A theorem on semi-simple \( p \)-adic groups

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References

Introduction

Let \( k \) be a non-archimedean local field and \( G \) a connected, semi-simple algebraic group defined over \( k \). If \( G = G(k) \) denotes the group of \( k \)-rational points of \( G \), then \( G \), with its natural topology, is locally compact. Let \( G' \) and \( V \) denote respectively the sets of regular and unipotent elements of \( G \), and let \( C_c^\infty(G) \) denote the space of locally constant, complex valued functions on \( G \) having compact support. For \( x \in G' \cup V \), let \( G(x) \) denote the conjugacy class of \( G \) containing \( x \). \( G(x) \) carries an essentially unique \( G \)-invariant measure \( \mu \) (§1.2). Fix a normalization of \( \mu \). For \( f \in C_c^\infty(G) \) and \( x \) as above, let

\[
I_f(x) = \int_{G(x)} f \, d\mu.
\]

It is known [5(c)] that the integral converges for \( x \) regular. It has been conjectured by Harish-Chandra that the integral also converges for \( x \) unipotent. We verify this by direct computation for \( G = SL_2 \) (§1.2). The general results described below depend on this conjecture and we assume it throughout.\(^1\)

The main purpose of this paper is the study of the function \( I_f \). We consider the following problem.

For \( x \in G' \), determine the behavior of \( I_f(x) \) as \( x \) approaches the singular set (that is, the complement of \( G' \) in \( G \)). We show in the present paper that \( I_f \) has, in the precise sense defined below, an asymptotic expansion in terms of the integrals

\[
\Lambda_C(f) = \int_C f \, d\mu,
\]

\(^1\) Since the preparation of this manuscript, the author has been informed that this conjecture has been proved in characteristic zero, independently by Deligne and Ranga Rao.
of $f$ over the unipotent conjugacy classes $\mathcal{O}$. (For $\mathcal{O} = \{1\}$, the trivial class, we may assume that $\Lambda_\mathcal{O}(f) = f(1)$.) The coefficients $\Gamma_\mathcal{O}(x)$ ($x \in G'$) in this expansion are independent of $I$. The functions $\Gamma_\mathcal{O}$ are determined explicitly only in the case $G = SL_2$. We believe that the explicit determination of these functions in the general case is desirable for the following reasons.

(a) Firstly, one expects a formula for $f(1)$ obtained from the values of $L_f$ on $G'$. We present some heuristic evidence (§2.2), involving the Steinberg representation of $G$, to the effect that, for $\mathcal{O} = \{1\}$ and $x$ elliptic, $\Gamma_\mathcal{O}(x)$ may be expressed in a simple fashion in terms of invariants of the affine Weyl group, $W_\mathcal{O}$, of $G$. The importance of such a formula as a step in proving the Plancherel formula for real groups is well known. It was suggested by Harish-Chandra that a similar formula could play an important role in the $p$-adic case as well.

(b) Secondly, by taking $f$ to be a matrix coefficient of a representation $\Pi$ of $G$ of compact support, one expects to obtain an explicit formula for the character of $\Pi$ near the identity on each compact torus. The leading term in this formula should be expressed in a simple fashion in terms of the formal degree of $\Pi$ and invariants of $W_\mathcal{O}$. This, of course, agrees with the explicit form of such characters when $G = SL_2$ [12(a)].

We outline how explicit expressions for the functions $\Gamma_\mathcal{O}$ lead to a proof of the Plancherel formula for $G = SL_2$, (assuming $k$ of odd characteristic). Let $G$ denote the set of equivalence classes of irreducible, unitary representations of $G$. For $\Pi \in \hat{G}$ acting on a Hilbert space $\mathcal{H}$, let $\Pi(f)$ denote the operator on $\mathcal{H}$ associated to $f \in C_c(G)$. It is known [5(c)], [6], [12(a)], that $\Pi(f)$ is of trace class. Let $\hat{\Pi}(f) = \text{trace } \Pi(f)$. $\hat{f}$ is called the Fourier transform of $f$. For $\Pi \in \hat{G}$, let $h = h(\Pi)$ denote the conductor of $\Pi$ [12(a)]. Let $d\Pi$ denote the Plancherel measure on $\hat{G}$. Put

$$\Lambda_h(f) = \int_{\hat{\mathcal{O}}_h} \hat{f} d\Pi,$$

where $\hat{\mathcal{O}}_h$ denotes the set of $\Pi \in \hat{G}$ having conductor less than or equal to $h$. A simple formula for $\Lambda_h$, which is a locally integrable function on $G$, is given in [12(b)]. By using Weyl's integration formula (§1.1), it is easily seen that the formula

$$\lim_{h \to \infty} \Lambda_h(f) = f(1)$$

is equivalent to the formula

$$\sum_{I \mathcal{O}}, \int_{I} |D(t)| \Gamma_\mathcal{O}(t) \Lambda_h(t) dt = 0$$

for $\mathcal{O}$ unipotent of positive dimension and $h$ sufficiently large, together with
the knowledge of $\Gamma_0$ for $\emptyset = \{1\}$. In this last formula, the sum is extended over the equivalence classes of maximal tori in $G$, and $D(t)$ is the well-known discriminant. For the case of general $G$, one may have to guess the formula for $\Lambda_\alpha$. The above identity (*) then gives a necessary condition that such functions converge to the delta distribution.

The results of this paper will be applied elsewhere to a detailed study of the Fourier transform on $SL_2$ over a non-archimedean local field [13].

The author would like to take this opportunity to express his sincere appreciation to Harish-Chandra for his encouragement and suggestions over the last several years. Many of the results described here grew out of conversations concerning his work on invariant distributions in both the real and $p$-adic cases. I refer especially to his lectures on $p$-adic groups [5(c)].

Let $G$ be a connected semi-simple algebraic group defined over a non-archimedean local field $k$. Let $V$ denote the set of $k$-rational unipotent elements of $G$.

Throughout the paper we will make various assumptions concerning the set $V$. They have been verified, by several authors, for $\text{char}(k) = 0$. The assumptions will be stated in this paragraph and referred to when appropriate. We will also discuss under what conditions they are known to be true. The assumptions are as follows:

(U1) For all $x_0 \in V$, the morphism

$$\varphi(x_0): G \rightarrow G$$

defined by $\varphi(x_0)g = gx_0g^{-1}$ ($g \in G$) is separable.

This is obvious for $\text{char}(k) = 0$. It is false for $SL_2$ in characteristic two. (U1) will be assumed throughout the paper.

Let $G = G(k)$ denote the group of $k$-rational points of $G$. $G$ has the natural structure of a locally compact group. For $x_0 \in V$, let $Z_G(x_0)$ denote the centralizer of $x_0$ in $G$. $Z_G(x_0)$ is a closed subgroup of $G$.

(U2) $Z_G(x_0)$ is unimodular.

