ON CERTAIN SUM OF AUTOMORPHIC $L$-FUNCTIONS

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In Tate’s thesis \[18\], the local factor at $p$ of abelian $L$-function is constructed as the integral of the characteristic function of $\mathbb{Z}_p^\times$ against a quasi-character of $\mathbb{Q}_p^\times$. This construction has been generalized by Tamagawa, Godement and Jacquet in \[17, 8\] to principal $L$-functions of $GL_n$ with the characteristic function of the space of integral matrices as test function. General automorphic $L$-function depends on a representation of the dual group. For each representation of the dual group, one can construct a function depending on a complex parameter $s$ whose trace on each unramified representation is its local $L$-factor. The main contribution of this paper is a description of the support of this test function via Vinberg’s theory of algebraic monoids \[19\]. Prior to us, efforts have been made in this direction, notably by Braverman and Kazhdan \[4\]. By inserting into the trace formula the product of these functions at almost every place, we would get, at least formally, the sum of $L$-functions considered in \[6\]. We will explain a conjectural geometric interpretation of this sum of $L$-functions, similar to \[7\], improved by the use of monoids.

I’m not fortunate enough to know Professor Piatetski-Shapiro personally, though I have met him once or twice. In a conference in Luminy sometimes around 2004, his wife told me that Professor Piatetski-Shapiro appreciated my works. I keep the fond memory of this comment as one of my proudest mathematical prize. It is with deep admiration for his ideas and his courage that I dedicated this work to his memory.

1. AUTOMORPHIC $L$-FUNCTIONS AS A TRACE

Let $G$ be a split reductive group defined over a global field $K$. Let $\pi$ be a discrete automorphic representation of $G$ i.e. an irreducible sub-representation of $L^2(G(K)\backslash G(\mathbb{A}_K), \chi)$ where $\chi$ is a central character. It is known that $\pi$ can factorized as tensor product $\pi = \bigotimes_v \pi_v$. For every non-archimedean place $\nu$ of $K$, $\pi_v$ is an irreducible admissible representation of $G(K_v)$ which is unramified for almost all $\nu$.

Let me recall that $\pi_v$ is unramified means it contains a non-zero fixed vector of the compact open subgroup $G(\mathfrak{O}_v)$. The space of all such fixed vectors is an irreducible module for the algebra

$$\mathcal{H}_v = C_c^\infty(G(\mathfrak{O}_v) \backslash G(K_v)/G(\mathfrak{O}_v))$$
of compactly supported functions on $G(K_v)$ that are left and right $G(O_v)$-invariant. The multiplication is given by the convolution product with respect to the Haar measure on $G(K_v)$ normalized in the way that $G(O_v)$ has volume one. This is a commutative algebra whose structure is best described with the aid of the Satake isomorphism, [16]. Let $\hat{G}$ denote the Langlands dual group of $G$ that is the complex reductive group whose root data is obtained from the root data of $G$ by exchanging the role of roots and coroots, [10]. With this definition, the algebra $H_v$ can be identified with the algebra $C(\hat{G})$ of regular algebraic functions on $\hat{G}$ that are invariant with respect to its adjoint action. For every Hecke function $\phi \in H_v$, we will denote $\tilde{\phi} \in C(\hat{G})$ the corresponding regular algebraic function on $\hat{G}$.

As a $H_v$-irreducible module, $\pi_v G(O_v)$ has at most one dimension as $C$-vector space. If it is non zero, it defines an algebra homomorphism $\sigma_v : H_v \to C$ that can be identified with a semi-simple conjugacy class in $\hat{G}$ according to the Satake isomorphism. This identification was made so that for every $\phi \in H_v$, the equality

$$\text{tr}(\pi_v(\phi)) = \tilde{\phi}(\sigma_v)$$

holds.

Let $\rho : \hat{G} \to \text{GL}(V_\rho)$ be an irreducible algebraic representation of $\hat{G}$. Following Langlands, we attach to each pair $(\rho, \pi)$ a $L$-function with local factor at unramified places

$$L(s, \rho, \pi_v) = \det(1 - \rho(\sigma_v) q_v^{-s})^{-1}$$

where $q_v$ is the cardinal of the residue field $k_v$ and $s$ is a complex number. We will consider $\det(1 - \rho(\sigma_v) q_v^{-s})^{-1}$ as a rational function of $\sigma \in \hat{G}$. We set

