1. Review on reductive semi-groups

The reference for this material is the paper "Very flat reductive monoids" of Rittatore and "On reductive algebraic semi-groups" of Vinberg.

Let $M'$ be an algebraic normal irreducible monoid over an algebraically closed field $k$. The unit group $G'$ acts be left and right translation on $M$. The orbit of 1 is an open dense subset of $M'$ isomorphic to $G' \times G'/G'$ with respect to the diagonal embedding $G' \to G' \times G'$. Suppose that $G$ is reductive and $G$ is the commutator of $G'$ which is a semi-simple group. The quotient $A' = M'/G \times G$ exists and is a commutative monoid of unit group the torus $G'/G$. The quotient morphism $M' \to A'$ is called the abelianisation of $M'$. The monoid $M'$ is said to very flat if the abelinization is a flat morphism with reduced irreducible fibers. The universal monoid is often attached to the name of Vinberg.

Given a semi-simple group $G$, the category $\mathcal{M}(G)$ of very flat monoids $M'$ who has $G$ as the commutator of unit group, admit a final object $M^+$. The abelianization of this universal monoid $M^+ \to A^+$ is an affine space of dimension equal to the rank of $G$. Every every flat semi-group $M' \in \mathcal{M}(G)$ is a fibered product of $A' \to A^+$ and the universal semi-group $M^+$.

The notion of normal affine monoids of unit group $G'$ is equivalent to normal affine embedding of $G'$. Recall that affine embedding is an open embedding of $G'$ into a normal affine variety $X$ such that the action of $G'$ by left and right translation can be extended to $X$. In that case the morphism $(G' \times X) \cup (X \times G') \to X$ can be extended to $X \times X \to X$ thus makes $X$ a monoid. Therefore the notion of normal affine monoid is a particular instance of normal spherical embeddings. This observation was intended for those people who are particularly interested to spherical embeddings.

2. The Vinberg semi-group

There is no harm to restrict first to the case where $G$ is simply connected. The general case can be deduced from the simply connected case as we will explain later.
Let $G$ be a semi-simple simply connected group. Let $(T, B)$ be a Borel pair and let $Z$ denote the center of $G$. We write $G^+ = (T \times G)/Z$ where $Z \to T \times G$ is the diagonal embedding. The center of $G^+$ is $T$, and we have the exact sequence

$$1 \to T \to G^+ \to G^{\text{ad}} \to 1$$

where $G^{\text{ad}} = G/Z$. The commutator group of $G^+$ is $G$, and we have the exact sequence

$$1 \to G \to G^+ \to T^{\text{ad}} \to 1$$

where $T^{\text{ad}} = T/Z$.

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots with respect to $B$ and $\omega_1, \ldots, \omega_r$ the fundamental weights satisfying the relation $\langle \omega_i, \alpha^\vee_i \rangle = 1$, $\alpha^\vee_i$ being the coroot attached with the root $\alpha_i$. Let $\rho_i : G \to \text{GL}(V_i)$ denote the irreducible representation of highest weight $\omega_i$. This can be extended to a representation $\rho^+_i : G^+ \to \text{GL}(V_i)$ given by the formula

$$\rho^+_i(t, g) = \omega_i(w_0 t^{-1}) \rho_i(g)$$

where $w_0$ is the long element in the Weyl group $W$ of $G$. The root $\alpha_i : T \to \mathbb{G}_m$ will also be extended to $G^+$

$$\alpha^+_i : G^+ \to \mathbb{G}_m$$

given by $\alpha^+_i(t, g) = \alpha_i(t)$. Altogether, these map define

$$(\alpha^+, \rho^+) : G^+ \to \mathbb{G}_m^r \times \prod_{i=1}^r \text{GL}(V_i).$$

Following Vinberg, we will consider the closure of the image of $(\alpha^+, \rho^+)$ in $\mathbb{G}_m^r \times \prod_{i=1}^r \text{End}(V_i)$ as well as in $\mathbb{G}_m^r \times \prod_{i=1}^r (\text{End}(V_i) - \{0\})$ that we will denote by $M^+$ and $M^{str+}$ respectively. By construction $M^+$ is an affine scheme equipped with action of $G^+ \times G^+$ extending its action of $G^+$ on itself by left and right translation. This action can be extended to a structure of semi-group on $M^+$.

It is worth to recall that $M^{str+}$ is a smooth open subset of $M^+$. In fact, the quotient of $M^{str+}$ by the free action of $T$ is the wonderful compactification of the adjoint group $G^{\text{ad}}$.