For $\text{char}(k) = 0$, this has been observed by Harish-Chandra. For a proof and a detailed discussion of when the second assumption has been verified in general, we refer to the article of Springer-Steinberg in [3]. For further discussion see also [11], [16].

For $Z_G(x_0)$ unimodular, the homogeneous space $G/Z_G(x_0)$ carries an invariant measure. Denote this measure by $dx^*$. Let $x^*$ denote the canonical image of an element $x$ of $G$ in $G/Z_G(x_0)$.

(U3) The integral
\[
\int_{\gamma(x_0 \in \gamma)} f(x^* x_0 (x^{-1})^*) dx^* \quad (x_0 \in V)
\]

converges for all \( f \in \mathcal{C}_c^\omega(G) \).

As stated in the Introduction, the third assertion has been proved in characteristic zero, independently by Deligne and Ranga Rao [10]. (U3) is readily verified for \( G = SL_n \) in all characteristics (see Prop. 1.2.2 below).

Finally, we assume

(U4) The number of unipotent conjugacy classes in \( G \) is finite.

It follows directly from Kostant [7] and Borel-Serre [2], that this assumption is true in characteristic zero. It is false for \( SL_n \) in characteristic two.

1. Invariant distributions

1.1. Invariant distributions on \( G \).

In this section, \( G = G(k) \), the group of \( k \)-rational points of a connected semi-simple algebraic group \( G \) defined over a non-archimedean local field \( k \). For \( O \) an open subset of \( G \), denote by \( \mathcal{C}_c^\omega(O) \) the vector space of locally constant functions defined on \( O \) which have compact support. \( \mathcal{C}_c^\omega(O) \) may be regarded in a natural way as a subspace of \( \mathcal{C}_c^\omega(G) \). If \( \Lambda \) is a distribution on \( G \), we say that an element \( x \) of \( G \) is in the non-support of \( \Lambda \) if there exists a neighborhood \( O \) of \( x \) such that the restriction of \( \Lambda \) to \( \mathcal{C}_c^\omega(O) \) is zero. The support of \( \Lambda \), denoted by \( \text{Supp} \, \Lambda \), is then the complement in \( G \) of the non-support.

Let \( V \) denote the set of unipotent elements in \( G \) and \( G' \) the set of regular elements of \( G \). Let \( \Lambda \) be an invariant distribution on \( G \), that is, a linear functional on \( \mathcal{C}_c^\omega(G) \) invariant by the action of \( G \) induced by inner automorphisms. In the following we assume that \( \text{Supp} \, \Lambda \) is contained in the closed set \( V \) or in the open set \( G' \) and "determine" all such \( \Lambda \). For this, we need some general results of Harish-Chandra [5(a)] from the theory of distributions on real manifolds whose analogues are true for all local fields [5(c)].

For any manifold \( M \) over a non-archimedean local field, let \( \mathcal{C}_c^\omega(M) \) denote the vector space of locally constant functions on \( M \) which have compact support. (For the standard terminology concerning manifolds over local fields see [14].) Suppose that \( M_1 \) and \( M_2 \) are analytic manifolds over \( k \) of dimension \( m_1 \) and \( m_2 \) respectively. Let \( \omega_1 \) and \( \omega_2 \) be analytic differential forms on \( M_1 \) and \( M_2 \) of degrees \( m_1 \) and \( m_2 \), and \( |\omega_1| \) and \( |\omega_2| \) the corresponding positive Borel measures [17]. Let \( \pi \) be a surjective, analytic submersion from \( M_1 \) to \( M_2 \).

**Theorem.** (Harish-Chandra) Suppose that \( \omega_2 \) is nowhere zero. Then, for every \( \alpha \in \mathcal{C}_c^\omega(M_1) \), there exists a unique function \( f_\alpha \in \mathcal{C}_c^\omega(M_2) \) such that

\[(1.1.1) \quad \int_M (F \circ \pi) \alpha |\omega_1| = \int_M F f_\alpha |\omega_2| \]
for all $F \in C_c^\infty(M)$. Moreover

(1) $\text{Supp } f_\alpha \subset \pi(\text{Supp } \alpha)$,

(2) if $\omega_1$ is nowhere zero, $\alpha \mapsto f_\alpha$ is surjective,

(3) if $F$ is a Borel measurable function on $M$, then $F$ is locally integrable (with respect to $|\omega_2|$) if and only if $F \circ \pi$ is locally integrable (with respect to $|\omega_1|$) and (1.1.1) continues to hold in this case.

We apply the above result in two cases.

First application. Let $G$ be a connected semi-simple algebraic group defined over $k$ and let $\mathcal{L} = \text{Lie } (G)$. Let $x_0$ be a $k$-rational unipotent element of $G$. Let the morphism $\varphi(x_0) : G \to G$ be defined by $\varphi(x_0)(g) = gx_0g^{-1}$ ($g \in G$).

The kernel, $Z(x_0)$, of the tangent map $d\varphi(x_0)$ at any point of $G$ is seen to be the set of $D \in \mathcal{L}$ satisfying $\text{Ad}(x_0) \cdot D = D$. Let $Z_\alpha(x_0)$ denote the centralizer of $x_0$ in $G$. Then (assuming (U1)) one has [1]

\begin{equation}
(1.1.2) \quad \text{Lie } Z_\alpha(x_0) = Z(x_0).
\end{equation}

Let $\varphi(x_0)$ also denote the induced map of $G = G(k)$ into itself, and let $Z_\alpha(x_0) = Z_\alpha(x_0)(k)$ denote the centralizer of $x_0$ in $G$. Then $Z_\alpha(x_0)$ is a closed subgroup (submanifold) of $G$, and the quotient $G/Z_\alpha(x_0)$ is an analytic manifold. It follows from (1.1.2) that the induced map of $G/Z_\alpha(x_0)$ into $G$ defined by $\varphi(x_0)$ is immersive. Let $G(x_0)$, the conjugacy class of $x_0$ in $G$, denote the image of this map. Then $G(x_0)$ is a submanifold of $G$ and is, in particular, locally closed. Let $\mathcal{G} = \text{Lie } (G)$. The tangent space at any point of $G(x_0)$ may be identified with $\mathcal{G}/\text{Lie } Z_\alpha(x_0)$.

Let $J$ be a coordinate neighborhood of $x_0$ in $G$, and let $x_1, \ldots, x_h$ be the coordinate functions. Suppose that $G(x_0)$ is defined in $J$ by the equations $x_1 = x_2 = \cdots = x_r = 0$. Choose $U$ a local analytic submanifold of $G$ containing the identity so that $x_0U$ is defined by $x_{r+1} = \cdots = x_h = 0$. Consider the map $\pi$ of $G \times U$ into $G$ defined by $\pi(x, u) = x(x_0u)x^{-1}$ ($x \in G, u \in U$). We may identify the tangent space at any point of $U$ with a subspace $\mathcal{U}$ of $\mathcal{G}$. With these identifications, we find that $d$, the jacobian of $\pi$, evaluated at $(x, u)$ in $G \times U$ is given by

\[d(D, \omega) = \text{Ad}(x)[(\text{Ad}(x_0u)^{-1} - 1)D + \omega], \quad (D \in \mathcal{G}, \omega \in \mathcal{U}).\]

As in [5(a)], we have

Proposition 1.1.3. There exists a local submanifold $U'$ of $U$ so that $d$ is surjective for $x \in G, u \in U'$.