$$\psi_s(\sigma) = \det(1 - \rho(\sigma) q_v^{-s})^{-1}$$

and ask the question whether $\psi_s$ is the Satake transform of some function $\psi_s$ on $G(K_v)$. This function would satisfy the equality

$$\text{tr}(\pi_v(\psi_s)) = L(s, \rho, \pi_v)$$

and would lead to the possibility of expressing $L$-function as trace of certain operator. We can expand $\det(1 - \rho(\sigma) q_v^{-s})^{-1}$ as formal series

$$\det(1 - \rho(\sigma) q_v^{-s})^{-1} = \sum_{n=0}^{\infty} \text{tr}(\text{Sym}^n \rho(\sigma)) q_v^{-ns}$$

which is absolutely convergent for large $\Re(s)$. Let us denote $\psi_n$ be the element of $H_v$ whose Satake transform is the regular invariant function on $\hat{G}$

$$\psi_n(\sigma) = \text{tr}(\text{Sym}^n \rho(\sigma)).$$
Thus the function $\psi$ we seek to define must have the form

$$\psi_s = \sum_{n=0}^{\infty} \psi_n q_v^{-ns}$$

which is absolutely convergent for large $\Re(s)$.

It will be convenient to restrict ourselves to particular cases in which the sum $\sum_{n=0}^{\infty} \psi_n q_v^{-ns}$ is well defined for all $s$ and in particular, the sum $\sum_{n=0}^{\infty} \psi_n$ is well defined. This restriction will not deprive us much of generality.

2. Geometric construction of $\psi$

We seek to define a function $\psi$ on $G(K_v)$ such that

$$\text{tr}(\psi_s, \pi) = \text{tr}(\psi, \pi \otimes \det |^s)$$

where $\det : G \to G_m$ a kind of determinant. We will assume that the kernel $G'$ of $\det : G \to G_m$ is a semi-simple group. Dualizing the exact sequence

$$0 \to G' \to G \to G_m \to 0$$

we get an exact sequence

$$0 \to G_m \to \hat{G} \to \hat{G}' \to 0$$

of groups defined over the field of complex numbers. We will assume that the representation $\rho : \hat{G} \to \text{GL}(V)$ induces on the central $G_m$ an isomorphism upon the central $G_m$ of $\text{GL}(V)$. Under this assumption, the $n$-th symmetric power $\text{Sym}^n \rho : \hat{G} \to \text{GL}(\text{Sym}^n V)$ induces on the central $G_m$ of $\hat{G}$ the $n$-th power of the identity character on the central $G_m$ of $\text{GL}(\text{Sym}^n V)$. It follows that the Hecke function $\psi_n$ whose Satake transform is $\psi_n(\sigma) = \text{tr}(\text{Sym}^n \rho(\sigma))$ is supported on the subset of $G(K_v)$ of elements satisfying $\text{val}(\det(g)) = n$. In particular, different functions $\psi_n$ have disjoint support and therefore, the infinite sum $\psi = \sum_{n=0}^{\infty} \psi_n$ is well defined as function on $G(K_v)$.

All this sounds of course very familiar because it is modeled on the case of principal $L$-function of Tamagawa, Godement and Jacquet [17, 8]. In this case $G = \text{GL}_n$ and $\rho$ is the standard representation of $\hat{G} = \text{GL}_n(\mathbb{C})$. According to Tamagawa, Godement and Jacquet, the function $\psi$ is very simple to describe: it is the restriction of the characteristic function $1_{\text{Mat}_n(\mathbb{O}_v)}$ of $\text{Mat}_n(\mathbb{O}_v)$ from $\text{Mat}_n(K_v)$ to $\text{GL}_n(K_v)$, which generalizes of the function $1_{\mathbb{O}_v}$ appearing in Tate’s thesis.

We may ask whether it is possible to describe our function $\psi$ in a similar way. To put the question differently, we may ask what would play the
role of Mat_n for an arbitrary representation of $\hat{G}$. We will show that Vinberg's theory of algebraic monoids provide a nice conjectural answer to this question.

In Vinberg’s theory, the semi-simple group $G'$ is going to be fixed and $G$ is allowed to be any reductive group of which $G'$ is the derived group. A monoid is an open embedding $G \hookrightarrow M$ into a normal affine scheme $M$ such that the actions of $G$ on itself by left and right translation can be extended as actions on $M$. It can be proved that these two commuting action can be merged into a multiplicative structure on $M$, under the assumption that $M$ is affine and normal, or in other words $M$ is an algebraic monoid. Following Vinberg, we call the GIT quotient $A = M// (G' \times G')$ the abelianization of $M$. The monoid $M$ is said to be flat if the quotient map $M \to A$ is flat and its geometric fibers are reduced. The upshot of Vinberg’s theory is that there is a universal flat monoid for a given derived group $G'$ and every flat monoid with the same derived group $G'$ can be obtained from the universal one by base change over its abelianization $A$.

