On automorphic $L$-functions

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The Riemann zeta function

The series

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

converges absolutely for $\Re(s) > 1$ and uniformly on $\Re(s) > 1 + \epsilon$ for every $\epsilon > 0$. 
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- The function \( s \mapsto \zeta(s) \) is holomorphic on the half-plane \( \Re(s) > 1 \).

- Over that domain, it has a development as Eulerian product

\[ \zeta(s) = \prod_{p} (1 - p^{-s})^{-1}. \]
Meromorphic continuation and functional equation

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It satisfies a functional equation $\zeta^*(1-s) = \zeta^*(s)$ where $\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$. 

Zeros and special values of the Riemann zeta function have deep arithmetic significance. It is conjectured by Riemann that except the “trivial” zeros at even negative integers, all others zeros lie on the “critical” line $\text{Im}(s) = 1/2$. 

The non-vanishing of $\zeta$ on the line $\text{Im}(s) = 1$ is a crucial ingredient in the proof of prime number theorem.
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Dirichlet’s $L$-function

For any Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$, we can form the series

$$L(s, \chi) = \sum_{(n,m)=1} \chi(n) n^{-s}.$$
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- This series converges absolutely on the domain $\Re(s) > 1$. For $\chi \neq 1$, $L(s, \chi)$ can be continued as holomorphic function of $s \in \mathbb{C}$ satisfying a functional equation relating its values at $s$ and $1 - s$. 
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- The non vanishing $L(1, \chi) \neq 0$ is a critical ingredient in the proof of infiniteness of prime numbers in an arithmetic progression.
Completions of $\mathbb{Q}$

- The ring of integers $\mathbb{Z}$ and the field of rational numbers $\mathbb{Q}$ can be endowed with the $p$-adic topology in which two numbers are closed if their difference is divisible by a high power of $p$. 

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- The ring of \( p \)-adic integers \( \mathbb{Z}_p \) is the completion of \( \mathbb{Z} \) with respect to the \( p \)-adic topology. Idem, \( \mathbb{Q}_p \) is the completion of \( \mathbb{Q} \).
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- The ring of $p$-adic integers $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the $p$-adic topology. Idem, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$.
- Natural completions of $\mathbb{Q}$ include the field of real numbers $\mathbb{R}$ as well as the fields of $p$-adic numbers $\mathbb{Q}_p$ for different primes $p$. It is convenient to look at $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to the infinite prime $p = \infty$. 
Adèles

- An adèle is a sequence \((x_p)\) with \(x_p \in \mathbb{Q}_p\) if \(p\) is a finite prime and \(x_\infty \in \mathbb{R}\) if \(p = \infty\) such that \(x_p \in \mathbb{Z}_p\) for almost all finite primes \(p\).
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- \(\mathbb{Q}\backslash\mathbb{A}\) is the pro-universal covering of the circle \(\mathbb{R}/\mathbb{Z}\).
- More precisely, \(\mathbb{Q}\backslash\mathbb{A}/\hat{\mathbb{Z}}\) can be identified with \(\mathbb{R}/\mathbb{Z}\), and for every integer \(m\), \(\mathbb{Q}\backslash\mathbb{A}/m\hat{\mathbb{Z}}\) can be identified with the covering \(\mathbb{R}/m\mathbb{Z}\) of \(\mathbb{R}/\mathbb{Z}\).
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► There is a decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}_+^\times \times \hat{\mathbb{Z}}$, each idele can be written uniquely as $x = \alpha tu$ with $\alpha \in \mathbb{Q}^\times$, $t \in \mathbb{R}_+^\times$, $u \in \hat{\mathbb{Z}}^\times$. 

▶ Every quasi-character $\omega: \mathbb{A}^\times \to \mathbb{C}^\times$, trivial on $\mathbb{Q}^\times$, can be written as $\omega(\alpha tu) = ts\chi(u)$ where $s \in \mathbb{C}$ and $\chi: \mathbb{Z}^\times \to \mathbb{C}^\times$ is a i.e. factoring through a Dirichlet character $((\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$.

