1. In this note, we shall describe a classification for automorphic representations of GSp(4), the group of similitudes of four-dimensional symplectic space. The results are part of a project [A3] on the automorphic representations of general classical groups. The monograph [A3] is still in preparation. When complete, it will contain a larger classification of representations, subject to a general condition on the fundamental lemma.

In the case of GSp(4), the standard fundamental lemma for invariant orbital integrals has been established [Ha], [W]. However, a natural variant of the standard fundamental lemma is also needed. To be specific, the theorem we will announce here is contingent upon a fundamental lemma for twisted, weighted, orbital integrals on the group GL(4) \times GL(1) (relative to a certain outer automorphism). This has not been established. However, it seems likely that by methods of descent, perhaps in combination with other means, one could reduce the problem to known cases of the standard fundamental lemma. (The papers [BWW], [F], and [Sc] apply such methods to the twisted analogue of the fundamental lemma, but not its generalization to weighted orbital integrals.)

The general results of [A3] are proved by a comparison of spectral terms in the stabilized trace formula. It is for the existence of the stabilized trace formula [A2] (and its twisted analogues) that the fundamental lemma is required. However, any discussion of such methods would be outside the scope of this paper. We shall be content simply to state the classification for GSp(4) in reasonably elementary terms. The paper will in fact be somewhat expository. We shall try to motivate the classification by examining the relevant mappings from a Galois group (or some extension thereof) to the appropriate L-groups.

Representations of the group GSp(4) have been widely studied. The papers [HP], [Ku], [Y], [So], and [Ro] contain results that were established directly for GSp(4). Results for groups of higher rank in [CKPS] and [GRS] could also be applied (either now or in the near future) to the special case of GSp(4).

2. Let \( F \) be a local or global field of characteristic zero. If \( N \) is any positive integer, the general linear group GL(\( N \)) has an outer automorphism

\[
g \rightarrow g^\vee = {}^t g^{-1}, \quad g \in \text{GL}(N),
\]

over \( F \). Standard classical groups arise as fixed point groups of automorphisms in the associated inner class. In this paper, we shall be concerned with classical groups
of similitudes. We therefore take the slightly larger group
\[ \tilde{G} = \text{GL}(N) \times \text{GL}(1) \]
over $F$, equipped with the outer automorphism
\[ \alpha : (x, y) \mapsto (x^\vee, \det(x)y), \quad x \in \text{GL}(N), \quad y \in \text{GL}(1). \]
The corresponding complex dual group
\[ \hat{\tilde{G}} = \text{GL}(N, \mathbb{C}) \times \mathbb{C}^* \]
comes with the dual outer automorphism
\[ \hat{\alpha} : (g, z) \mapsto (g^\vee z, z), \quad g \in \text{GL}(N, \mathbb{C}), \quad z \in \mathbb{C}^*. \]
Motivated by Langlands’s conjectural parametrization of representations, we consider homomorphisms
\[ \tilde{\psi} : \Gamma_F \to \hat{\tilde{G}}, \]
from the Galois group $\Gamma_F = \text{Gal}(\bar{F}/F)$ into $\hat{\tilde{G}}$. Each $\tilde{\psi}$ is required to be continuous, which is to say that it factors through a finite quotient $\Gamma_E/F = \text{Gal}(E/F)$ of $\Gamma_F$, and is to be taken up to conjugacy in $\hat{\tilde{G}}$. Any $\tilde{\psi}$ may therefore be decomposed according to the representation theory of finite groups. We first write
\[ \tilde{\psi} = \psi \oplus \chi : \sigma \mapsto \psi(\sigma) \oplus \chi(\sigma), \quad \sigma \in \Gamma_F, \]
where $\psi$ is a (continuous) $N$-dimensional representation of $\Gamma_F$, and $\chi$ is a (continuous) 1-dimensional character on $\Gamma_F$. We then break $\psi$ into a direct sum
\[ \psi = \ell_1 \psi_1 \oplus \cdots \oplus \ell_r \psi_r, \]
for inequivalent irreducible representations
\[ \psi_i : \Gamma_F \to \text{GL}(N_i, \mathbb{C}), \quad 1 \leq i \leq r, \]
and multiplicities $\ell_i$ such that
\[ N = \ell_1 N_1 + \cdots + \ell_r N_r. \]
We shall be interested in maps $\tilde{\psi}$ that are $\hat{\alpha}$-stable, in the sense that the homomorphism $\hat{\alpha} \circ \tilde{\psi}$ is conjugate to $\tilde{\psi}$. It is clear that $\tilde{\psi}$ is $\hat{\alpha}$-stable if and only if the $N$-dimensional representation
\[ \psi^\vee \otimes \chi : \sigma \mapsto \psi(\sigma)^\vee \chi(\sigma), \quad \sigma \in \Gamma_F, \]
is equivalent to $\psi$. This in turn is true if and only if there is an involution $i \leftrightarrow i^\vee$ on the indices such that for any $i$, the representation $\psi_i^\vee \otimes \chi$ is equivalent to $\psi_i$, and $\ell_i$ equals $\ell_i^\vee$. We shall say that $\tilde{\psi}$ is $\hat{\alpha}$-discrete if it satisfies the further constraint that for each $i$, $i^\vee = i$ and $\ell_i = 1$. 