The universal flat monoid $M^+$ is an affine embedding of $G^+$ where $G^+$ is an extension of a torus $T^+$ by $G'$, $r$ being the rank of $G'$

$$0 \to G' \to G^+ \to T^+ \to 0.$$ 

Let $T'$ be a maximal torus of $G'$. Following Vinberg, we set $G^+ = (G' \times T')/Z'$ where $Z'$ is the center of $G'$ acting diagonally on $G'$ and $T'$. It follows that $T^+ = T'/Z'$ is the maximal torus of the adjoint group that can be identified with $G_{m}^r$ with aid of the set of simple roots $\{\alpha_1, ..., \alpha_r\}$ associated with the choice of a Borel subgroup of $G'$ containing $T'$. The universal abelianization $A^+$ is the obvious toric variety $A_r$ of torus $G_{m}^r$.

For simplicity, we will assume that $G'$ is simply connected from now on. Let $\omega_1, ..., \omega_r$ denote the fundamental weights dual to the simple coroots $\alpha_1^\vee, ..., \alpha_r^\vee$ and let $\rho_i : G \to GL(V_i)$ denote the irreducible representation of highest weight $\omega_i$. This can be extended to $G^+$

$$\rho_i^+ : G^+ \to GL(V_i)$$

by the formula $\rho_i^+(t, g) = \omega_i(w_0 t^{-1}) \rho_i(g)$ where $w_0$ is the long element in the Weyl group $W$ of $G$. The root $\alpha_i : T \to G_m$ will also be extended to $G^+$

$$\alpha_i^+ : G^+ \to G_m$$

given by $\alpha_i^+(t, g) = \alpha_i(t)$. Altogether, these map define a homomorphism

$$(\alpha^+, \rho^+) : G^+ \to G_m^r \times \prod_{i=1}^r GL(V_i).$$
In good characteristics, Vinberg’s universal monoid is defined as the closure of \( G^+ \) in \( \mathbb{A}^r \times \prod_{i=1}^{r} \text{End}(V_i) \).

Back to our group \( G \) which is an extension of \( G_m \) by \( G' \). We are looking for flat monoid \( M \), with \( G \) as the group of invertibles, and whose abelianization is the affine line \( \mathbb{A}^1 \) as toric variety of \( G_m \). By universal property, this amounts to the same as a homomorphism \( \lambda : G_m \rightarrow T^+ \) which can be extended as a regular map \( \mathbb{A}^1 \rightarrow \mathbb{A}^r \) such that

\[
(2.1) \quad G = G^+ \times_{T^+} G_m.
\]

The co-character is the highest weight of the representation \( \rho : \hat{G}' \rightarrow \text{GL}(V) \). The relation \( G = G^+ \times_{T^+} G_m \) derives from the hypothesis that \( \rho \) restricted on the center, induces identity map from the central \( G_m \) of \( \hat{G} \) on the central \( G_m \) of \( \text{GL}(V) \).

The set \( M_n \) of elements \( g \in M(\mathcal{O}_v) \cap G(K_v) \) such that \( \text{val} (\det(g)) = n \) is a compact subset of \( G(K_v) \) which is invariant under left and right actions of \( G(\mathcal{O}_v) \).

**Proposition.** The support of the Hecke function \( \psi_n \) whose Satake transform is \( \tilde{\psi}_n(\sigma) = \text{tr}(\text{Sym}^n \rho(\sigma)) \), is contained in \( M_n \).

Assume from now on that \( K \) is the field of rational functions of certain smooth projective curve \( X \) over a finite field \( \kappa \). For every closed point \( v \in |X| \), we denote \( K_v \) the completion of \( K \) at \( v \), \( \mathcal{O}_v \) its ring of integers and \( \kappa_v \) its residue field. In this case, the set \( M_n \) can be seen as the set of \( \kappa_v \)-points of certain infinite dimensional algebraic variety \( \mathcal{M}_n \). Adequate modification of the geometric Satake theory, due to Ginzburg, Mirkovic and Vilonen \[12\], allows us to define a perverse sheaf \( \mathcal{A}_n \) on \( \mathcal{M}_n \) for which \( \psi_n \) is the function on \( \kappa_v \)-points given by the Frobenius trace. In an appropriate sense, \( \mathcal{A}_n \) is the \( n \)-th symmetric convolution power of \( \mathcal{A}_1 \) just as its Satake transform is the \( n \)-th symmetric tensor power of \( \rho \).