Let $M_1 = G \times U'$ and let $\pi$ also denote the restriction of $\pi$ to $M_1$. Let $\omega$ be an (analytic) form on $G$ so that $|\omega|$ is a Haar measure $dx$ on $G$. We set
Put \( du = dx_1 \wedge \cdots \wedge dx_n \), and \( \omega_\nu = \omega \times du \). If \( M_\nu = \Omega \) denotes the image of \( \pi \), then, by Proposition 1.1.3, \( M_\nu \) is an open submanifold of \( G \). We let \( \omega_\nu \) denote the restriction of \( \omega \) to \( M_\nu \).

**Second application.** Let \( T \) be a maximal \( k \)-torus in \( G \), and let \( T = T(k) \). Put \( T' = T \cap G' \). For \( t \in T \), let \( D(t) \) denote the determinant of \( 1 - \text{Ad}(t) \) acting on \( G/\text{Lie}(T) \). \( T' \) is defined by the equation \( D(t) \neq 0 \), and hence is an open submanifold of \( T \). Let \( N(T) \) denote the normalizer of \( T \) in \( G \), and let \( W_T \) denote the finite group \( N(T)/T \). Let \( M_\nu = G/T \times T' \). For \( x \in G \), let \( x^* \) denote the image of \( x \) under the canonical map from \( G \) to \( G/T \).

Now define \( \pi \) from \( M_\nu \) to \( G \) by setting \( \pi(x^*, t) = xtx^{-1} \). \( \pi \) is well-defined. Choose \( \omega_\nu^* \) and \( \omega_\tau \), differential forms on \( G/T \) and \( T' \) respectively, so that \( |\omega_\nu^*| = dx^* \) is an invariant measure on \( G/T \), and \( |\omega_\tau| \) is the restriction to \( T' \) of a Haar measure \( dt \) on \( T \). We may assume that \( dx = dx^*dt \). In this case, it is known and easily verified that the jacobian of \( \pi \) at \((x^*, t)\) is \( |D(t)| \). Thus \( \pi \) is submersive. The image, \( M_\nu \), of \( M \) under \( \pi \), which we also denote by \((T')^\nu \), is thus an open submanifold of \( G \). Let \( \omega_\nu = \omega_\nu^* \times \omega_\tau \) and let \( \omega_\nu \) be the restriction of \( \omega \) to \( M_\nu \). With these definitions, as in [5(a), 5(c)], we have the following corollary to the above theorem.

**Corollary 1.1.4.** (1) For all \( \alpha \in C^\infty_c(G \times U') \), there exists a unique function \( f_\alpha \in C^\infty_c(\Omega) \) such that

\[
\int_{G \times U'} F(x(x_0u)x^{-1}) \alpha(x, u) dx du = \int_{U'} F(x) f_\alpha(x) dx .
\]

The map \( \alpha \mapsto f_\alpha \) of \( C^\infty_c(G \times U') \) into \( C^\infty_c(\Omega) \) is surjective and \( \text{Supp} \ f_\alpha \subset \pi(\text{Supp} \ \alpha) \). Moreover, if \( F \) is locally integrable on \( \Omega \), then the map \( (x, u) \mapsto F(x(x_0u)x^{-1}) \), \((x \in G, u \in U')\), defines a locally integrable function on \( G \times U' \) and (1.1.5) holds in this case.

(2) For all \( \alpha \in C^\infty_c(G/T \times T') \), there exists a unique function \( f_\alpha \in C^\infty_c((T')^\nu) \) such that

\[
\int_{G/T \times T'} F(x^*x(t^{-1})^*) \alpha(x^*, t) dx^* dt = \int_{(T')^\nu} F(x) f_\alpha(x) dx .
\]

The map \( \alpha \mapsto f_\alpha \) of \( C^\infty_c(G/T \times T') \) into \( C^\infty_c((T')^\nu) \) is surjective and \( \text{Supp} \ f_\alpha \subset \pi(\text{Supp} \ \alpha) \). Moreover, if \( F \) is locally integrable on \( (T')^\nu \), then the map \( (x^*, t) \mapsto F(xtx^{-1}) \), \((x \in G, t \in T')\), defines a locally integrable function on \( G/T \times T' \) and (1.1.6) holds in this case.

As in [5(a)], one deduces the following.

**Theorem 1.1.7.** (1) Let \( \Lambda \) be an invariant distribution on \( \Omega \). Then there exists a unique distribution \( \sigma_\Lambda \) on \( U' \) such that
\[ \Lambda(f_a) = \sigma_\lambda(\beta_a), \]

where \( \beta_a \in C^*_c(U') \) is defined by \( \beta_a(u) = \int_\mathcal{O} \alpha(x, u) dx, \ (u \in U') \).

(2) Let \( \Lambda \) be an invariant distribution on \((T')^0\). Then there exists a unique distribution \( \sigma_\lambda \) on \( T' \) such that

\[ \Lambda(f_a) = \sigma_\lambda(\beta_a), \]

where \( \beta_a \in C^*_c(T') \) is defined by \( \beta_a(t) = \int_{\mathcal{O} / T} \alpha(x^*, t) dx^*, \ (t \in T') \).

In both (1) and (2) above, \( \sigma_\lambda = 0 \) implies that \( \Lambda = 0 \).

The expression for \( \beta_a \) may be made more explicit by using Weyl’s Lemma. Suppose that \( dx, dx^* \), and \( dt \) are normalized so that \( dx = dx^* dt \). Then, for any integrable function \( f \) on \((T')^0\),

\[ \int_{(T')^0} f(x) dx = [W_T : 1]^{-1} \int_T |D(t)| \int_{\mathcal{O} / T} f(x^* t(x^{-i})^*) dx^* dt. \]

The proof follows from the explicit form of the jacobian of \( \pi \) in this case.

Using Weyl’s Lemma and taking \( F \) to be a locally integrable class function in part (2) of Corollary 1.1.4, we see immediately that, for \( \alpha \in C^*_c(G/T \times T') \)

\[ \beta_a(t) = [W_T : 1]^{-1} |D(t)| \int_{\mathcal{O} / T} f_a(x^* t(x^{-i})^*) dx^*, \ (t \in T') \]

For any \( f \in C^*_c(G) \) and any maximal torus \( T \) in \( G \), we set

(1.1.8) \[ F_f^w(t) = |D(t)|^{1/2} \int_{\mathcal{O} / T} f(x^* t(x^{-i})^*) dx^*, \ (t \in T'), \]

where \( dx^* \) is the measure appearing in Weyl’s Lemma.