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- Every quasi-character \(\omega : \mathbb{A}_\times \to \mathbb{C}_\times\), trivial on \(\mathbb{Q}_\times\), can be written as \(\omega(\alpha tu) = t^s \chi(u)\) where \(s \in \mathbb{C}\) and \(\chi : \mathbb{Z}_\times \to \mathbb{C}_\times\) is a i.e. factoring through a Dirichlet character \((\mathbb{Z}/m\mathbb{Z})_\times \to \mathbb{C}_\times\).
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The Mellin transform

- If \( \phi \in C^\infty(\mathbb{R}_+^\times) \) has rapid decay at 0 and \( \infty \), its Mellin transform

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\tilde{\phi}(s) = \int_{\mathbb{R}_+^\times} \phi(t)t^s d^\times t
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  defines a holomorphic function of variable $s$.

- If $\phi$ has rapid decay only at $\infty$, the above integral converges only for $\Re(s) > 0$. Nevertheless, the function $s \mapsto \tilde{\phi}(s)$ can be continued as meromorphic function of $s \in \mathbb{C}$. 

- If $\phi = e^{-t}$ then $\tilde{\phi}(s)$ is the Gamma function $\Gamma(s) = \int_{\mathbb{R}_+^\times} e^{-t} t^s \, d^\times t$ that has simple poles at negative integers.
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For every locally constant with compact support $\phi \in C_c^\infty(\mathbb{Q}_p^\times)$, the integral

$$\tilde{\phi}(\omega) = \int_{\mathbb{Q}_p^\times} \phi(x)\omega(x)d^\times x$$

for every quasi-character $\omega : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$. 

The function $\omega \mapsto \tilde{\phi}(\omega)$ is analytic.
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The function $\omega \mapsto \tilde{\phi}(\omega)$ is analytic.

For instance, if $\phi = 1_{p\mathbb{Z}_p^\times}$, then for unramified quasi-character $\omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ (i.e. trivial on $\mathbb{Z}_p^\times$), $\tilde{\phi}(\omega) = \omega(p)$. 
Let us allow the test function \( \phi \) to have compact support in \( \mathbb{Q}_p \) instead of \( \mathbb{Q}_p^\times \). Then the integral
\[
\int_{\mathbb{Q}_p^\times} \phi(x) \omega(x) d^\times x
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may not converge.
$p$-adic Mellin transform

- Let us allow the test function $\phi$ to have compact support in $\mathbb{Q}_p$ instead of $\mathbb{Q}_p^\times$. Then the integral $\int_{\mathbb{Q}_p^\times} \phi(x) \omega(x) d^\times x$ may not converge.

- If $\omega$ is unitary, the integral

$$\tilde{\phi}(s, \omega) = \int_{\mathbb{Q}_p^\times} \phi(x) \omega(x) |x|^s d^\times x$$

converge for $\Re(s) > 0$. Moreover it can be meromorphically continued to all $\omega$. 
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For instance, if $\phi = 1_{\mathbb{Z}_p}$, and for unramified character $\omega$, we have
$$\tilde{\phi}(s, \omega) = \sum_{i=0}^{\infty} \omega(p)^i p^{-is} = (1 - \omega(p)p^{-s})^{-1}$$
Adelic Mellin transform

We consider functions on $\mathbb{A}$ of the form $\phi = \bigotimes p \phi_p$ with $\phi_\infty \in S(\mathbb{R})$ and $\phi_p \in C_c^\infty(\mathbb{Q}_p)$, $\phi_p = 1_{\mathbb{Z}_p}$ for almost all $p$. 

For every character $\omega: \mathbb{A} \times \to \mathbb{C} \times$, the integral $\tilde{\phi}(s, \omega) = \int_{\mathbb{A} \times} \phi(x) \omega(x) |x|^s dx$ converges for $\Re(s) > 1$. On this region, the above integral defines a holomorphic function with development as Euler product $\tilde{\phi}(s, \omega_p) = \prod p \tilde{\phi}_p(s, \omega_p)$ with $\tilde{\phi}_p(s, \omega_p) = (1 - \omega_p(p)^{-s})^{-1}$ for almost all $p$. 