The union \( M = \bigcup_{n=0}^{\infty} M_n \) is the set of \( \kappa_v \)-points of some algebraic variety \( \mathcal{M} \) which is an open subset of the loop space \( LM \) of \( M \). It is tempting to try to glue the perverse sheaves \( \mathcal{A}_n \) on different strata \( M_n \) into a single object. Hints in this directions have been given in \[15\]. In that paper, Sakellaridou also pointed out links between local \( L \)-factors and geometry of spherical varieties. Although Vinberg’s monoids are special instances of spherical varieties, the relation between his construction and ours is not clear for the moment.

3. **Certain Sum of \( L \)-functions**

Let \( X \) be a smooth projective curve defined over the finite field \( \kappa = \mathbb{F}_q \). Let us choose a point \( \infty \in X \), a uniformizing parameter \( \epsilon_\infty \in K_\infty^\times \). Let us...
also choose a nontrivial central homomorphism $\mathbb{G}_m \to G$. The uniformizing parameter $\epsilon_\infty$ defines a central element of $G(A_K)$, still denoted by $\epsilon_\infty$, which satisfies $|\det(\epsilon_\infty)| = q^{-m}$, for some positive integer $m$.

We are interested in automorphic representations $\pi = \otimes_v \pi_v$ as irreducible subrepresentation of

$$L^2(G(F)\epsilon_\infty^\mathbb{Z} \backslash G(A_K) / G(\mathfrak{O}_K))$$

with $\mathfrak{O}_K = \prod_{v \in |X|} \mathfrak{O}_v$. Its partial $L$-function can be represented as a trace

$$L'(s, \rho, \pi) = \prod_{v \neq \infty} L(s, \rho, \pi_v) = \text{tr}(\prod_v \psi_v, \pi \otimes |\det|^s).$$

If we are willing to ignore the continuous spectrum, we would have the equality

$$\sum \pi L'(s, \rho, \pi) + \cdots = \text{tr}(\prod_{v \neq \infty} \psi_v, L^2(G(F)\epsilon_\infty^\mathbb{Z} \backslash G(A_K) / G(\mathfrak{O}_K)) \otimes |\det|^s).$$

We expect that the right hand side can be expanded geometrically as integration of $\prod_{v \neq \infty} \psi_v$ over a diagonal by an appropriate form of the trace formula, and we hope that the geometric side would provide us insights for understanding this sum of $L$-functions. The intended trace formula is beyond the application range of Arthur’s trace formula as our test functions do not have compact support [2]. It seems however in the application range of the trace formula that is being developed by Finis, Lapid and Muller [5].

Assume that $\pi$ is tempered and that it corresponds in the sense of Langlands’ reciprocity to a homomorphism

$$\sigma_\pi : W_K \to \hat{G}$$

where $W_K$ denotes the Weil group of $K$. If we denote $H(\pi)$ the closure of the image of $\sigma_\pi$, the order of the pole at $s = 1$ of $L(s, \rho, \pi)$ will be equal to the multiplicity of the trivial representation of the representation of $H(\pi)$ obtained by restricting $\rho$. Following [11][6], we hope that an investigation of the sum

$$\sum \pi L'(s, \rho, \pi) + \cdots$$

will eventually allow us to break the set of $\pi$ into subsets in which $H(\pi)$ is a given reductive subgroup of $\hat{G}$, up to conjugacy, and therefore will lead us towards Langlands’ functoriality conjecture. It seems reasonable to seek to understand this phenomenon from the geometric expansion of the trace formula for the test function $\prod_{v \neq \infty} \psi_v$. This strategy has been carried out successfully by S. Ali Altug in his PhD thesis [1] for $G = \text{GL}_2$ and $\rho$ the symmetric square.
The geometric expansion can be given a geometric interpretation with aid of certain moduli space of bundles with additional structures. Recall that the moduli stack of principal $G$-bundles over a $X$ is an algebraic stack locally of finite type which will be denoted by $\text{Bun}(G)$. Let $\mathcal{Z}_\phi \backslash \text{Bun}(G)$ be the moduli stack of $G$-bundles over $X$ modulo the equivalence relation $V \sim V(e_\infty)$ where $V(e_\infty)$ is the central twisting of a $G$-bundle $V$ by the line bundle $\mathcal{O}_X(\infty)$ via the chosen central homomorphism $\mathbb{G}_m \to G$. We will denote $[V]$ the point in $\mathcal{Z}_\phi \backslash \text{Bun}(G)$ represented by a $G$-bundle $V$.