Throughout the remainder of the paper, we assume that the measures \( dx \), \( dx^* \), and \( dt \) are normalized as in Weyl’s Lemma.

1.2. MEASURES ON CONJUGACY CLASSES

In this paragraph, we associate to each regular or unipotent conjugacy class in \( G \) (see assumptions below) an essentially unique distribution on \( G \). We also determine all invariant distributions on \( G \) with support in the set \( V \) of unipotent elements.

Let \( x_0 \) be a regular or unipotent element in \( G \). As in § 1.1, let \( G(x_0) \) denote the conjugacy class in \( G \) containing \( x_0 \), and let \( Z_\alpha(x_0) \) denote the centralizer of \( x_0 \) in \( G \).

\( G(x_0) \) is locally closed (§ 1.1) and is therefore locally compact. Hence, by Arens’ Theorem [9], \( G/Z_\alpha(x_0) \) and \( G(x_0) \) are homeomorphic. Let \( \mu \) be the Borel measure on \( G(x_0) \) obtained by transporting an invariant measure \( dx^* \) on \( G/Z_\alpha(x_0) \). We quote a result of Harish-Chandra [7(c)] which states that, for
$x_0 \in G'$, the integral

$$\int_{G(x_0)} f d\mu = \int_{GL_G(x_0)} f(x^*x_0(x^{-1})^*) dx^*$$

(1.2.1)

converges for all $f \in C_\infty^G(G)$.

For later purposes, we prove the following.

**PROPOSITION 1.2.2.** Let $G = SL_2$. For $x_0 \in V$, the integral

$$\int_{G(x_0)} f d\mu$$

converges for all $f \in C_\infty^G(G)$.

**Proof.** For $x_0 = 1$, the proposition is trivial. We may therefore assume by conjugation that $x_0$ is of the form $\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$ where $\zeta \in k^\times$. Let $r$ denote the ring of integers in $k$ and put $K = SL_2(r)$. Let $A$ denote the group of diagonal matrices in $G$ and $N$ the subgroup of elements of the form $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ for $\xi \in k$. Let $dk$, $da$, and $dn$ denote Haar measures on $K$, $A$, and $N$ respectively. It is well known that $G = KAN$, and, for $f$ integrable on $G$,

$$\int_G f(x) dx = \int_{K \times A \times N} f(\kappa a n) |\lambda|^2 d\kappa da dn,$$

provided $\alpha$ is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in k^\times$, and $dx$ is some Haar measure on $G$. It follows readily that, if $\alpha$ is a function on $G$ satisfying $\alpha(xn) = \alpha(x)$, $x \in G$, $n \in N$, and $\alpha$ is integrable on $G/N$, then

$$\int_{G/N} \alpha(x^*) dx^* = \int_{K \times A} \alpha(\kappa a) |\lambda|^2 d\kappa da .$$

(1.2.3)

Now, for $f \in C_\infty^G(G)$, define $\alpha$ by $\alpha(x) = f(x x_0 x^{-1})$, $x \in G$. Using (1.2.3) and making a change of variables, we obtain

$$\int_{G(x_0)} f d\mu = \int_K \tilde{f}(x) dx d\lambda,$$

where $\tilde{f}(x) = \int_K f(\kappa x x^{-1}) d\kappa$, and $d\lambda$ is a suitable Haar measure on $k$. The proposition is now clear.

For any unipotent or regular conjugacy class $\mathcal{O}$, fix $x_0 \in \mathcal{O}$ and $\mu$ an invariant measure on $\mathcal{O}$ as above. Assuming (U2) and (U3) we define, for $f \in C_\infty^G(G)$,

$$\Lambda_\mathcal{O}(f) = \int_{GL_G(x_0)} f(x^*x_0(x^{-1})^*) dx^* .$$

$\Lambda_\mathcal{O}$ is an invariant distribution on $G$ with support in the closure $\mathcal{O}'$ of $\mathcal{O}$. Assume (U1) and (U4). We then have the following result.

**PROPOSITION 1.2.4.** Let $\mathcal{O}_0$ be a unipotent conjugacy class. If $\Lambda$ is an invariant distribution on $G$ with support in the closure $\mathcal{O}_0'$ of $\mathcal{O}_0$, then, for suitable constants $C_\mathcal{O}$,

$$\Lambda = \sum_\mathcal{O} C_\mathcal{O} \Lambda_\mathcal{O}$$
where the summation is extended over all conjugacy classes \( \mathcal{O} \) contained in \( \mathcal{O}_0 \).

Let \( \mathcal{O}_0 \) be a regular class. If \( \Lambda \) is an invariant distribution on \( G \) with support in \( \mathcal{O}_0 \), then

\[
\Lambda = c \Lambda_{\mathcal{O}_0}
\]

for a suitable constant \( c \).

The proof of the proposition is contained in the following lemmas.

**Lemma 1.2.5.** (Localization) Let \( M \) be a manifold over \( k \), and let \( f \in C^\infty_c(M) \). Suppose that \( \{ V_i \}_{i \in I} \) is an open covering of \( \text{Supp} f \). Then there exist functions \( f_i \in C^\infty_c(M) \) with \( \text{Supp} f_i \subset V_i \) such that

1. \( f_i = 0 \) for all but finitely many \( i \in I \),
2. \( f = \sum_{i \in I} f_i \).

**Proof.** We may assume without loss of generality that \( f \) is the characteristic function of a compact open set \( S \), that \( I \) is finite, and that each \( V_i \) is contained in \( S \). \( S \) is a compact manifold and, hence, is the disjoint union of a finite number of balls. We may therefore assume that \( S \) is a ball in Euclidean space.

Suppose that \( S = V_1 \cup V_2 \) where \( V_1 \) and \( V_2 \) are open. Take \( x \in S \). If \( x \in V_1 \), let \( B(x) \) be a ball containing \( x \) and contained in \( V_1 \). If \( x \in V_2 \), let \( B(x) \) be a ball containing \( x \) and contained in \( V_2 \). We have \( \bigcup_{x \in S} B(x) = S \). Select a finite subcover \( B_1, \ldots, B_s \). We may assume that \( B_i \cap B_j = \emptyset \) for \( i \neq j \). Thus \( S = \bigcup_{i=1}^s B_i \) (disjoint union). Suppose that \( B_1, \ldots, B_r \) are contained in \( V_1 \) and \( B_{r+1}, \ldots, B_s \) are not. Let \( \alpha_i \) denote the characteristic function of \( B_i \), \( 1 \leq i \leq s \). Let \( f_1 = f \sum_{i=1}^s \alpha_i \), and \( f_2 = f \sum_{i=r+1}^s \alpha_i \). Then \( f_1 \) and \( f_2 \) are in \( C^\infty_c(S) \), \( f = f_1 + f_2 \), and \( f_1 \) has support in \( V_1 \) \((i = 1, 2)\). This proves the lemma in this case. The general case follows by a simple induction.