### 3. Groupoid whose abelinazation is $\mathbb{G}_a$

The abelinazation of the universal monoid $M^+$ is an $A^+ = \mathbb{G}_a^r$. The embedding $T^{\text{ad}}$ into $A^+$ is of course given by

$$t \mapsto (\alpha_1(t), \ldots, \alpha_r(t))$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots.
We are particularly interested in very flat monoid \( M \in \mathcal{M}(G) \) whose abelianization if \( G \) as toric embedding of \( \mathbb{G}_m \). By universal property, these monoids are given by homomorphism

\[
\lambda : \mathbb{G}_m \to T^{ad}
\]

that can be extended to a morphism of monoids \( \mathbb{G}_a \to A^+ \). This latter condition means \( \langle \alpha_i, \lambda \rangle \geq 0 \) i.e. \( \lambda \) is a dominant cocharacter of \( T^{ad} \). Let \( P^\vee \) denote the group of cocharacter of \( T^{ad} \) and \( P^\vee_+ \) its cone of dominant cochararacters. For every \( \lambda \in P^\vee_+ \), we have a monoid \( M^\lambda \in \mathcal{M}(G) \) whose fits into a commutative diagram

\[
\begin{array}{ccc}
M^\lambda & \longrightarrow & M^+ \\
\downarrow & & \downarrow \\
\mathbb{G}_a & \longrightarrow & A^+
\end{array}
\]

The homomorphism \( \lambda : \mathbb{G}_m \to T^{ad} \) induces an extension

\[
1 \to G \to G^\lambda \to \mathbb{G}_m \to 1
\]

and \( M^\lambda \) is an affine embedding of \( G^\lambda \). We also have \( M^\lambda_{str} \) defined as pullback of \( M^+_{str} \). (Question : is \( M^\lambda_{str} \) smooth ?)

Since \( G \) is assumed simply connected, its dual group \( \hat{G} \) is adjoint. The dual group of \( G^{ad} \) is a simply connected group that we will denote by \( \hat{G}^{sc} \). The maximal torus \( \hat{T}^{sc} \) of \( \hat{G}^{sc} \) is dual to the torus \( T^{ad} \). We have exact sequence

\[
1 \to Z \to T \to T^{ad} \to 1
\]

and

\[
1 \to \hat{Z} \to \hat{T}^{sc} \to \hat{T} \to 1
\]

where \( \hat{Z} \), the Cartier dual of \( Z \), is the center of \( \hat{G}^{sc} \).

An element \( \lambda \in P^\vee_+ \) can be seen as a dominant character \( \lambda : \hat{T}^{sc} \to \mathbb{G}_m \), thus defines an irreducible representation \( \rho_\lambda : \hat{G}^{sc} \to \text{GL}(V_\lambda) \) of \( \hat{G}^{sc} \) with highest weight \( \lambda \). The irreducibility implies that the center \( \hat{Z} \) of \( \hat{G}^{sc} \) acts on \( V_\lambda \) by scalars thus induces a homomorphism \( \hat{Z} \to \mathbb{G}_m \). Using this homomorphism, we can construct a central extension

\[
1 \to \mathbb{G}_m \to \hat{G}^\lambda \to \hat{G} \to 1
\]

equipped with a homomorphism \( \rho_\lambda^\lambda : \hat{G}^\lambda \to \text{GL}(V_\lambda) \) where the central \( \mathbb{G}_m \) acts as homothety in \( V_\lambda \). If \( \lambda : T^{sc} \to \mathbb{G}_m \) factors through \( \hat{T} \), we have \( \hat{G}^\lambda = \mathbb{G}_m \times \hat{G} \) and \( \rho_\lambda^\lambda \) is nothing but the product the homothety on \( \mathbb{G}_m \) and the representation of highest weight \( \lambda \) of \( \hat{G} \). This group is
quite a convenient host for the definition of the local $L$-factor. We can check that $\hat{G}^\lambda$ is the Langlands dual group of $G^\lambda$.

The most well-known example for this construction is the case $G = \text{SL}_n$. In that case $\hat{G} = \text{PGL}_n$ and $\hat{G}^\text{sc} = \text{SL}_n$. Let $\lambda$ be the highest weight of the standard $n$-dimensional representation of $\hat{G}^\text{sc}$. In this case $\hat{G}^\lambda = \text{GL}_n$. I expect of course that $G^\lambda = \text{GL}_n$ and $M^\lambda = \text{Mat}_n$ but the needed calculation remains to be done. Let us call the case Godement-Jacquet’s case.

4. Relation with the Cartan decomposition

Let $F$ be a non-archimedean local field and $O$ its ring of integers. Let $\pi$ be an uniformizing parameter of $F$. We will describe the set $M^+(O) \cap G^+(F)$ and $M^+_{\text{str}}(O) \cap G^+(F)$ in terms of the Cartan decomposition of $G^{ad}(F)$ into double cosets of $K = G^{ad}(O)$. In fact, with some efforts, we should be able to prove the Cartan decomposition in this way which is probably done by Gaitsgory and Nadler.