Suppose that $\tilde{\psi}$ is $\tilde{\alpha}$-discrete. Then

$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$

for distinct irreducible representations $\psi_i$ of degree $N_i$ that are $\chi$-self dual, in the sense that $\psi_i$ is equivalent to $\psi_i^\vee \otimes \chi$. We write

$$\psi_i(\sigma) \otimes \chi(\sigma) = A_i^{-1} \psi_i(\sigma) A_i, \quad \sigma \in \Gamma_F, \quad 1 \leq i \leq r,$$

for fixed intertwining operators $A_i \in \text{GL}(N_i, \mathbb{C})$. Applying the automorphism $g \mapsto g^\vee$ to each side of the last equation, we deduce from Schur’s lemma that

$$A_i = c_i A_i,$$

for some complex number $c_i$ with $c_i^2 = 1$. The operator $A_i$ can thus be identified with a bilinear form on $\mathbb{C}^n$ that is symmetric if $c_i = 1$ and skew-symmetric if $c_i = -1$. We are of course free to replace any $\psi_i$ by a conjugate

$$B_i^{-1} \psi_i(w) B_i, \quad B_i \in \text{GL}(N_i, \mathbb{C}).$$

This has the effect of replacing $A_i$ by the matrix

$$B_i A_i^{-1} B_i.$$

We can therefore assume that the intertwining operator takes a standard form

$$A_i = \begin{pmatrix} 0 & 1 \\ \vdots & \ddots & \ddots \\ 1 & 0 \end{pmatrix}, \quad \text{if } c_i = 1,$$

and

$$A_i = \begin{pmatrix} 0 & 1 \\ \vdots & \ddots & \ddots \\ -1 & 0 \end{pmatrix}, \quad \text{if } c_i = -1,$$

We shall say that $\psi_i$ is **orthogonal** or **symplectic** according to whether $c_i$ equals 1 or $-1$.

We have shown that the image of the homomorphism $\tilde{\psi}_i = \psi_i \oplus \chi$ is contained in the subgroup

$$\{(g, z) \in \text{GL}(N_i, \mathbb{C}) \times \mathbb{C}^* : g^\vee z = A_i^{-1} g A_i\}$$

of $\text{GL}(N_i, \mathbb{C}) \times \mathbb{C}^*$. If $(g, z)$ belongs to this subgroup, $z$ is the image

$$g \mapsto z = \Lambda(g)$$
of $g$ under a rational character $\Lambda$. The subgroup of $\text{GL}(N_i, \mathbb{C}) \times \mathbb{C}^*$ therefore projects isomorphically onto the subgroup

$$\{ g \in \text{GL}(N_i, \mathbb{C}) : A_i^t g A_i^{-1} g = \Lambda(g) I \}$$

of $\text{GL}(N_i, \mathbb{C})$. This subgroup is, by definition, the group $\text{GO}(N_i, \mathbb{C})$ of orthogonal similitudes if $c_i = 1$, and the group $\text{GSp}(N_i, \mathbb{C})$ of symplectic similitudes if $c_i = -1$. In each case, the rational character $\Lambda$ is called the similitude character of the group. Set

$$N_{\pm} = \sum_{i \in I_{\pm}} N_i, \quad I_{\pm} = \{ i : c_i = \pm 1 \}.$$ 

We then obtain a decomposition

$$\psi = \psi_+ \oplus \psi_-,$$

where $\psi_+$ takes values in a subgroup of $\text{GL}(N_+, \mathbb{C})$ that is isomorphic to $\text{GO}(N_+, \mathbb{C})$, while $\psi_-$ takes values in a subgroup of $\text{GL}(N_-, \mathbb{C})$ that is isomorphic to $\text{GSp}(N_-, \mathbb{C})$. The original representation takes a form

$$\tilde{\psi} = \psi_+ \oplus \psi_- \oplus \chi,$$

in which the two similitude characters satisfy

$$\Lambda(\psi_+(\sigma)) = \Lambda(\psi_-(\sigma)) = \chi(\sigma), \quad \sigma \in \Gamma_F.$$ 

The complex group $\text{GSp}(N_-, \mathbb{C})$ is connected. It is therefore isomorphic to the complex dual group $\hat{G}_-$ of a split group $G_-$ over $F$. If $N_+$ is odd, $\text{GO}(N_+, \mathbb{C})$ is also connected. It is again isomorphic to a complex dual group $\hat{G}_+$, for a split group $G_+$ over $F$. If $N_+ = 2n_+$ is even, however, the mapping

$$\nu : g \mapsto \Lambda(g)^{-n_+} \det(g), \quad g \in \text{GO}(N_+, \mathbb{C}),$$

is a nontrivial homomorphism from $\text{GO}(N_+, \mathbb{C})$ to the group $\{ \pm 1 \}$, whose kernel $\text{SGO}(N_+, \mathbb{C})$ is connected. (See [Ra, §2].) The composition of $\psi_+$ with $\nu$ then provides a homomorphism from $\Gamma_F$ to a group of outer automorphisms of $\text{GO}(N_+, \mathbb{C})$. In this case, we take $G_+$ to be the quasisplit group over $F$ whose dual group is isomorphic to the group $\text{SGO}(N_+, \mathbb{C})$, equipped with the given action of $\Gamma_F$. Having defined $G_+$ and $G_-$ in all cases, we write $G$ for the quotient of $G_+ \times G_-$ whose dual group is isomorphic to the subgroup

$$\hat{G} = \{ (g_+, g_-, z) \in \hat{G}_+ \times \hat{G}_- \times \mathbb{C}^* : \Lambda(g_+) = \Lambda(g_-) = \chi \}$$

of $\hat{G}_+ \times \hat{G}_- \times \mathbb{C}^*$.

The quasisplit groups $G$ over $F$, obtained from $\alpha$-discrete homomorphisms $\tilde{\psi}$ as above, are called the (elliptic, $\alpha$-twisted) endoscopic groups for $\hat{G}$. Any such $G$ is determined up to isomorphism by a partition $N = N_+ + N_-$, and an extension $E$ of $F$ of degree at most two (with $E = F$ unless $N_+$ is even). One sees easily that
there is a natural $L$-embedding
\[ L G = \hat{G} \times \Gamma_F \hookrightarrow L \tilde{G} = (\text{GL}(n, \mathbb{C}) \times \mathbb{C}^*) \times \Gamma_F \]
of $L$-groups. Given the $\hat{\alpha}$-discrete parameter $\tilde{\psi}$, we conclude that the mapping $\sigma \mapsto \tilde{\psi}(\sigma) \times \sigma$ from $\Gamma_F$ to $L \tilde{G}$ factors through a subgroup $L G$, for a unique endoscopic group $G$. The discussion above can also be carried out for $\hat{\alpha}$-stable maps $\tilde{\psi}$ that are not $\hat{\alpha}$-discrete. In this case, however, the mapping $\sigma \mapsto \tilde{\psi}(\sigma) \times \sigma$ could factor through several subgroups $L G$ of $L \tilde{G}$.