Assume \((U1)\) and \((U4)\). Then we have

**Lemma 1.2.6.** Let \( \mathcal{O}_0 \) be a unipotent conjugacy class. Let \( \mathcal{O}_0' \) denote the closure of \( \mathcal{O}_0 \). If \( \mathcal{O} \) is a conjugacy class in \( G \) contained \( \mathcal{O}_0' \setminus \mathcal{O}_0 \), then \( \dim_k \mathcal{O} < \dim_k \mathcal{O}_0 \).

**Proof.** Fix \( x_0 \in \mathcal{O}_0 \). Let \( G(x_0) \) denote the image of \( x_0 \) under \( \varphi(x_0) \), that is, the \( G \) conjugacy class containing \( x_0 \). Let \( \text{CL} G(x_0) \) denote the Zariski-closure of \( G(x_0) \) in \( G \). Then \( G(x_0) \) is a smooth variety which is open in \( \text{CL} G(x_0) \) [1]. Moreover, the algebraic dimension of any \( G \) orbit in \( \text{CL} G(x_0) \setminus G(x_0) \) is smaller than that of \( G(x_0) \).

Decompose \( G(x_0)(k) \) into \( G \) orbits:

\[
G(x_0)(k) = G(x_0) \cup G(x_1) \cup \cdots \cup G(x_r),
\]

\(^2\) The author is indebted to W. Casselman for the proof of this lemma. See also [5(c)].
where \( \mathcal{O}_0 = G(x_0) \). By (U4), the (disjoint) union is finite. From the fact that \( \varphi(x_0) \) is submersive, it follows that the induced map \( G(k) \hookrightarrow G(x_0)(k) \) is submersive and thus the image \( G(x_0) \) is open in \( G(x_0)(k) \). Similarly, each \( G(x_i), 1 \leq i \leq r, \) is open and hence closed in \( G(x_0)(k) \). It follows that \( \mathcal{O} \subset \mathcal{O}' \setminus \mathcal{O} \subset \text{CL} G(x_0) \setminus G(x_0) \). Thus \( \text{dim}_k \mathcal{O} < \text{dim} G(x_0) = \text{dim} G(x_0) \). This proves the lemma.

Now let \( V_\ast = \bigcup_{\text{dim} \mathcal{O} \leq s} \mathcal{O} \), where \( \mathcal{O} \) is a unipotent class. By Lemma 1.2.6, \( V_\ast \) is closed and each class of dimension \( s \) is open in \( V_\ast \). Fix \( x_0 \in V_\ast \). Since \( G(x_0) \) is open in its closure \( G(x_0)' \), we can choose \( \Omega_0 \) open in \( G \), invariant by inner automorphisms, so that \( \Omega_0 \cap G(x_0)' = G(x_0) \). Then we can choose \( U_0 \) open in \( U \) containing 1 so that \( \pi(G \times U_0) = \Omega_0 \). With this notation, we have (assuming (U1) and (U4))

**Lemma 1.2.7.** Suppose that \( \Lambda \) is an invariant distribution on \( \Omega_0 \) with support in \( \mathcal{O} = G(x_0) \). Then

\[
\Lambda = c \Lambda_0
\]

for a suitable constant \( c \).

**Proof.** Let \( \sigma_\Lambda \) be the distribution on \( U_0 \) associated to \( \Lambda \) by Theorem 1.1.7. Clearly, \( xx_0 x^{-1} \in x_0 U, \ x \in G \), implies that \( xx_0 x^{-1} = x_0 \). Proceeding as in Lemma 23 of [5(a)], we have \( \text{Supp} \Lambda \subset \{1\} \). Thus, \( \sigma_\Lambda = c \Delta_1 \), where \( \Delta_1 \) is the delta function at 1 in \( U_0 \). Since \( \Lambda_0 \) clearly satisfies the assumptions of the lemma, we are through.

We now prove Proposition 1.2.4 for unipotent classes \( \mathcal{O}_0 \). Suppose \( \text{dim} \mathcal{O}_0 = s \). Let \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N \) be all the unipotent classes of dimension \( s \). Then, by the above, each \( \mathcal{O}_i, \ 1 \leq i \leq N \), is open in \( V_\ast \). As above, choose \( \Omega_1, \ldots, \Omega_N \) open in \( G \), invariant by inner automorphisms, of the form \( \pi(G \times U_0) \), and such that \( \Omega_j \cap V_\ast = \mathcal{O}_j, \ 1 \leq j \leq N \). Let \( \Lambda_j \) denote the restriction of \( \Lambda \) to \( \Omega_j \). Then, by Lemma 1.2.7,

\[
\Lambda_j = c_j \Lambda_{\mathcal{O}_j}, \quad 1 \leq j \leq N
\]

for suitable constants \( c_j \). Fix \( f \in C_c^\infty(G) \) so that the restriction of \( f \) to \( V_\ast \setminus (\bigcup_{j=1}^N \mathcal{O}_j) \) is zero. Then \( \text{Supp} f \subset \bigcup_{j=1}^N \Omega_j \cup (G \setminus V_\ast) \). Therefore, by Lemma 1.2.5, we may write

\[
f = \sum_{j=1}^N f_j + g,
\]

where \( \text{Supp} f_j \subset \Omega_j, \ 1 \leq j \leq N \), and \( \text{Supp} g \subset G \setminus V_\ast \). In this case, since \( f - f_j \) vanishes on the closure of \( \mathcal{O}_j \),

\[
\Lambda(f) = \sum_{j=1}^N \Lambda(f_j) = \sum_{j=1}^N c_j \Lambda_{\mathcal{O}_j}(f_j) = \sum_{j=1}^N c_j \Lambda_{\mathcal{O}_j}(f).
\]

Thus, \( \Lambda = \sum_{j=1}^N c_j \Lambda_{\mathcal{O}_j} \) has support in \( V_\ast \setminus (\bigcup_{j=1}^N \mathcal{O}_j) = V_{\ast -} \). The proposition
follows easily by induction.

2. Asymptotic expansion of $F_f$

In this section we obtain a limiting relation between distributions supported on $G'$ and those supported on $V$. In the case when $G = SL_r$ and $k$ has odd residual characteristic this result will be applied elsewhere to derive the Fourier transform of a distribution with support in $V$ from that of a distribution with support in $G'$. In particular, one finds an expression for $f(1)$ from the values of $I_f$ on $G'$ [13]. This is similar in spirit to the result of Harish-Chandra for real semi-simple Lie groups.