**Proposition 4.1.** Let $g^+ \in M^+(O) \cap G^+(F)$. We denote $a^+$ its image in

$$A^+(O) \cap T^{ad}(F) = \bigsqcup_{\lambda \in P^+_+} \pi^\lambda T^{ad}(O).$$

Assume $a^+ \in \pi^\lambda T^{ad}(O)$ and denote $g^{ad}$ the image of $g^+$ in $G^{ad}(F)$. Then

$$g^{ad} \in \bigsqcup_{\lambda' \leq \lambda} K \pi^{\lambda'} K$$

where $\lambda' \leq \lambda$ if and only if $\lambda - \lambda'$ is a sum of positive coroots. If we assume $g^+ \in M^+_{\text{str}}(O) \cap G^+(F)$, then we have a more precise relation

$$g^{ad} \in K \pi^\lambda K.$$

**Proof.** This is essentially the formula (2.1).

We will derive similar description for the monoid $M^\lambda$ of unit group $G^\lambda$. Recall that we have an exact sequence

$$1 \to G \to G^\lambda \to G_m \to 1.$$

**Proposition 4.2.** Let $g \in M^\lambda(O) \cap G^\lambda(F)$. We denote $a$ its image in

$$\mathbb{G}_a(O) \cap \mathbb{G}_m(F) = \bigsqcup_{n \in \mathbb{N}} \pi^n \mathbb{O}^\times.$$
Assume \( a \in \pi^n \mathcal{O}^\times \) and denote \( g^{ad} \) the image of \( g \) in \( G^{ad}(F) \). Then
\[
g^{ad} \in \bigsqcup_{\lambda' \leq n\lambda} K\pi^{\lambda'}K
\]
where \( \lambda' \leq \lambda \) if and only if \( \lambda - \lambda' \) is a sum of positive coroots. If we assume \( g \in M_{str}^\lambda(\mathcal{O}) \cap G^\lambda(F) \), then we have a more precise relation
\[
g^{ad} \in K\pi^n\lambda K.
\]

5. Basic function attached to a monoid

Following a proposal of Sakellaridis, for each monoid \( M' \) of unit group \( G' \) we should use the following function on \( G'(F) \) at almost every places. Let consider \( L^+M' = M'(\mathcal{O}) \) as a projective limit of affine schemes and let consider the intersection complex \( IC(L^+M') \). It is convenient to normalize \( IC(L^+M') \) so that its restriction to the smooth locus of \( L^+M' \) is the constant sheaf \( \mathbb{Q}_\ell \) on degree 0. In particular the trace function attached to this takes values 1 on the smooth locus of \( L^+M' \).

We obtain a function
\[
\text{Sake} : M'(\mathcal{O}) \cap G'(F) \to \mathbb{Q}_\ell.
\]

I expect (first conjecture) that \( M'_{str} \) is smooth and so is \( L^+M'_{str} \) by Hensel’s lemma. In other words \( L^+M'_{str} \) is contained in the smooth locus of \( L^+M' \). Therefore, the restriction of \( \text{Sake} \) to \( M'_{str}(\mathcal{O}) \) should be 1. This is true for the universal monoid \( M^+ \) as we know that \( M^+_{str} \) is smooth.

The second conjecture is somehow more surprising. Consider the exact sequence
\[
1 \to G \to G' \to T' \to 1.
\]
Then we have an application
\[
M'(\mathcal{O}) \cap G'(F) \to A'(\mathcal{O}) \cap T'(F).
\]
This allows us to stratify \( L^+M' \) as
\[
L^+M' = \bigsqcup_{\lambda' \in X^+_*(T')} L^+M'_{\lambda'}
\]
over the set of dominant cochararacters \( \lambda' : \mathbb{G}_m \to T' \) i.e. those characters that can be extended to a morphism of semi-groups \( \mathbb{G}_a \to A' \). We expect that the restriction of \( IC(L^+M') \) to each stratum \( L^+M'_{\lambda'} \) is still a direct sum of perverse sheaves of course with suitable normalization.

The third conjecture is concerned with the universal monoid \( M^+ \). In this case the set \( X^+_*(T') \) is the set \( P_+^\vee \). It is probably true that the restriction of \( IC(L^+M^+) \) to \( L^+M^+_{\lambda'} \) is nothing but the intersection
complex of $L^+M_\lambda^+$. I have no evidences to offer but the lack of other reasonable answer.

The fourth conjecture is concerned with our favorite monoid $M^\lambda$. In this case $X^+_\tau(T')$. It would be wonderful if the restriction of $IC(L^+M^+)$ to $L^+M_n^+$ is the equivariant sheaf that correspond via the geometric Satake equivalence to $\operatorname{Sym}^n(V_\lambda)$. This should be the "right answer" anyway. If the fourth conjecture is not true, we need to modify the definition of Sake so that it becomes true!

6. EPIGRAPH

Dear Yannis,

I have in mind some reasons to believe in this but it would take me more time to write it down in a good way. For the moment, I prefer to share my optimistic thoughts with you and look forward to your further comments.

Best regards, Bao Chau