The classification of $\hat{\alpha}$-discrete maps $\tilde{\psi}$ has been a simple exercise in elementary representation theory. The group $\Gamma_F$ plays no special role, apart from the property that its quotients of order two parametrize quadratic extensions of $F$. The discussion would still make sense if $\Gamma_F$ were replaced by a product of the group $\text{SL}(2, \mathbb{C})$ with the Weil group $WF$, or more generally, the Langlands group $L F$ of $F$. We recall [Ko, §12] that $L F$ equals $WF$ in the case that $F$ is local archimedean, and equals the product of $WF$ with the group $\text{SU}(2)$ if $F$ is local nonarchimedean. If $F$ is global, $L F$ is a hypothetical group, which is believed to be an extension of $WF$ by a product of compact, semisimple, simply connected groups. We assume its existence in what follows. Then in all cases, $L F$ comes with a projection $w \mapsto \sigma(w)$ onto a dense subgroup of $\Gamma_F$.

Having granted the existence of $L F$, we consider continuous homomorphisms
\[ \tilde{\psi} = \psi \oplus \chi : L F \times \text{SL}(2, \mathbb{C}) \to \hat{G} = \text{GL}(n, \mathbb{C}) \times \mathbb{C}^*. \]
In this context, we also impose the condition that the restriction of $\tilde{\psi}$ to $L F$ be unitary, or equivalently, that the image of $L F$ in $\hat{G}$ be relatively compact. Assume that $\tilde{\psi}$ is $\hat{\alpha}$-stable and $\hat{\alpha}$-discrete. The discussion above then carries over verbatim. We obtain a decomposition
\[ \psi = \psi_1 \oplus \cdots \oplus \psi_r, \]
for distinct irreducible representations
\[ \psi_i : L F \times \text{SL}(2, \mathbb{C}) \to \text{GL}(N_i, \mathbb{C}), \]
such that $\psi_i$ is equivalent to $\psi_i^\vee \otimes \chi$. We shall again say that $\psi_i$ is symplectic or orthogonal, according to whether its image is contained in the subgroup $\text{GSp}(N_i, \mathbb{C})$ or $\text{GO}(N_i, \mathbb{C})$ of $\text{GL}(N_i, \mathbb{C})$. Combining the symplectic and orthogonal components as before, we see that $\tilde{\psi}$ factors through a subgroup $L G$ of $L \tilde{G}$, for a unique (elliptic, $\alpha$-twisted) endoscopic group $G$ for $\hat{G}$.

We are working with a product $L F \times \text{SL}(2, \mathbb{C})$, in place of the original group $\Gamma_F$. This means that the irreducible components of $\psi$ decompose into tensor products
\[ \psi_i = \mu_i \otimes v_i, \quad 1 \leq i \leq r, \]
for irreducible representations $\mu_i : L F \to \text{GL}(m_i, \mathbb{C})$ and $v_i : \text{SL}(2, \mathbb{C}) \to \text{GL}(n_i, \mathbb{C})$ such that $N_i = m_i n_i$. Any irreducible representation
of $\text{SL}(2, \mathbb{C})$ is automatically self dual. This means that for any $i$, $\mu_i$ is equivalent to $\mu_i \otimes \chi$. Moreover, the representation $\psi_i$ of $\text{SL}(2, \mathbb{C})$ is symplectic or orthogonal according to whether it is even or odd dimensional. It follows that $\psi_i$ is symplectic if and only if either $\mu_i$ is symplectic and $\psi_i$ is odd dimensional, or $\mu_i$ is orthogonal and $\psi_i$ is even dimensional.

3. We have classified the $\hat{\alpha}$-discrete representations of the group $L_F \times \text{SL}(2, \mathbb{C})$ in order to motivate a classification of symplectic automorphic representations. At this point, we may as well specialize to the case that $\psi$ is purely symplectic, which is to say that the corresponding group $\hat{G}$ is purely symplectic. We assume henceforth that $N$ is even, and that $G$ is the split group over $F$ such that $\hat{G}$ is isomorphic to $\text{GSp}(N, \mathbb{C})$. Then $G$ is isomorphic to the general spin group

$$\text{GSpin}(N+1) = (\text{Spin}(N+1) \times \mathbb{C}^*)/\{\pm 1\}$$

over $F$. Our ultimate concern will in fact be the special case that $N$ equals 4. In this case, there is an exceptional isomorphism between $\text{GSpin}(N+1)$ and $\text{GSp}(N)$, so that $G$ is isomorphic to the group $\text{GSp}(4)$ of the title.