2.1. Germs of Functions

In order to state the main result of this paper, we must first define the germ of a function near 1 on a maximal torus in $G$. As above, let $G$ be a semi-simple group defined over $k$. If $\rho$ is a fixed $k$-rational faithful linear representation of $G$, and $t$ is an indeterminate, then

$$\det(t - (\rho(x) - 1)) = t^n + P_{n-1}(x)t^{n-1} + \cdots + P_0(x).$$

Each $P_j$ is a regular function on $G$ defined over $k$, and $P_{n-1}(x) = P_{n-2}(x) = \cdots = P_0(x) = 0$ if and only if $x$ is unipotent.

Let $P$ denote the map of $G$ into $k^\times$ defined by

$$x \mapsto (P_{n-1}(x), \ldots, P_0(x)), \quad x \in G.$$  

Now suppose that $d$ is a positive integer and that $\mathfrak{p}$ is the prime ideal in $k$. We set $k^*_d = \mathfrak{p}^d \times \cdots \times \mathfrak{p}^N \subset k \times \cdots \times k$, where there are $N$ terms in each product. Let $G_d$ denote the complete inverse image of $k^*_d$ in $G$ under the mapping $P$. Then, since $k^*_d$ is open and closed in $k^\times$, $G_d$ is open and closed in $G$, invariant by inner automorphisms, and $\bigcap_{d \geq 0} G_d = V$.

Let $T$ be a maximal torus in $G$, $T' = G' \cap T$, and put $T'_d = T' \cap G_d$. Two complex valued functions, $\Gamma_1$ and $\Gamma_2$, defined on $T'_d$ and $T'_{d_2}$ respectively, are called equivalent if they have the same restriction to $T'_{d_1} \cap T'_{d_2}$. An equivalence class defined by this last relation is called the germ of a function on $T'$. If $f$ is a complex valued function on $T'$, we denote the germ of $f$ by $\{f\}$.

In the remainder of this paragraph we will assume (U1) through (U4).

Theorem 2.1.1. Let $T$ be a maximal torus in $G$. Then there exist unique germs $\{\Gamma_0\}$ on $T'$ in one-to-one correspondence with the unipotent classes $\mathcal{O}$, and depending only on $T$, such that, for all $f \in C^\infty_c(G)$,

$$\{F_f^T\} = \sum_{\mathcal{O}} \{\Gamma_0\} \Lambda_{\mathcal{O}}(f).$$

\[\text{The theorem is valid without reservation in characteristic zero.}\]
Let $s_0 > s_1 > \cdots > s_n$ be the dimensions of the various unipotent classes. Let $s$ be one of these integers. In the following, we shall deal with a family of distributions $\Lambda_t^s$, $t \in T'$, satisfying the following condition:

\begin{equation}
\text{for all } f \in C_c^\infty(G) \text{ vanishing on } V_s, \text{ there exists an integer } d, \text{ possibly depending on } f, \text{ so that } \Lambda_t^s(f) = 0 \text{ for all } t \in T'_d.
\end{equation}

Let $\emptyset$ be a unipotent class of dimension $s$. Choose $\Omega_\emptyset$ open in $G$ of the form $\pi(G \times U_0)$ so that $\Omega_\emptyset \cap V_s = \emptyset$. Let $\sigma_t^s$, $t \in T'$, be the distribution on $U_0$ associated with $\Lambda_t^s$.

**Lemma 2.1.3.** Fix $\beta \in C_c^\infty(U_0)$ such that $\beta(1) = 0$. Then there exists an integer $d_\emptyset$, possibly depending on $\beta$, so that $\sigma_t^s(\beta) = 0$ for all $t \in T'_d$.

**Proof.** Take $\beta$ as in the hypothesis. Since $\beta$ is locally constant, $1 \in \text{Supp} \beta$. Take $\gamma \in C_c^\infty(G)$ so that $\int_G \gamma \, dx = 1$. Put $\alpha = \gamma \times \beta$ and set $f = f_\alpha$. Then

\begin{align}
\Lambda_t^s(f) &= \sigma_t^s(\beta), \\
\text{Supp } f &\subset \pi(\text{Supp } \alpha).
\end{align}

As in the proof of Lemma 23 of [7(a)], we have $\text{Supp } f \cap \emptyset = \emptyset$. Since $f$ has support in $\Omega_\emptyset$, it follows that $f$ vanishes on $V_s$. Therefore, by (2.1.2) and (2.1.4), the proof is immediate.

For $t \in T'$, $f \in C_c^\infty(G)$, define a function $\varphi_{s,f}$ on $T'$ by

\[ \varphi_{s,f}(t) = \Lambda_t^s(f), \quad t \in T'. \]

**Corollary 2.1.6.** Fix a unipotent class $\emptyset$. Then there exists a unique germ $\{\Gamma_\emptyset\}$ on $T'$, depending only on $T$, such that, for all $f \in C_c^\infty(G_\emptyset)$,

\[ \{\varphi_{s,f}\} = \{\Gamma_\emptyset\} \Lambda_\emptyset(f). \]

**Proof.** Choose $\alpha \in C_c^\infty(G \times U_0)$ so that $f = f_\alpha$. Let $\beta = \beta_\emptyset$. Take $L$ to be a fixed open compact set in $U_0$ containing $1$, and let $C$ denote the characteristic function of $L$. Put $\beta_\emptyset = \beta - \beta(1)C$. Then $\beta_\emptyset(1) = 0$. Therefore, by Lemma 2.1.3, there exists an integer $d$ so that $\sigma_t^s(\beta_\emptyset) = \sigma_t^s(\beta) - \beta(1)\sigma_t^s(C) = 0$, for $t \in T'_d$. By the proof of Lemma 1.2.7, $\beta(1) = c\Lambda_\emptyset(f)$, for some constant $c$, independent of $f$. We may take $\Gamma_\emptyset(t) = c\sigma_t^s(C)$. The uniqueness is obvious.

**Corollary 2.1.7.** Let $s = s_k$. For each unipotent class $\emptyset$ of dimension $s$, there exists a unique germ $\{\Gamma_\emptyset\}$ on $T'$, depending only on $T$, such that, for all $f \in C_c^\infty(G)$ which vanish on $V_{s_k+1}$,

\[ \{\varphi_{s,f}\} = \sum_{\dim \emptyset = s} \{\Gamma_\emptyset\} \Lambda_\emptyset(f). \]

**Proof.** For each class $\emptyset$ choose $\Omega_\emptyset$ as above. If $f$ vanishes on $V_{s_k+1}$, then
Supp \( f \subset \bigcup_{\dim \mathcal{O} = s} \Omega \cup (G \setminus V_s) \). Hence, by Lemma 1.2.5, \( f = \sum_{\dim \mathcal{O} = s} f_0 + g \), where Supp \( f_0 \subset \Omega \), Supp \( g \subset G \setminus V_s \), and each \( f_0 \) belongs to \( C^\infty_c(G) \). Since \( g \) vanishes on \( V_s \), by (2.1.2) there is an integer \( d_0 \) so that \( \Lambda_i^s(g) = 0 \), for all \( t \in T_{d_0} \). Therefore, by Corollary 2.1.6,
\[
\{ \varphi_{s,f} \} = \sum_{\dim \mathcal{O} = s} \varphi_{s,f_0} = \sum_{\dim \mathcal{O} = s} \{ \Gamma_0 \} \Lambda_0(f_0).
\]
Since \( f - f_0 \) is zero on \( \mathcal{O} \), this last sum is equal to \( \sum_{\dim \mathcal{O} = s} \{ \Gamma_0 \} \Lambda_0(f) \).

