## Mathematics 102 — Fall 1999

## Polynomials

After linear functions, the next simplest functions are polynomials. Examples are

$$1 + 2x, \quad x^2 + 3x + 2, \quad x^3 - 2 .$$

Here $x$ is a variable. Recall that we have defined a function to be a rule that tells you how to obtain one number from another. How is a polynomial a function in that sense? You substitute the first number for $x$ and calculate the value of the polynomial for that value of $x$.

The **degree** of a polynomial is the highest exponent of $x$ appearing. Thus, the polynomials above have degrees $1$, $2$, $3$ respectively. Since

$$x^0 = 1$$

a constant function is a polynomial of degree $0$, and a linear function is one of degree $1$. The coefficients of the powers of $x$ in a polynomial expression are called the **coefficients** of the polynomial. We will index the coefficients by degree:

$$P(x) = c_0 + c_1 x + \cdots + c_n x^n .$$

Notice that a polynomial of degree $n$ might have as many as $n + 1$ non-zero coefficients.

### Roots and factoring

If $P(x)$ is a polynomial, a **root** of the polynomial is a number $a$ such that $P(a) = 0$. For example, let

$$P(x) = x^2 - 3x + 2 .$$

You know how to factor it as

$$x^2 - 3x + 2 = (x - 2)(x - 1) .$$

If $a$ is a root, then

$$(a - 2)(a - 1) = 0 .$$

Now the product of two numbers can be $0$ only if one of them is $0$. Thus if $a$ is a root, then either $a - 2$ or $a - 1$ must be zero, and consequently either $a = 2$ or $a = 1$.

This is a special case of a general relationship betwen roots and factors of a polynomial. The first part is very simple—if $(x - a)$ is a factor of $P(x)$ then $a$ is a root of $P(x)$. The other way is slightly more complicated to see, but still true—if $a$ is a root, then $(x - a)$ divides $P(x)$.

Here's an example. Let

$$P(x) = 2x^3 - 3x^2 + 2x - 1 .$$

Then

$$P(1) = 2 - 3 + 2 - 1 = 0$$

so $1$ is a root. But then we can divide $P(x)$ by $x - 1$, as the following calculation indicates:

$$
\begin{array}{r}
2x^2 \quad - \quad x \quad + \quad 1 \phantom{xxxxx} \\
x \quad - \quad 1 \;\big)\; 2x^3 \quad - \quad 3x^2 \quad + \quad 2x \quad - \quad 1 \\
2x^3 \quad - \quad 2x^2 \phantom{xxxxxxxxxxxx} \\
- \quad x^2 \quad + \quad 2x \phantom{xxxx} \\
- \quad x^2 \quad + \quad x \phantom{xxxx} \\
x \quad - \quad 1 \\
x \quad - \quad 1
\end{array}
$$

so that

$$P(x) = (x - 1)(2x^2 - x + 1)$$

as you can check. This doesn't tell you how to find roots, but it does tell you that if you have one root, you can divide and then look for roots of a polynomial of smaller degree.

**Tables**

When looking for roots of $P(x)$, it is almost always good to be able to look at the graph of $P(x)$ in order to understand exactly what you're in for. Later on, we shall learn all kinds of tricks to do this. but for now we just want to explain to you the simplest (and also perhaps most tedious) method. We shall show what happens by an example. Let

$$P(x) = x^2 + x - 1 \ .$$

The first thing we shall do is calculate the value of $P(x)$ for $x = -5, -4, -3, \ldots, 4, 5$. This choice of range is somewhat arbitrary, but large enough to get a good feeling for this polynomial. We can clearly, with some work, just calculate all these values explicitly. We get the table

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 19 | 11 | 5 | 1 | $-1$ | $-1$ | 1 | 5 | 8 | 19 | 26 |

and then plot the values of $P(x)$ in this table:



Now it is simple enough to just calculate the values of $P(x)$ in this table in the ordinary way, but there is a useful trick to know about that some find easier. Suppose we add to the table above the **differences** in the values of $P(x)$, and then the difference of the differences. We get

