Mathematics 102 — Fall 1999

Summing arithmetic progressions

Summing the odd numbers

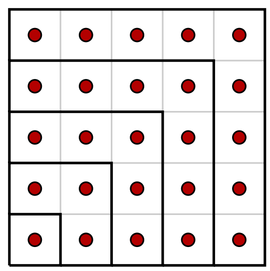
If we write down the first several odd numbers

$$1,\ 3,\ 5,\ 7,\ \ldots$$

and then their sums

$$1,\ 4,\ 9,\ 16,\ \ldots$$

we see that these are apparently the squares of all the integers. This is explained by the fact that we can partition a square of side n into n L-shaped regions of sizes $1, 3, \ldots, 2n-1$:

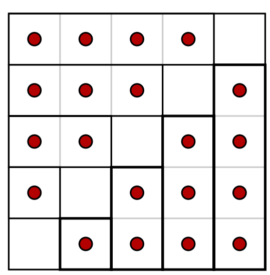


Summing all the integers

With a few modifications, the same figure can tell us how to find the sum of the first few integers

$$1 + 2 + 3 + \cdots + n\ .$$

Here is the new figure:



Here we have an $n \times n$ square containing two copies of the sum $1 + \cdots + n - 1$ as well as the n blocks along the diagonal. Therefore

$$\begin{aligned} 2(1 + 2 + \cdots + n - 1) + n &= n^2 \\ 1 + 2 + \cdots + n - 1 &= \frac{n^2 - n}{2} \\ &= \frac{n(n-1)}{2}\ . \end{aligned}$$

Changing $n - 1$ to n this gives

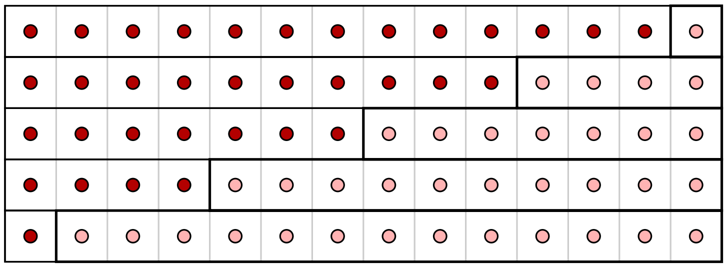
$$1 + \cdots + n = \frac{n(n+1)}{2}\ .$$

Summing an arbitrary arithmetic progression

Given any integer constants a and b we can construct the arithmetic progression

$$b,\ a+b,\ 2a+b,\ 3a+b,\ \ldots$$

with initial term b, increment a, and n-th term $b+(n-1)a$. In the following figure $b=1$, $a=3$, and $n=5$.



From that figure we can see that the sum of the first n terms of this sequence is one-half the area of the rectangle. If the height is n then the width is

$$\big(b+(n-1)a\big)+b=(n-1)a+2b$$

and the area equal to

$$n(n-1)a+2nb\ .$$

Therefore

$$b+\big(b+a\big)+\cdots\big(b+(n-1)a\big)=nb+\frac{n(n-1)a}{2}\ .$$

There is another way to see this argument without the picture. Let's see how it works in a simple example. Suppose we want to find the sum

$$1+2+3+\cdots+99+100\ .$$

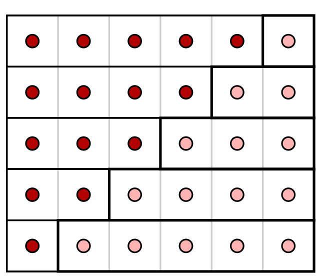
What we do is line up the sum under itselfand add the columns:

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \ldots & + & 3 & + & 2 & + & 1 \\ 101 & + & 101 & + & 101 & + & \ldots & + & 101 & + & 101 & + & 101 \end{array}$$

We have 100 terms altogether, so we see that the sum plus itself is equal to $100\cdot 101$.

$$2s=100\cdot 101,\quad 2s=10100,\quad s=5050\ .$$

Here is a picture that goes with this particular example:



Exercise 1. *Find*

$$1+4+7+10+13+\cdots+2002\ .$$

in this way. (Note that in order to apply the method, you have to decide how many terms there are in this sum.)

Now some algebra

You may have already realized that the second formula for $1 + \cdots + n - 1$ actually implies the last, since

$$b + \big(b + a\big) + \cdots \big(b + (n-1)a\big) = nb + (0 + 1 + \cdots + n - 1)a \ .$$

Exercise 2. *Find the sum*

$$2 + 5 + 8 + \cdots + 1001$$