For the given group $G \cong \text{GSpin}(N+1)$, we write $\Psi(G) = \Psi(G/F)$ for the set of continuous homomorphisms $\psi$ from $L_F \times \text{SL}(2, \mathbb{C})$ to $\hat{G}$, taken up to conjugacy in $\hat{G}$, such that the image of $L_F$ is relatively compact. For any such $\psi$, we set

$$\chi(w) = \Lambda(\psi(w, u)), \quad w \in L_F, \quad u \in \text{SL}(2, \mathbb{C}),$$

where $\Lambda: \hat{G} \rightarrow \mathbb{C}^*$ is the similitude character on $\hat{G}$. Then $\chi = \chi_\psi$ is a one-dimensional unitary character on $L_F$. The correspondence $\psi \mapsto \tilde{\psi} = \psi + \chi_\psi$ gives a bijection from $\Psi(G)$ to the subset of (equivalence classes of) $\hat{\alpha}$-stable representations $\tilde{\psi}$ such that the mapping

$$(w, u) \rightarrow \tilde{\psi}(w, u) \times \sigma(w), \quad w \in L_F, \quad u \in \text{SL}(2, \mathbb{C}),$$

factors through the subgroup $L_G$ of $L_F$. We shall write $\Psi_2(G)$ for the subset of elements $\psi \in \Psi(G)$ such that $\tilde{\psi}$ is $\hat{\alpha}$-discrete. For any unitary 1-dimensional character $\chi$ on $L_F$, we also write $\Psi(G, \chi)$ and $\Psi_2(G, \chi)$ for the subsets of elements $\psi$ in $\Psi(G)$ and $\Psi_2(G)$, respectively, such that $\chi_\psi = \chi$.

If $\psi$ belongs to $\Psi(G)$, we set

$$S_\psi = \text{Cent}_{\hat{G}}(\im(\psi)) = \{ s \in \hat{G} : s \psi(w, u) = \psi(w, u)s, (w, u) \in L_F \times \text{SL}(2, \mathbb{C}) \},$$

and also

$$S_\psi = S_\psi/Z_{\psi}(\hat{G}),$$

where $Z(\hat{G}) \cong \mathbb{C}^*$ is the center of $\hat{G}$. Then $\psi$ belongs to $\Psi_2(G)$ if and only if the connected group $S_\psi^0$ equals $Z(\hat{G})$, which is to say that the group $S_\psi$ is finite modulo $Z(\hat{G})$. It is not hard to compute $S_\psi$ directly in terms of the irreducible components $\psi_i$ of $\psi$. For example, if $\psi$ belongs to the subset $\Psi_2(G)$, and has $r$ components, then $S_\psi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1} \times \mathbb{C}^*$, while $S_\psi$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{-1}$. 
We assume from now on that $F$ is global. We write $V_F$ for the set of valuations of $F$, and $V_{F,\infty}$ for the finite subset of archimedean valuations in $V_F$. The Langlands group $L_F$ is supposed to come with an embedding $L_{F_v} \hookrightarrow L_F$ for each $v \in V_F$. This embedding is determined up to conjugacy in $L_F$, and extends the usual conjugacy classes of embeddings $W_{F_v} \hookrightarrow W_F$ and $\Gamma_{F_v} \hookrightarrow \Gamma_F$. It gives rise to a restriction mapping

$$\psi \rightarrow \psi_v = \psi |_{L_{F_v} \times \text{SL}(2, \mathbb{C})}, \quad \psi \in \Psi(G),$$

from $\Psi(G) = \Psi(G/F)$ to $\Psi(G/F_v)$, which in turn provides an injection $S_{\psi_v} \rightarrow S_\psi$, and a homomorphism $S_{\psi_v} \rightarrow S_\psi$. Consider the special case that $\psi$ is unramified at $v$. This means that $v$ lies in the complement of $V_{F,\infty}$, and that for each $u \in \text{SL}(2, \mathbb{C})$, the function $w_v \rightarrow \psi_v(w_v, u), \quad w_v \in L_{F_v}$, depends only on the image of $w_v$ in the quotient $W_{F_v}/I_{F_v} \cong F_v^*/\mathcal{O}_v^*$

of $W_{F_v} = L_{F_v}$. Following standard notation, we write $\sigma_v$ for a fixed uniformizing element in $F_v^*$. Then $\sigma_v$ maps to a generator of the cyclic group $W_{F_v}/I_{F_v}$, and can also be mapped to the element

$$\begin{pmatrix} |\sigma_v|^{-\frac{1}{2}} & 0 \\ 0 & |\sigma_v|^{-\frac{1}{2}} \end{pmatrix}$$

in $\text{SL}(2, \mathbb{C})$. Composed with $\psi_v$, these mappings yield a semisimple conjugacy class

$$c_v(\psi) = c(\psi_v) = \psi_v \left( \sigma_v, \begin{pmatrix} |\sigma_v|^{-\frac{1}{2}} & 0 \\ 0 & |\sigma_v|^{-\frac{1}{2}} \end{pmatrix} \right)$$

in $\hat{G}$.

The Langlands group is assumed to have the property that its finite dimensional representations are unramified almost everywhere. It follows that any $\psi \in \Psi(G)$ determines a family

$$c(\psi) = \{c_v(\psi) = c(\psi_v): \quad v \notin V_\psi\}$$

of semisimple conjugacy classes in $\hat{G}$, indexed by the complement of a finite subset $V_\psi \supset V_{F,\infty}$ of $V_F$. We note that if $\psi$ belongs to a subset $\Psi_2(G, \chi)$ of $\Psi(G)$, the family of complex numbers $c(\chi) = \{c_v(\chi)\}$ is equal to the image $\Lambda(c(\psi)) = \{\Lambda(c_v(\psi))\}$ of $c(\psi)$ under the similitude character. In general the relationships among the different elements in any family $c(\psi)$ convey much of the arithmetic information that is wrapped up in the homomorphism $\psi$.

There is of course another source of semisimple conjugacy classes in $\hat{G}$, namely automorphic representations. If $\pi$ is an automorphic representation of $G$,
the Frobenius–Hecke conjugacy classes provide a family
\[ c(\pi) = \{ c_v(\pi) = c(\pi_v) : v \notin V_\pi \} \]
of semisimple conjugacy classes in \( \hat{G} \), indexed by the complement of a finite subset \( V_\pi \supset V_{F,\infty} \) of \( V_F \). The elements in \( c(\pi) \) are constructed in a simple way from the inducing data attached to unramified constituents \( \pi_v \) of \( \pi \). (See [B], for example.)