**Proposition 2.1.8.** There exist unique germs \( \{ \Gamma_0 \} \) on \( T_U \) so that, for all \( f \) vanishing on \( V_s \),
\[
\{ F^U_f \} = \sum_{\dim \mathcal{O} > s} \{ \Gamma_0 \} \Lambda_0(f).
\]

**Proof.** First suppose that \( f \) vanishes on \( V = V_{0} \). Then, since \( \bigcap_d G_d = V \) and \( f \) has compact support, Supp \( f \subset G \setminus G_d \) for \( d \geq d_0 \). In this case, \( F^U_f(t) = 0 \) for \( t \in T_{d_0} \). Suppose the statement is true for \( s = s_k \), and suppose \( f_0 \in C^\infty_c(G) \) vanishes on \( V_{s_k+1} \). For \( t \in T' \), \( f \in C^\infty_c(G) \), put
\[
\Lambda_i^s(f) = F^U_f(t) - \sum_{\dim \mathcal{O} > s} \Gamma_0(t) \Lambda_0(f).
\]
By induction, \( \Lambda^s \) satisfies (2.1.2). Hence, by Corollary 2.1.7, there exists an integer \( d_i \) so that \( \Lambda^s_i(f_0) = \sum_{\dim \mathcal{O} = s} \Gamma_0(t) \Lambda_0(f_0) \), for \( t \in T_{d_i} \). Combining these two equalities proves the assertion for \( s_k+1 \).

The proof of Theorem 2.1.1 is now obvious.

**2.2. Determination of \( \Gamma_0 \) for \( SL_2 \)**

In this paragraph, we determine explicitly the germs \( \{ \Gamma_0 \} \), defined in §2.1, in the case \( G = SL_2 \). We assume that the characteristic of \( k \) is not two.

Let \( N \) and \( N^- \) denote the subgroups of \( G \) consisting of elements of the form \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \), \( x \in k \), respectively. Then \( V = N^\circ \). For \( x_0 \) in \( N \), it is easily seen that we may take the local submanifold \( U' \) (§1.1) to be \( N^- \). As above, let \( \Omega = \Omega(x_0) \) denote the set of elements in \( G \) of the form \( xx_0nx^{-1} \) (\( x \in G \), \( n \in N^- \)). Let \( T_0 = T' \cap \Omega \).

**Proposition 2.2.1.** \( T_0 \) depends only on the conjugacy class \( \Theta \) of \( G \) containing \( x_0 \) (\( x_0 \in N \)).

The proof follows immediately from the fact that two non-identity elements of \( N \) are conjugate if and only if they are conjugate by an element of \( A \) together with the fact that \( A \) normalizes \( N^- \). Thus we may write \( T_0 \) for \( T_0 \).

We normalize \( \Lambda_0 \) as follows. If \( \Theta = \{ 1 \} \), we take \( \Lambda_0(f) = f(1) \), \( f \in C^\infty_c(G) \).

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4 In this case all of the assumptions (U1)-(U4) are valid.
If $\mathcal{O}$ contains an element of the form $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$, we take

$$\Lambda_{\mathcal{O}}(f) = \int_{\xi \in k \times \xi^2} \bar{f} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} d\xi$$

(see § 1.2).

With these normalizations, we have

**Theorem 2.2.2.** Let $T$ be a compact torus in $G$. Let $C_{\mathcal{O}}$ denote the characteristic function of $T_{\mathcal{O}}$. Then, for $f \in C^\infty_c(G)$,

$$\{Ff\} = -A_T\left[|D|^{1/2}\right]f(1) + B_T \sum_{\dim \mathcal{O} > 0} \{C_{\mathcal{O}}\} \Lambda_{\mathcal{O}}(f),$$

where $A_T$ and $B_T$ are positive constants.

The proof is contained in the following lemmas.

For $t \in T_{\mathcal{O}}$, we may write $t = xx_0nx_n^{-1} (x \in G, n \in N^-)$. It is easy to see that, given $t$, $n$ is unique. Let $n = n(t)$. By direct computation, one sees that for $t_1, t_2 \in T_{\mathcal{O}}, n(t_2) = n(t_1)$ if and only if there exists $w \in W_T$ such that $t_1 = wt_2w^{-1}$ (see the proof of the lemma below). Thus we obtain an analytic map from $T_{\mathcal{O}}/W_T$ into a subset $N_T$ of $N$. The Jacobian at $t \in T_{\mathcal{O}}$ is seen to be $c_T |D(t)|^{1/2}$, where $c_T$ is a positive constant [13]. Therefore, the above mapping is an analytic isomorphism.

**Lemma 2.2.3.** Suppose that $T$ is a fixed compact torus in $G$. Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class. Then

$$\{\Gamma_{\mathcal{O}}\} = c_{\mathcal{O}}\{C_{\mathcal{O}}\},$$

where $c_{\mathcal{O}}$ is a constant possibly depending on $\mathcal{O}$.

**Proof.** Choose $\Omega_{\mathcal{O}}$ as above. Fix $\beta \in C^\infty_c(N^-)$ so that $\beta(1) \neq 0$. Choose $\gamma \in C^\infty_c(G)$ so that $\int_{\mathcal{O}} \gamma dx = 1$. Let $\alpha = \gamma \times \beta$. Let $f = f_\alpha$ be the corresponding function on $\Omega_{\mathcal{O}}$. By Corollary 2.1.6, there exists an integer $d$ so that $Ff_{\gamma}(t) = \Gamma_{\mathcal{O}}(t)\Lambda_{\mathcal{O}}(f)$ for $t \in T''_d$. Since $\beta$ is locally constant, we may choose $d$ so large that $\beta(n) = \beta(1)$ for all $n \in N$ satisfying $x_0n \in G_d$.

Let $F$ be any locally integrable class function with support in $(T''_d)^0$. Then, by Corollary 1.1.4,

$$\int_{\mathcal{O}} F(x)f(x)dx = \int_{N^-} F(x_0n)\beta(n)dn = \beta(1)\int_{N_T^{-1}} F(x_0n)dn.$$

On the other hand, by Weyl's Lemma, since $f$ has support in $\Omega_{\mathcal{O}}$

$$\int_{\mathcal{O}} F(x)f(x)dx = \int_{\mathcal{O}/W_T} |D(t)|^{1/2} F(t)Ff_{\gamma}(t)dt.$$

By changing variables, we can write the last integral as $(c_T)^{-1}\int_{N_T^{-1}} F(x_0n)Ff_{\gamma}(x_0n)dn$.