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 19 | 11 | 5 | 1 | $-1$ | $-1$ | 1 | 5 | 8 | 19 | 26 |
| $\Delta P$ | $-8$ | $-6$ | $-4$ | $-2$ | 0 | 2 | 4 | 6 | 8 | 10 | |
| $\Delta^2 P$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | | |

The differences form an arithmetic progression, and the second order differences are constant. Once we know this, we can use it to carry out the calculation of the table in a very simple-minded way. First, we calculate the first three values in the table and their differences:

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 19 | 11 | 5 | | | | | | | | |
| $\Delta P$ | $-8$ | $-6$ | | | | | | | | | |
| $\Delta^2 P$ | 2 | | | | | | | | | | |

Then since we know the second order differences are constant, we can fill in the bottom row, and then the one above it.

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | 19 | 11 | 5 | | | | | | | | |
| $\Delta P(x)$ | $-8$ | $-6$ | $-4$ | $-2$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| $\Delta^2 P(x)P$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | |

But then from this in turn we can fill in the row for $P(x)$. The charm of this method is that the only multiplications we have to do are at the beginning. After that, all we need is addition or subtraction.

**Exercise 1.** *Make up a similar table and graph for $x^2 - x - 1$ in the range $-5$ to $5$, with $\Delta x = 1$. Use the difference method.*

**Exercise 2.** *Make up a similar table and graph for $x^3 - x + 1$ in the range $-5$ to $5$, with $\Delta x = 1$. Use the difference method. Here, the third order differences are constant, and you will have to calculate explicitly the first 4 values of $P(x)$.*

### Finding roots

We can see from the table of values of $x^2 + x - 1$ that $P(-2) = 1$ while $P(-1) = -1$. Therefore the graph of $P(x)$ must cross the $x$-axis in between $-2$ and $-1$, or in other words a root lies in the interval $[-2, -1]$. Similarly, a root lies in the interval $[0, 1]$. This agrees with what the graph shows us, too. We can now continue making tables to locate a root to considerable accuracy. Let's look for the root between $0$ and $1$, for example. We make up a table of values of $P(x)$ for $x = 0.0$, $x = 0.1, 0.2, \ldots, 0.9, 1.0$. We get this table

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | $-1.00$ | $-0.89$ | $-0.76$ | $-0.61$ | $-0.44$ | $-0.25$ | $-0.04$ | 0.19 | 0.44 | 0.71 | 1.00 |
| $\Delta P(x)$ | 0.11 | 0.13 | 0.15 | 0.17 | 0.19 | 0.21 | 0.23 | 0.25 | 0.27 | 0.29 | |
| $\Delta^2 P(x)$ | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | | |

and the graph in the middle here (with the previous one next to it showing how we have zoomed in):
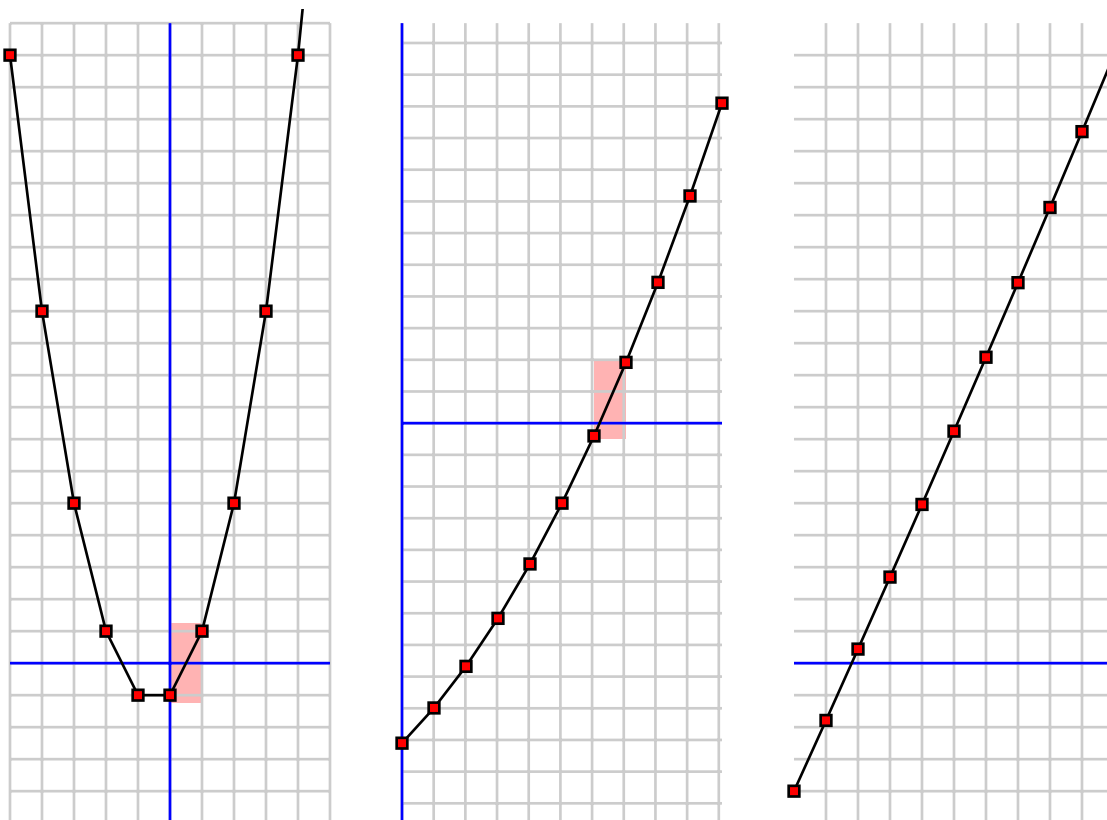
Then we do this once more, this time to the interval from $0.6$ to $0.7$. This time we see that the root is between $0.61$ and $0.62$.

| $x$ | 0.60 | 0.61 | 0.62 | 0.63 | 0.64 | 0.65 | 0.66 | 0.67 | 0.68 | 0.69 | 0.70 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | $-0.0400$ | $-0.0179$ | 0.0044 | 0.0269 | 0.0496 | 0.0725 | 0.0956 | 0.1189 | 0.1424 | 0.1661 | 0.1900 |
| $\Delta P(x)$ | 0.0221 | 0.0223 | 0.0225 | 0.0227 | 0.0229 | 0.0231 | 0.0233 | 0.0235 | 0.0237 | 0.0239 | |
| $\Delta^2 P(x)$ | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | | |

**Pascal's triangle**

We are now going to digress a bit in order to introduce some important particular polynomials. Let's start with one of the simplest of all sequences:

$$f: \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots$$

and then write its sum sequence next to it:

$$
\begin{array}{rccccccccc}
f_0 = f: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
f_1 = \Sigma f: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots
\end{array}
$$

These will be the first sequences of several. These sequences will all be indexed; the term indexed by $j$ in the sequence indexed by $i$ will be $f_{i,j}$. Thus $f_{0,j} = 1$ for all $j \geq 0$, and $f_{1,j} = j$ for all $j \geq 0$.

Now we add to this the sum sequence of the second row, which is $\Sigma\Sigma f = \Sigma^2 f$.

$$
\begin{array}{rccccccccc}
f_0: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
f_1: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
f_2: & 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & \ldots
\end{array}
$$

We don't have to calculate the sum all over again each time in this row. Any term in the new row is the sum of the terms (1) just to its left and (2) just up and to its left. For example, $15 = 5 + 10$:

$$
\begin{array}{llllllllll}
f_0: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
f_1: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
f_2: & 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & \ldots
\end{array}
$$

We can write this recipe like this:

$$f_{2,n} = f_{2,n-1} + f_{1,n-1}$$

for all $n \geq 1$. This givs us the whole row, if we take into account that the row must start with $0$.