Now a one dimensional character \( \chi \) of \( L_F \) amounts to an idèle class character of \( F \), and this in turn can be identified with a character on the center of \( G(\mathbb{A}) \). Let \( L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A}), \chi) \) be the space of \( \chi \)-equivariant, square integrable functions on \( G(F) \setminus G(\mathbb{A}) \) that decompose discretely under the action of \( G(\mathbb{A}) \). We write \( \Pi_2(G, \chi) \) for the set of equivalence classes of automorphic representations of \( G \) that are constituents of \( L^2_{\text{disc}}(G(F) \setminus G(\mathbb{A}), \chi) \). If \( \pi \) belongs to \( \Pi_2(G, \chi) \), the family \( c(\chi) \) is equal to the image \( \Lambda_1(\Lambda(c(\pi))) = \{ \Lambda(c_v(\pi)) \} \) of \( c(\pi) \) under \( \Lambda \). In general, the relationships among the different elements in any family \( c(\pi) \) convey much of the arithmetic information that is wrapped up in the automorphic representation \( \pi \).

The following conjecture was an outgrowth of Langlands’s conjectural theory of endoscopy. We have stated it here somewhat informally. A more precise assertion, which applies to any group, is given in [A1] and [AG].

**Conjecture.** (i) For any \( \psi \), there is a canonical mapping \( \pi \to \psi \) from \( \Pi_2(G, \chi) \) to \( \Psi_2(G, \chi) \) such that
\[ c(\pi) = c(\psi), \quad \pi \in \Pi_2(G, \chi). \]

(ii) Any fiber of the mapping is of the form
\[ \{ \pi \in \Pi_2(G, \chi) : c(\pi) = c(\psi) \}, \quad \psi \in \Psi_2(G, \chi), \]
and can be characterized explicitly in terms of the groups \( S_{\psi_v} \), the diagonal image of the map
\[ S_\psi \to \prod_v S_{\psi_v}, \]
and a character
\[ \epsilon_\psi : S_\psi \to \{ \pm 1 \} \]
attached to certain symplectic root numbers [A1, §8].

4. The conjecture describes a classification of the automorphic \( \chi \)-discrete spectrum of \( G \) in terms of mappings
\[ \psi = \psi_1 \oplus \cdots \oplus \psi_r = (\mu_1 \otimes v_1) \oplus \cdots \oplus (\mu_r \otimes v_r) \]
in \( \Psi_2(G, \chi) \). It is the simplest way to motivate what one might try to prove. The conjecture would actually be very difficult to establish in the form stated above (and in [A1] and [AG]). Indeed, one would first have to establish the existence
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and fundamental properties of the global Langlands group $L_F$. However, there is a natural way to reformulate the conjecture as a classification of automorphic representations of $G$ in terms of those of general linear groups. The idea is to reinterpret the “semisimple” constituents $\mu_i$ of $\psi$.

We may as well keep the id`ele class character $\chi$ fixed from this point on. In the formulation above, $\mu_i$ stands for an irreducible unitary representation of $L_F$ of dimension $m_i$ that is $\chi$-self dual. The main hypothetical property of $L_F$ is that its irreducible unitary representations of dimension $m$ should be in canonical bijection with the unitary cuspidal automorphic representations of GL($m$). We could therefore bypass $L_F$ altogether by interpreting $\mu_i$ as an automorphic representation. From now on, $\mu_i$ will stand for a unitary, cuspidal automorphic representation of GL($m_i$) that is $\chi$-self dual, in the sense that the representation

$$x \to \mu_i(x^\vee)\chi(\det x), \quad x \in \text{GL}(m_i, \mathbb{A}),$$

is equivalent to $\mu_i$. As an automorphic representation of GL($m_i$), $\mu_i$ comes with a family

$$c(\mu_i) = \{c_v(\mu_i): \quad v \notin V_{\mu_i}\}$$

of Frobenius–Hecke conjugacy classes in GL($m_i$, $\mathbb{C}$). The family satisfies

$$c_v(\mu_i)^{-1}c_v(\chi) = c_v(\mu_i), \quad v \notin V_{\mu_i},$$

since $\mu_i$ is $\chi$-self dual, and it determines $\mu_i$ uniquely, by the theorem of strong multiplicity one.

As an example, we shall describe the classification of $\chi$-self dual, unitary, cuspidal automorphic representations of GL(2).

**Example.** Suppose that $E$ is a quadratic extension of $F$, and that $\theta$ is an id`ele class character of $E$. We assume that $\theta$ is not fixed by $\text{Gal}(E/F)$. Then there is a (unique) $\chi$-self dual, unitary, cuspidal automorphic representation $\mu = \mu(\theta)$ of GL(2) such that

$$c(\mu) = \{c_v(\mu) = \rho(c_v(\theta)): \quad v \notin V_{\theta}\},$$

where $\theta$ is regarded as an automorphic representation of the group $K_E = \text{Res}_{E/F}(\text{GL}(1))$, and $\rho$ is the standard two dimensional representation of $E^*$. Conversely, suppose that $\mu$ is a $\chi$-self dual, unitary, cuspidal automorphic representation of GL(2). We write $\chi_{\mu}$ for the central character of $\mu$. It follows from the definitions that $\chi_{\mu}^2 = \chi^2$, or in other words, that the id`ele class character $\eta_{\mu} = \chi_{\mu}\chi^{-1}$ of $F$ has order one or two. If $\eta_{\mu} \neq 1$, it is known that $\mu$ equals $\mu(\theta)$, for an id`ele class character $\theta$ of the class field $E$ of $\eta_{\mu}$. In this case, we shall say that $\mu$ is of orthogonal type. If $\eta_{\mu} = 1$, $\mu$ is to be regarded as symplectic, for the obvious reason that GL(2) $\cong$ GSp(2). This is really the generic case, since if $\mu$ is any automorphic representation with central character $\chi_{\mu}$ equal to $\chi$, $\mu$ is automatically $\chi$-self dual.
For the group $GL(4)$, some $\chi$-self dual, unitary, cuspidal automorphic representations can also be described in relatively simple terms. They are given by the following theorem of Ramakrishnan.