A $M$-morphism between two $G$-bundles $V$ and $V'$ is a global section of the twisted space obtained by twisting $M$ by $V$ and $V'$ via the left and right action of $G$ on $M$. We will denote the set of $M$-morphisms from $V$ to $V'$ by $M(V, V')$. The multiplication structure on $M$ allows us to define a composition $M(V, V') \times M(V', V'') \to M(V, V'')$ and thus a category $\text{Bun}(G, M)$ of $G$-bundles with $M$-morphisms. If $G = \text{GL}_n$ and $M = \text{Mat}_n$, $M(V, V')$ is nothing but the pace of linear morphisms of vector bundles from $V$ to $V'$.

We observe that the identity transformation of $V|_{X_{-\infty}}$ can be extended to a $M$-morphism $i_d : V \to V(d e_\infty)$ for all $d \geq 0$. The groupoid $\mathcal{Z}_\phi \backslash \text{Bun}(G)$ can be constructed by formally inverting the $M$-morphisms $i_d$ in the category $\text{Bun}(G, M)$. In the localized category, we have

$$M([V], [V]) = \lim_{d \to \infty} M(V, V(d e_\infty)).$$

Since the determinant $M \to \mathbb{A}^1$ is invariant with respect to $G$-conjugation, the determinant of $\phi \in M([V], [V])$ is well defined as a meromorphic function on $X$ which is regular at $X_{-\infty}$. For every $d$, let $M_d$ denote the moduli stack of pairs $([V], \phi)$ where

$$[V] \in \mathcal{Z}_\phi \backslash \text{Bun}(G) \text{ and } \phi \in M([V], [V])$$

with $\det(\phi) \in H^0(X, \mathcal{O}_X(d \infty))$. This is a algebraic stack locally of finite type. As $d$ varies, these stacks forms an injective system

$$M_0 \hookrightarrow M_1 \hookrightarrow \cdots$$

over the injective system of finite dimensional vector spaces

$$\cdots \hookrightarrow H^0(X, \mathcal{O}_X(d \infty)) \hookrightarrow H^0(X, \mathcal{O}_X((d + 1) \infty)) \hookrightarrow \cdots$$

Let $M$ denote the limit of the inductive system $M_d$.

We expect that the sum of $L$-functions $\Sigma_{d=0}^\infty a_d q^{-d^2}$ where for large enough $d$, the "number" $a_d$ is

$$a_d = \text{tr}(\text{Frob}_q, \Gamma_c(M_d - M_{d-1}, \text{IC}))$$

where IC is denote the intersection complex on $M_d - M_{d-1}$. The word number has been put into quotation marks because it may be infinite as
stated. However, according to the lemma, the "number" $a_d$ can be formally expressed as a sum over $\mathbb{F}_q$-points of $M_d - M_{d-1}$. In order to prove that the trace of Frobenius of the fiber of IC over a $\mathbb{F}_q$-point is equal to the term indexed by that point, we need to prove that singularities of $M_d - M_{d-1}$ are equivalent to singularities in Schubert cells of the affine Grassmannian. Note that similar results have been obtained by Bouthier in his PhD thesis [3]. It is also very interesting to understand how the IC of the strata $M_d - M_{d-1}$ be glued together. To summarize, we expect that the sum of $L$-functions (3.1) can be calculated from the injective system (3.2).

Under the assumption that the derived group $G'$ is simply connected, the ring of invariants functions $k[M]^G$ is a polynomial algebra. Using the generators of $k[M]^G$, we can construct a morphism

$$f_d : M_d \to B_d$$

similar to the Hitchin fibration [9]. It seems to be possible to study the direct image $(f_d)_* Q_\ell$ in a similar way as [13]. Whether the direct images $(f_d)_* Q_\ell$ form an inductive system as $d$ varies, depend on the possibility of glueing the IC on $M_d - M_{d-1}$ together. At any rate, it is desirable to study the asymptotic of the sum of $L$-functions (3.1) via $(f_d)_* Q_\ell$ as $d \to \infty$.

REFERENCES