Add similarly a third row, the sum sequence of the second row:

$$
\begin{array}{llllllllll}
f_0: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
f_1: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
f_2: & 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & \ldots \\
f_3: & 0 & 0 & 0 & 1 & 4 & 10 & 20 & 35 & \ldots
\end{array}
$$

with the rule

$$f_{3,n} = f_{3,n-1} + f_{2,n-1} \ .$$

Continuing on in this way, we obtain an unlimited number of rows determined completely by the rules

$$
\begin{aligned}
f_{0,0} &= 1 \\
f_{m,0} &= 0 \ (m > 0) \\
f_{0,n} &= 1 \ (n > 0) \\
f_{m,n} &= f_{m,n-1} + f_{m-1,n-1} \ (m > 0, n > 0) \ .
\end{aligned}
$$

This figure is called **Pascal's triangle** because it is often written without bothering to write the initial zeroes in each row:

$$
\begin{array}{llllllllll}
f_0: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
f_1: &   & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
f_2: &   &   & 1 & 3 & 6 & 10 & 15 & 21 & \ldots \\
f_3: &   &   &   & 1 & 4 & 10 & 20 & 35 & \ldots \\
f_3: &   &   &   &   & 1 & 5 & 15 & 35 & \ldots
\end{array}
$$

**Exercise 3.** *Write the next terms in all five rows.*

**Exercise 4.** *Write the first nine terms of the next row (including the zeroes).*

**Exercise 5.** *Write the term indexed by $99$ for each of the first three rows.*

**Exercise 6.** *Write down a formula for the term $c_{2,n}$, recalling how to sum an arithmetic progression.*

**Exercise 7.** *The numbers in this table occur in other contexts as well. Find $(a + b)^2$, $(a + b)^3$. and then write down a guess for $(a + b)^6$ without direct calculation.*

We have a way to find any number in the table, but it is not so practical, since in effect if we want to find $f_{m,n}$ we must find a whole chain of the $f_{i,j}$ with $i < m$ and $j < n$. For example, the number at the lower right here depends on all the other numbers in the table.

$$
\begin{array}{lllllllll}
f_0: & 1 & 1 & 1 & & & & \\
f_1: & 0 & 1 & 2 & 3 & & & & \ldots \\
f_2: & 0 & 0 & 1 & 3 & 6 & & & \ldots \\
f_3: & 0 & 0 & 0 & 1 & 4 & 10 & & \ldots \\
f_4: & 0 & 0 & 0 & 0 & 1 & 5 & 15 & \ldots
\end{array}
$$

This involves calculating roughly $mn/2$ other numbers to get it. Not so good. Ideally, we would like a more efficient formula for $f_{m,n}$ that does not have this drastic **recursive** (meaning going backwards) quality. And we are going to find it.

**The binomial theorem**

In Pascal's triangle, we know immediately formulas for the terms in the first two rows:

$$f_{0,n} = 1, \quad f_{1,n} = n$$

and know from an earlier calculation that

$$f_{2,n} = \frac{n(n-1)}{2} \ .$$

We now *guess* that $f_{m,n}$ is a polynomial in $n$ of degree $m$. For example, we would expect that $f_{3,m}$ is a cubic polynomial in $n$. Which cubic polynomial? The best hint we have is that the first three numbers in the row indexed by 3 are 0, so that this polynomial must have 0, 1, and 2 as roots. This means that $n$, $n-1$, and $n-2$ must be factors. So the polynomial we are looking for must be a constant times $n(n-1)(n-2)$. What constant? The first non-zero enry in the row is $f_{3,3} = 1$, and on the other hand $3(3-1)(3-2) = 3 \cdot 2 \cdot 1$. So we will guess that

$$f_{3,n} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \ .$$

Similarly, we can guess that

$$f_{m,n} = \frac{n(n-1)\ldots(n-m+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots m} \ .$$

So wWe now have a plausible guess for a formula, but how can we verify that it is true?