**Theorem:** [Ra]. Let $E$ be an extension of $F$ of degree at most two, and set

$$H_E = \begin{cases} GL(2) \times GL(2), & \text{if } E = F, \\ Res_{E/F}(GL(2)), & \text{if } E \neq F. \end{cases}$$

Let $\rho$ be the homomorphism from $^k H_E$ to the group $GL(4, \mathbb{C}) \cong GL(M_2(\mathbb{C}))$ defined by setting

$$\rho(g_1, g_2)X = g_1X^t g_2, \quad g_1, g_2 \in GL(2, \mathbb{C}),$$

and

$$\rho(\sigma)X = \begin{cases} X, & \text{if } \sigma_E = 1, \\ X^t, & \text{if } \sigma_E \neq 1, \end{cases}$$

for any $X \in M_2(\mathbb{C})$, and for any $\sigma \in \Gamma_F$ with image $\sigma_E$ in $\Gamma_{E/F}$. Suppose that $\tau$ is a unitary, cuspidal automorphic representation of $H_E$ that is not a transfer from $GL(2)$ (relative to the natural embedding of $GL(2, \mathbb{C})$ into $^t H_E$), and whose central character is the pullback of $\chi$ (relative to the natural mapping from the center of $H_E$ to $GL(1)$). Then there is a unique $\chi$-self dual, unitary, cuspidal automorphic representation $\mu$ of $GL(4)$ such that

$$c(\mu) = \{c_v(\mu) = \rho(c_v(\tau)): v \notin V_1\}.$$
To complete the definition, one would of course have to be able to say what it means for $\mu_i$ to be of symplectic or orthogonal type. In general, this must be done in terms of whether a certain automorphic $L$-function for $\mu_i$ (essentially the symmetric square or skew-symmetric square) has a pole at $s = 1$. The necessary consistency arguments for such a characterization are inductive, and require higher cases of the fundamental lemma, even when $N = 4$. However, if $m_i$ equals either 2 or 4, we can give an ad hoc characterization. In these cases, we have already defined what it means for $\mu_i$ to be of orthogonal type. We declare $\mu_i$ to be of symplectic type simply if it is not of orthogonal type. This expedient allows us to construct the family $/Psi_2(G, \chi)$ in the case that $N = 4$.

Suppose that $N$ is such that the set $\Psi_2(G, \chi)$ has been defined, and that

$$\psi = (\mu_1 \boxtimes v_1) \boxplus \cdots \boxplus (\mu_r \boxtimes v_r)$$

is an element in this set. For any $i$, $\mu_i$ comes with a family $c(\mu_i)$ of Frobenius–Hecke conjugacy classes in $GL(m_i, \mathbb{C})$. The representation $v_i$ of $SL(2, \mathbb{C})$ gives rise to its own family

$$c(v_i) = \left\{ c_v(v_i) = v_i \begin{pmatrix} |\sigma_v|^\frac{i}{2} & 0 \\ 0 & |\sigma_v|^{-\frac{i}{2}} \end{pmatrix} : v \notin V_{F,\infty} \right\}$$

of conjugacy classes in $GL(n_i, \mathbb{C})$. The tensor product family

$$c(\psi_i) = \{ c_v(\psi_i) = c_v(\mu_i) \otimes c_v(v_i) : v \notin V_{\mu_i} \}$$

is then a family of semisimple conjugacy classes in $GL(N, \mathbb{C})$. This is of course parallel to the family of conjugacy classes constructed with the earlier interpretation of $\psi$ as a representation of $L_F \times SL(2, \mathbb{C})$. We also set

$$S_\psi = (\mathbb{Z}/2\mathbb{Z})^{r-1},$$

as before. We can then define a character

$$\epsilon_\psi : S_\psi \to \{ \pm 1 \}$$

in terms of symplectic root numbers by copying the prescription in [A1, §8].

The local Langlands conjecture has now been proved for the general linear groups $GL(m_i)$ [HT], [He]. We can therefore identify any local component

$$\psi_v = (\mu_{1,v} \boxtimes v_1) \boxplus \cdots \boxplus (\mu_{r,v} \boxtimes v_r)$$

of an element $\psi \in \Psi_2(G, \chi)$ with an $N$-dimensional representation of the group $L_{F_v} \times SL(2, \mathbb{C})$. In the process of proving the classification theorem stated below,
one shows that if $\mu_i$ is either symplectic or orthogonal (in the sense alluded to above), the same holds for the local components $\mu_{i,v}$ (as representations of $L_{F_v}$). It follows that $\psi_v$ can be identified with a homomorphism from $L_{F_v} \times \text{SL}(2, \mathbb{C})$ into $\hat{G}$. In particular, we can define the groups $S_{\psi_v}$ and $S_{\psi_v}$ as before, in terms of the centralizer of the image of $\psi_v$. Moreover, there is a canonical homomorphism $s \to s_v$ from $S_{\psi}$ to $S_{\psi_v}$. The groups $S_{\psi}$ and $S_{\psi_v}$ are always abelian, and in fact are products of groups $\mathbb{Z}/2\mathbb{Z}$.

5. We shall now specialize to the case that $N = 4$. Thus, $\hat{G}$ is isomorphic to $\text{GSp}(4, \mathbb{C})$, and

$$G \cong \text{GSpin}(5) \cong \text{GSp}(4).$$

As we have noted, the set $\Psi_2(G, \chi)$ can be defined explicitly in this case in terms of certain cuspidal automorphic representations of general linear groups.

The object of this article has been to announce the following classification theorem for automorphic representations of $G$. The theorem is contingent upon cases of the fundamental lemma that are in principle within reach, and I should also admit, the general results in [A3] that have still to be written up in detail.

