$. That's a lot of terms. Not to be summed explicitly in my lifetime or yours.

This technique shows equally that the series

$$1 + 1/2^p + 1/3^p + \cdots$$

converges for all $p > 1$.