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- For every character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$, the integral
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with $\tilde{\phi}_p(s, \omega_p) = (1 - \omega_p(p) p^{-s})^{-1}$ for almost all $p$. 
Characters of the idèle class group

- If $\omega$ is a character of the idèle class group $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, the function $\tilde{\phi}(s, \omega)$ can be meromorphically continued to $s \in \mathbb{C}$. 

It is closely related to the Dirichlet $L$-function and the Riemann zeta function. It satisfies the functional equation $\tilde{\phi}(s, \omega) = \hat{\phi}(1-s, \omega^{-1})$ where $\hat{\phi}$ is the Fourier transform of $\phi$. The above formula derives from the Poisson summation formula.
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Tate’s thesis: upshot

- Tate’s proof is not fundamentally different from Riemann’s proof of the functional equation satisfied by ζ and L-functions. Both relied ultimately on the Poisson summation formula.
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- However, the adelic language allows us to express the value of \( L \)-functions as the trace of certain operator on one-dimensional representation of idèle class group.
- The operator \( 1_{\mathbb{Z}_p} \) is the shadow of of the affine line \( \mathbb{A}^1 \) acted on by the multiplicative group \( \mathbb{G}_m \).
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The operator \( 1_{\mathbb{Z}_p} \) is the shadow of of the affine line \( \mathbb{A}^1 \) acted on by the multiplicative group \( \mathbb{G}_m \).

The functional equation relies on the additive structure of the affine line: the Fourier transform and the Poisson summation formula.
Automorphic representations

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- $\pi$ decomposes as tensor product $\pi = \otimes_p \pi_p$ for finite and infinite primes $p$, $\pi_p$ is an irreducible unitary representation of $G(\mathbb{Q}_p)$. 
A prime $p$ is said to be unramified prime for $G$ if $G$ has a reductive model over $\mathbb{Z}_p$. 
Unramified representation at an unramified prime

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A representation $\pi_p$ of $G(\mathbb{Q}_p)$ is unramified if $G(\mathbb{Z}_p)$ has a non-zero fixed vector.
Classification of unramified representation and the Langlands dual group

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- By inverting the role of characters and cocharacters, roots and coroots, we define a semi-simple group $\hat{G}$ over $\mathbb{C}$.
- There is natural 1-1 correspondence between unramified representations of $G(\mathbb{Q}_p)$ and semi-simple conjugacy classes of $\hat{G}(\mathbb{C})$

$$\pi_v \leftrightarrow \alpha(\pi_v) \in \hat{G}/\sim$$
Langlands’ automorphic $L$-function

Let $r$ be a finite dimensional representation of $\hat{G}$. 
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- Local factors at ramified places and infinite places can also be properly defined.
- The automorphic $L$-function is defined as an infinite product
  \[
  L(s, \pi, r) = \prod_p L_v(s, \pi, r)
  \]
  which is absolutely convergent $\Re(s) \gg 0$. 

Langlands’ conjecture

- It is conjectured that $L(s, \pi, r)$ has meromorphic continuation and functional equation.
On automorphic $L$-functions

Ngô Bảo Châu

Classical zeta functions and $L$-functions

Adèles and Tate’s thesis

Automorphic $L$-functions

Vinberg’s theory of flat monoids

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- This statement is surprisingly deep. It contains a lot of arithmetic information about the arithmetic of automorphic representation.
- It is known in some cases where $r$ is closed to the standard representations by various method.
- There have been many attempts to generalize the method of Tate to general automorphic $L$-function. Tamagawa, Godement, Jacquet succeeded completely in the case of principal $L$-function.
Tamagawa-Godement-Jacquet principal $L$-function

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- Let $M = \mathfrak{gl}_n$ the space of $n \times n$ matrices equipped with action of $G$ by left and right translation.
Tamagawa-Godement-Jacquet principal \( L \)-function

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- Let \( M = gl_n \) the space of \( n \times n \) matrices equipped with action of \( G \) by left and right translation.
- The trace of \( 1_M(\mathbb{Z}_p) \) on the unramified representation \( \pi_v \otimes |\det|^s \) is the principal local \( L \)-factor