**Classification Theorem.** (i) There exist canonical local packets

$$\Pi_{\psi_v}, \quad \psi \in \Psi_2(G, \chi), \quad v \in V_{F_v},$$

of (possibly reducible) representations of the groups $G(F_v)$, together with injections

$$\pi_v \to \xi_{\pi_v}, \quad \pi_v \in \Pi_{\psi_v},$$

from these packets to the associated finite groups $\hat{S}_{\psi_v}$ of characters on $S_{\psi_v}$.

(ii) The automorphic discrete spectrum attached to $\chi$ has an explicit decomposition

$$L^2_{\text{disc}}(G(F) \backslash G(\mathbb{A}), \chi) = \bigoplus_{\psi \in \Psi_2(G, \chi)} \bigoplus_{\pi \in \Pi_{\psi}} \pi$$

in terms of the global packets

$$\Pi_{\psi} = \{\pi = \bigotimes_v \pi_v : \pi_v \in \Pi_{\psi_v}, \xi_{\pi_v} = 1 \text{ for almost all } v\}$$

of (possibly reducible) representations of $G(\mathbb{A})$, and corresponding characters

$$\xi_{\pi} : s \to \prod_v \xi_{\pi_v}(s_v), \quad s \in S_{\psi}, \quad \pi \in \Pi_{\psi},$$

on the groups $S_{\psi}$.

(iii) The global packets

$$\Pi_{\psi}, \quad \psi \in \Psi_2(G, \chi),$$
are disjoint, in the sense that no irreducible representation of \( G(\mathfrak{h}) \) is a constituent of representations in two distinct packets. Moreover, if \( \psi \) belongs to the subset \( \Psi_{ss,2}(G, \chi) \) of elements in \( \Psi_2(G, \chi) \) that are trivial on \( \text{SL}(2, \mathbb{C}) \), the packet \( \Pi_\psi \) contains only irreducible representations. Thus, for any \( \psi \in \Psi_{ss,2}(G, \chi) \), any representation \( \pi \in \Pi_\psi \) occurs in \( L^2_{\text{disc}}(G(F) \backslash G(\mathfrak{h}), \chi) \) with multiplicity 1 or 0.

Remarks. 1. The local packets \( \Pi_\psi \) in part (i) are defined by the endoscopic transfer of characters. More precisely, the characters of representations in \( \Pi_\psi \) are defined in terms of Langlands–Shelstad (and Kottwitz–Shelstad) transfer mappings of functions, and the groups \( S_\psi \). I do not know whether the representations in \( \Pi_\psi \) are generally irreducible. However, in the case that \( \psi \) is unramified at \( v \), one can at least show that the preimage of the trivial character in \( \hat{S}_\psi \) under the mapping \( \pi_v \rightarrow \xi_\pi_v \) is irreducible.

2. If \( \psi \) belongs to the complement of \( \Psi_{ss,2}(G, \chi) \) in \( \Psi_2(G, \chi) \), the representations in the packet \( \Pi_\psi \) are all nontempered. On the other hand, if \( \psi \) belongs to \( \Psi_{ss,2}(G, \chi) \), the generalized Ramanujan conjecture (applied to the groups \( \text{GL}(2) \) and \( \text{GL}(4) \)) implies that the representations in the packet \( \Pi_\psi \) are tempered. Thus, the multiplicity assertion at the end of the theorem pertains to what ought to be the tempered constituents of \( L^2_{\text{disc}}(G(F) \backslash G(\mathfrak{h}), \chi) \). If \( \psi \) is a more general element in \( \Psi_2(G, \chi) \), and if the direct sum of the representations in each local packet \( \Pi_\psi \) is multiplicity free, the irreducible constituents of the representations in \( \Pi_\psi \) also occur with multiplicity 1 or 0.

The multiplicity formula of the theorem is a quantitative description of the decomposition of the discrete spectrum. The general structure of the parameters \( \psi \) also provides useful qualitative information about the spectrum. We shall conclude with a list of the six general families of automorphic representations that occur in the discrete spectrum. In each case, we shall describe the relevant parameters \( \psi \), the corresponding families of Frobenius–Hecke conjugacy classes, the groups \( S_\psi \), and the sign characters \( \varepsilon_\psi \) on \( S_\psi \). (The characters \( \varepsilon_\psi \) are in fact trivial for all but one of the six families.) We shall write \( \nu(n) \) for the irreducible representation of \( \text{SL}(2, \mathbb{C}) \) of dimension \( n \). Observe that the Frobenius–Hecke conjugacy classes

\[
c(\nu(n)) = \left\{ \begin{array}{c} \nu(n) \\ \nu(n) \end{array} \right\} = \left( \begin{array}{ccc} |\sigma_v|^{-\frac{n}{2}} & 0 \\ |\sigma_v|^{-\frac{n+1}{2}} & \ddots \\ 0 & \ddots & |\sigma_v|^{-\frac{n}{2}} \end{array} \right)
\]

of \( \nu(n) \) have positive real eigenvalues. This is in contrast to the case of a unitary, cuspidal automorphic representation \( \mu \) of \( \text{GL}(m) \), which according to the generalized
Ramanujan conjecture, has Frobenius–Hecke conjugacy classes

\[ c(\mu) = \begin{cases} \displaystyle c_v(\mu) = \begin{pmatrix} c^1_v(\mu) & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c^m_v(\mu) \end{pmatrix}, \end{cases} \]

whose eigenvalues lie on the unit circle.

We list the six families according to how they behave with respect to stability (for the multiplicities of representations \( \pi \in \Pi_\psi \)) and the implicit Jordan decomposition (for elements \( \psi \in \Psi_2(G, \chi) \)). I have also taken the liberty of assigning proper names to some of the families, which I hope give fair reflection of their history.