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$^5$ For $k$ not of characteristic two, explicit values for $A_T$ and $B_T$ may be found in [13].
where the integration may be taken over those \( n \) satisfying \( x_n \in G_d \). Since \( \Gamma_\theta \) is invariant by the action of \( W_T \) on \( T' \), we may extend it to a function on \( (T'_d)^0 \); similarly for \( F'_f \). The last integral then becomes \( (c_T)\Lambda_\theta(f) \int_{x_n} \Gamma_\theta(x_n)F(x_n)dn \).

By Lemma 1.2.7, \( \beta(1) = c_\theta \Lambda_\theta(f) \), where \( c_\theta \) is a constant. Therefore, since \( \beta(1) \neq 0 \),

\[
c'_\theta c'_T \int_{x_n} F(x_n)dn = \int_{x_n} \Gamma_\theta(x_n)F(x_n)dn .
\]

Changing variables again, we obtain

\[
c'_\theta c'_T \int_{T_\theta/W_T} F(t) |D(t)|^{1/2}dt = \int_{T_\theta/W_T} F(t)\Gamma_\theta(t) |D(t)|^{1/2}dt .
\]

Since this last equality holds for any bounded function on \( T_\theta/W_T \) with support in \( T'_d/W_T \), we have \( \Gamma_\theta(t) = c_\theta c'_T \) for all \( t \) in \( T_\theta \cap T'_d \). To finish the proof, we observe that, for \( f \in C_\theta^\sigma(\Omega_\theta) \), \( F'_f \) has support in \( T_\theta \).

**Lemma 2.2.4.** For \( \Theta \) a non-trivial unipotent class, \( c_\Theta \) is independent of \( \Theta \).

**Proof.** Let \( g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \), \( a \in k^\times \). For \( x \in G \), let \( x^g = gxg^{-1} \). The map \( x \mapsto x^g \) induces a transitive action, \( \Theta \mapsto \Theta^g \), of \( k^\times \) on the non-trivial unipotent classes. For \( f \in C_\sigma^\sigma(G) \), define \( f^g \in C_\sigma^\sigma(G) \) by \( f^g(x) = f(x^g), \ x \in G \). Assume \( f \) satisfies \( f(\kappa x \kappa^{-1}) = f(x), \ x \in G, \ \kappa \in K \). Using the explicit formula for \( \Lambda_\Theta \), we have

\[
(2.2.5) \quad \Lambda_\Theta(f^g) = |a|^{-1} \Lambda_\Theta(f) .
\]

We also have \( \Omega_{\Theta^g} = g\Omega_\Theta g^{-1} \).

Now fix \( \beta \in C_\sigma^\sigma(N) \) so that \( \beta(1) \neq 0 \). Choose \( \gamma \in C_\sigma^\sigma(G) \) so that \( \int_\gamma dx = 1 \).

Put \( \alpha = \gamma \times \beta \) and let \( f = f_\alpha \) be the corresponding function on \( C_\sigma^\sigma(\Omega_\Theta) \). Then \( f^g \in C_\sigma^\sigma(g^{-1}\Omega_\Theta g) \). Define \( \alpha' \) on \( G \times N \) by

\[
(2.2.6) \quad \alpha'(x, n) = |a|^{-1} \alpha(gxg^{-1}, gng^{-1}) .
\]

Using the defining condition satisfied by \( f_\alpha \), we see that the function on \( g^{-1}\Omega_\Theta g \) corresponding to \( \alpha' \) is \( f^g \). Define \( c'_\Theta \) as in the proof of Lemma 2.2.3. It suffices to prove that \( c'_\Theta \) is independent of \( \Theta \). By (2.2.5) and (2.2.6), we have

\[
|a|^{-1} \beta(1) = c'_\Theta^{-1} \Lambda_\Theta^{-1}(f^g) = |a|^{-1} c'_\Theta \Lambda_\Theta(f) .
\]

Consequently, \( c'_\Theta = c'_\Theta^{-1} \), and the lemma is proved.

The fact that \( \{ \Gamma_\theta \} = -A_f \{ D|^{1/2} \} \) follows from the explicit form of \( F'_f \) for special \( f \in C_\sigma^\sigma(G) \) [12(b)], [13].

In the remainder of this paragraph, we give heuristic evidence to support a conjectural formula for \( \Gamma_\theta \) when \( G \) is connected, semi-simple and \( \Theta = \{1\} \) is the trivial class. For simplicity, we assume that \( G \) is split and refer to [8]
for the more general case. As in [15], let $\Pi = \Pi_\chi$ be the Steinberg representation of $G$ associated with the trivial character $\chi$. Then $d(\Pi)$, the formal degree of $\Pi$, is given by $d(\Pi)^{-1} = \operatorname{meas}(B) C_\sigma$, where $B$ is an Iwahori subgroup and $C_\sigma$ depends only on the affine Weyl group of $G$. Let $\varphi$ be a $B$-finite matrix coefficient of $\Pi(\varphi(1) = 1)$. In general $\varphi$ does not have compact support. It is reasonable to expect that $\Pi$ has a character $\Theta_n$ given by

$$\Theta_n(x) = d(\Pi) \int_{\sigma} \varphi(xy^{-1}) dy,$$

for $x$ elliptic. Moreover, by analogy with the formula of Curtis [4] for the Steinberg character of a finite $B$-$N$ pair, we expect that, for $x$ elliptic, $\Theta_n(x) = (-1)^r$, where $r$ is the rank of $G$. Thus, formally (assuming the asymptotic formula for $\varphi$), we have

$$F_n(x) = (-1)^r \operatorname{meas}(B) C_\sigma \mu(T)^{-1} |D(t)|^{1/2} = \Gamma_{11}(x) + \sum_{\dim \sigma > 0} \Gamma^{(1)}(x) \Lambda_\sigma(\varphi).$$

By using homogeneity properties of the functions $\Gamma^{(1)}$ under “stretching” (assuming $k$ has characteristic zero) as in [5c] we conclude that

$$\Gamma_{11}(x) = (-1)^r \operatorname{meas}(B) C_\sigma \mu(T)^{-1} |D(t)|^{1/2},$$

for $x$ elliptic and sufficiently small. One may check this directly for $G = \text{SL}_2(k)$, $k$ any non-archimedean local field (char $k \neq 2$), by using the results of [13].

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6. The author has been informed that recently A. Borel and J.-P. Serre have obtained an explicit formula for the Steinberg character associated with a reductive $p$-adic group.

(Received October 4, 1970)
(Revised April 28, 1971)