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L_v(s, \pi) = (1 - \alpha(\pi_v)p^{-s})^{-1}.
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  \[ L_v(s, \pi) = (1 - \alpha(\pi_v)p^{-s})^{-1}. \]
- The additive structure on $M$ permits to define the Fourier transform and obtain the Poisson summation formula. Using this Tamagawa, Godement, Jacquet were able to meromorphically continue the principal $L$-function and prove its functional equation.
Search for a general answer

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- In general, $M$ has no additive structure, and is even singular.
- The generalization of $1_{\mathbb{Z}_p}$ has to take into account to incorporate singularities of $M$. 

Monoids for a semi-simple group

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- Since $M'$ is affine, this is equivalent to saying that the group law of $G'$ can be extended as a monoid law on $M'$.
- Typical example: $G = \text{SL}_n$, $G' = \text{GL}_n$ and $M = \text{gl}_n$. 
Flat monoids

Let $A'$ be the invariant quotient $M'//G \times G$. This is an affine embedding of the torus $G'/G$. Following Vinberg, we call it the abelianization of $M'$. 
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According to Vinberg, the category of flat monoids of $G$ has a pleasant description.
Universal flat monoid

- Let $G^+ = (G \times T)/Z$ where $T$ is a maximal torus of $G$ and $Z$ is the center of $G$, acting diagonally on $G \times T$. 
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- There is only one reasonable way to combine the $\alpha_i$ and $\rho_i$ into a faithful representation of $G^+$

$$(\alpha^+, \rho^+) : (T \times G)/Z \to \mathbb{G}_m^r \times \prod_{i=1}^r \text{GL}(V_i).$$
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- The invariant quotient of $M^+$ by the double action of $G$ is $A^+ = \mathbb{A}^r$ which is an affine embedding of the torus $G^+/G = T^{ad}$. 

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- $M^+$ is the universal flat monoid in the sense that every other flat monoid $M'$ of $G$ can be obtained by forming fibered product over the abelinization $A' \to A^+$. 
1-dimensional centered flat monoid

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- Let the corresponding monoid $M^\lambda = M^+ \times_{\mathbb{A}^+} \mathbb{A}$ should play the role of the affine line in Tate's thesis.
- The group $G^\lambda$ has a determinant map $\det : G^\lambda \rightarrow \mathbb{G}_m$ whose kernel is $G$. 

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The formal arc space

Let us consider the arc space of $M^\lambda$. This is an infinite dimensional scheme $\mathcal{L}M^\lambda$ such that

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- According to Grinberg-Kazhdan and Drinfeld, singularities of $\mathcal{L}M^\lambda$ can be approximated by finite dimensional singularities i.e. for every point $x \in \mathcal{L}M^\lambda$, there is a scheme $Y$ of finite type, $y \in Y$ such that there is an isomorphism of formal schemes
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It makes sense to talk about the intersection complex of the formal arc space $\mathcal{LM}^\lambda$. The trace of Frobenius on stalks of $\mathcal{LM}^\lambda$ defines a function

$$m_\lambda : M^\lambda(\mathbb{F}_p[[t]]) \to \mathbb{Q}_\ell \sim \mathbb{C}.$$
Conjecture on local test function

Following ideas of Sakellaridis, it is tempting to conjecture that for every unramified representation $\pi_p$ of $G^\lambda(F_p((t)))$, the trace of $m_\lambda$ on $\pi_v \otimes |\det|^s$ is the local $L$-factor $L_p(s, \pi_p, r_\lambda)$. 
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The function $m_\lambda$ can be transferred to $G^\lambda(\mathbb{Q}_p)$ and the similar spectral statement should be true for $G^\lambda(\mathbb{Q}_p)$. 
Evidences

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Evidences

- It works in the case of Tate, and Godement, Jacquet.
- There is a closely related, but different geometric interpretation of the local test function as sheaf on the formal arc space of $M^\lambda$ that can actually be proved.
Problems in increasing order of difficulty

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- Is there a Poisson summation formula for $M^\lambda(\mathbb{Q})$ in $M^\lambda(\mathbb{A})$?