Let's look at the row indexed by 3 again. The numbers in this row are determined by these two properties: (1) Its initial term is 0 and (2) its difference series is the second row. This means

$$f_{3,0} = 0, \quad f_{3,n+1} - f_{3,n} = f_{2,n} \ .$$

In order to prove that our formula for $f_{3,n}$ is correct, we must verify that it, too, has these two properties. The first is clear. As for the second:

$$\frac{(n+1)(n)(n-1) - n(n-1)(n-2)}{6} = \frac{n(n-1)}{2}$$

sure enough. So we are done, at least for this case.

**Exercise 8.** *Let*

$$C_{m,n} = \frac{n(n-1)\ldots(n-(m-1))}{1 \cdot 2 \cdot 3 \cdot 4 \cdots m} \ .$$

*Verify that*

$$C_{m,n+1} - C_{m,n} = C_{m-1,n} \ .$$

*and*

$$C_{m,0} = 0 \ .$$

**Exercise 9.** *Explain why $C_{m,n} = C_{n-m,n}$. Find $C_{5,10}$.*

**Exercise 10.**

$$C_{m,n} = \frac{n!}{m!(n-m)!} \ .$$

*where $n! = 1 \cdot 2 \cdot 3 \cdots n$ is called the **factorial** of $n$*

**Exercise 11.** *Let $(a+b)^n = a^n + \cdots + b^n$. Let $P_{m,n}$ be coefficient of $a^m b^{n-m}$. Verify same recursion formula. Binomial theorem.*

$$(a+b)^N = a^N + Na^{N-1}b + \frac{N(N-1)}{2}a^{N-2}b^2 + \cdots + b^N$$

**Exercise 12.** *Coefficient of $x^5 y^5$ in $(x+y)^{10}$.*

**Exercise 13.** *Flip a red coin and a green coin. There are four possible outcomes. $HH, HT, TH, TT$. Just like $(H+T)^2$. Flip three coins, how many $3H, 2H1T$, etc.*

**Exercise 14.** *We know that $\Sigma C_m = C_{m+1}$. Express $n^2$ in terms of $C_1$ and $C_2$. Find $\Sigma n^2$. Same for $n^3$, in terms of $C_1, C_2, C_3$. Note that $\Sigma$ and $\Delta$ are linear.*

**Polynomial sequences**

A **polynomial sequence** is one where $a_n = P(n)$, $P$ a polynomial. If $a$ is a polynomial sequence of degree $m$, then $\Delta a$ is one of degree $m-1$.

Because $(n+1)^m - n^m = \ldots$ by the binomial theorem.

Thus $\Delta^k a$ has degree $m-k$, $\Delta^m P$ is constant, $\Delta^k a = 0$ for $k > m$.

Example. Suppose
$$a_n = n^3 - n + 1 .$$

Then:

| $n$: | 0 | 1 | 2 | 3 | 4... |
|---|---|---|---|---|---|
| $a$: | 1 | 1 | 7 | 25 | 61... |
| $\Delta\, a$: | 0 | 6 | 18 | 36 | ... |
| $\Delta^2 a$: | 6 | 12 | 18 | ... | |
| $\Delta^3 a$: | 6 | 6 | 6 | ... | |
| $\Delta^4 a$: | 0 | 0 | ... | | |

where we write $\Delta^2$ for the difference of the difference of $a$. Etc.

The converse is true: If $\Delta^k a = 0$ then $a_n = P(n)$ for some polynomial of degree less than $k$.

Write down the formula explicitly:

$$f(n) = f(0) + n\Delta f(0) + \frac{n(n-1)}{2}\Delta^2 f(0) + \cdots$$

**Exercise 15.** *Do some. Check by applying to $n^{(m)}$.*

**Exercise 16.** *Sample $y = x^2$, picture comb, find $\Delta f/\Delta x$. Take a limit?*