(a) Stable, semisimple (general type)

\[ \psi = \psi_1 = \mu \boxtimes 1, \]

where \( \mu \) is a \( \chi \)-self dual, unitary cuspidal automorphic representation of \( \text{GL}(4) \) that is not of orthogonal type,

\[ c(\psi) = c(\mu) = \begin{cases} \displaystyle c_v(\mu) = \begin{pmatrix} c^1_v(\mu) & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c^m_v(\mu) \end{pmatrix}, \end{cases} \]

\[ S_\psi = 1, \]

\[ \varepsilon_\psi = 1. \]

(b) Unstable, semisimple (Yoshida type \([Y]\))

\[ \psi = \psi_1 \boxplus \psi_2 = (\mu_1 \boxtimes 1) \boxplus (\mu_2 \boxtimes 1), \]

where \( \mu_1 \) and \( \mu_2 \) are distinct, unitary, cuspidal automorphic representations of \( \text{GL}(2) \) whose central characters satisfy \( \chi_{\mu_1} = \chi_{\mu_2} = \chi \),

\[ c(\psi) = c(\mu_1) \oplus c(\mu_2) = \begin{cases} \displaystyle c_v(\mu_1) = \begin{pmatrix} c^1_v(\mu_1) & 0 \\ \cdots & c^m_v(\mu_1) \end{pmatrix}, \end{cases} \]

\[ S_\psi = \mathbb{Z}/2\mathbb{Z}, \]

\[ \varepsilon_\psi = 1. \]

(c) Stable, mixed (Soudry type \([S]\))

\[ \psi = \psi_1 = \mu \boxtimes v(2), \]
where \( \mu = \mu(\theta) \) is a unitary, cuspidal automorphic representation of \( GL(2) \) of orthogonal type with \( \chi_{\mu}^2 = \chi \),

\[
c(\psi) = c(\mu) \otimes c(v(2))
\]

\[
= \begin{cases}
  \begin{pmatrix}
    c_1^1(\mu)|\sigma_v|^{\frac{1}{2}} & 0 \\
    c_2^2(\mu)|\sigma_v|^{\frac{1}{2}} & c_2^1(\mu)|\sigma_v|^{\frac{1}{2}} \\
    0 & c_1^2(\mu)|\sigma_v|^{-\frac{1}{2}}
  \end{pmatrix},
\end{cases}
\]

\[
S_\psi = 1,
\]

\[
\varepsilon_\psi = 1.
\]

(d) Unstable, mixed (Saito, Kurokawa type [Ku])

\[
\psi = \psi_1 \boxplus \psi_2 = (\lambda \boxtimes v(2)) \boxplus (\mu \boxtimes 1),
\]

where \( \lambda \) is an idèle class character of \( F \) and \( \mu \) is a unitary, cuspidal automorphic representation of \( GL(2) \), with \( \lambda^2 = \chi_{\mu} = \chi \),

\[
c(\psi) = (c(\lambda) \otimes c(v(2))) \oplus (c(\mu))
\]

\[
= \begin{cases}
  \begin{pmatrix}
    c_1(\lambda)|\sigma_v|^{\frac{1}{2}} & 0 \\
    c_2(\mu) & c_2(\lambda)|\sigma_v|^{\frac{1}{2}} \\
    0 & c_1(\lambda)|\sigma_v|^{-\frac{1}{2}}
  \end{pmatrix},
\end{cases}
\]

\[
S_\psi = \mathbb{Z}/2\mathbb{Z},
\]

\[
\varepsilon_\psi = \begin{cases}
  1, & \text{if } \varepsilon\left(\frac{1}{2}, \mu \otimes \lambda^{-1}\right) = 1, \\
  \text{sgn}, & \text{if } \varepsilon\left(\frac{1}{2}, \mu \otimes \lambda^{-1}\right) = -1,
\end{cases}
\]

where \( \text{sgn} \) is the nontrivial character on \( \mathbb{Z}/2\mathbb{Z} \).

(e) Unstable, almost unipotent (Howe, Piatetski-Shapiro type [HP])

\[
\psi = \psi_1 \boxplus \psi_2 = (\lambda_1 \boxtimes v(2)) \boxplus (\lambda_2 \boxtimes v(2)),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are distinct idèle class characters of \( F \) with \( \lambda_1^2 = \lambda_2^2 = \chi \),

\[
c(\psi) = (c(\lambda_1) \otimes c(v(2))) \oplus (c(\lambda_2) \otimes c(v(2)))
\]

\[
= \begin{cases}
  \begin{pmatrix}
    c_1(\lambda_1)|\sigma_v|^{\frac{1}{2}} & 0 \\
    c_2(\lambda_2)|\sigma_v|^{\frac{1}{2}} & c_2(\lambda_2)|\sigma_v|^{-\frac{1}{2}} \\
    0 & c_1(\lambda_1)|\sigma_v|^{-\frac{1}{2}}
  \end{pmatrix},
\end{cases}
\]

\[
S_\psi = \mathbb{Z}/2\mathbb{Z},
\]

\[
\varepsilon_\psi = 1.
\]
(f) Stable, almost unipotent (one dimensional type)

\[
\psi = \psi_1 = \lambda \otimes \nu(4),
\]

where \( \lambda \) is an idèle class character of \( F \) with \( \lambda^4 = \chi \),

\[
c(\psi) = c(\lambda) \otimes c(\nu(4)) = \begin{cases} 
  c_v(\lambda)|\sigma_v|^\frac{1}{2} & 0 \\
  0 & c_v(\lambda)|\sigma_v|^{-\frac{1}{2}} \\
  c_v(\lambda)|\sigma_v|^\frac{1}{2} & c_v(\lambda)|\sigma_v|^{-\frac{1}{2}} 
\end{cases},
\]

\[
S_\psi = 1,
\]

\[
\varepsilon_\psi = 1.
\]

REFERENCES

AUTOMORPHIC REPRESENTATIONS OF GSp(4)


