Afternoon Seminar

THE STRUCTURE OF TRACE FORMULAS AND THEIR COMPARISON

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1. Credo. There are now many results on the trace formula and many ideas, so that we can now begin to look beyond the analytic difficulties and attempt to put it in a form suitable for applications. Since obstacles remain, some of the ideas are only tentative. One purpose of the afternoon seminar is to test them in particular cases where the difficulties can be overcome.

In this part of the seminar I want to review the general ideas, at first briefly; state the result towards which it seems all efforts are tending; and then, in the context of $U(3)$ or $SU(3)$, explain the ideas in more detail, and at least partially justify them. However much will be left for later.

I begin by recalling what we have available in the way of results and ideas, introducing some catchwords whose meaning it is one of my purposes to explain.

1. First and foremost, the trace formula of Arthur, both the ordinary and the twisted forms.

2. The observation that the twisted formula can be used for the transfer of automorphic representations from one group to another, due to Saito-Shintani for base change and to Jacquet for the Gelbart-Jacquet transfer from $SL(2)$ to $PGL(3)$.

3. Stabilization.

4. Shelstad's formalism of endoscopic groups in the twisted case.

5. For (3) and (4) one needs the transfer of orbital integrals to endoscopic groups and the fundamental lemma. These represent perhaps the major obstacle at present, but results of Shelstad, Kottwitz, Rogawski, and Kazhdan permit considerable confidence that the transfer is possible and the lemma valid.

6. Hierarchical structure of the trace formula and decomposition of measures.

pairing?

7. This hierarchical structure will be obtained by paring off the contributions from proper parabolic subgroups by a procedure that I refer to as Flicker's trick. It is the necessity of utilizing this device, whose value was first emphasized by Flicker, that forces us to modify the basic identity, using ${}_{\varepsilon}\sigma_1^2$ rather than σ_1^2.

8. The principle of cancellation of singularities. This is a suggestion of Arthur, who may feel that its elevation to the status of a principle is premature.

Recall that to obtain the trace formula we start from the basic identity (modified) and integrate both sides over $G\backslash\mathbb{G}^1$, obtaining on the left $J^T(\phi)$ and on the right $\Theta^T(\phi)$. They both depend on the parameter T and are both distributions in ϕ, in general non-invariant.

The fine $\mathfrak{o}$-expansion will - it is hoped - allow us to decompose $J^T(\phi)$ as a sum

$$J^T(\phi) = \sum_M J_M^T(\phi) \quad ,$$

the sum being over conjugacy classes of Levi subgroups of ε-invariant standard parabolics, and thus over associate classes of ε-invariant parabolics.

We will go into this decomposition in more detail later. For now there are only two points to remark:

(a) Both $J^T(\phi)$ and all $J^T_M(\phi)$ are polynomials in T. The degree of $J^T_M(\phi)$ is $\dim \mathfrak{a}^{\varepsilon}_M/\mathfrak{a}^{\varepsilon}_G$. In particular $J_G(\phi) = J^T_G(\phi)$ is independent of T.

(b) The larger M is the closer J^T_M is to being ε-invariant. In particular J_G is ε-invariant.

The distribution J_G will have a simple form. To describe it, it may be best to fix once and for all a finite set of places S containing all infinite places and all places ramified for G and to assume that outside of S, ϕ_v is the characteristic function of K_v divided by the measure of K_v. Thus we are assuming that

$$\phi(g) = \prod_v \phi_v(g_v)$$

and ϕ is determined by

$$\phi_S = \prod_{v \in S} \phi_v .$$

Let $\mathcal{O} = \mathcal{O}_S$ be the set of conjugacy classes in $G(\mathbb{A}_S)$ with elliptic representatives in $G(\mathbb{Q})$. Then

$$J_G(\phi) = \sum_{\mathcal{O}} c_{\mathcal{O}} \int_{G_\gamma(\mathbb{A}_S \backslash G(\mathbb{A}_S)} \phi_S(g^{-1}\gamma g)\,dg \quad ,$$

γ in $G(\mathbb{Q})$ being a representative of the class $\mathcal{O}$.

There will be a similar decomposition

$$\theta^T(\phi) = \sum_M \theta_M^T(\phi)$$

and conditions (a) and (b) will be satisfied. To describe the form of $\theta_G = \theta_G^T$ we need some simple definitions.

Recalling the presence of ω in the definition of $R(\theta)$ (I now shift to the better afternoon notation replacing ε by θ) we agree to call an automorphic representation θ-invariant if it satisfies

$$(h \longrightarrow \pi(h)) \sim (h \longrightarrow \omega(\theta^{-1}(h))\pi(\theta^{-1}(h))$$

and of type ξ if

$$\pi(z) = \xi(z)I , \qquad z \in \mathbb{Z}_0 \quad .$$

Two θ-invariant automorphic representations π, π' of type ξ will be called projectively equivalent if

$$\pi' = \pi_\chi = \pi \otimes \chi \quad ,$$

χ being a character of G trivial on $\mathbb{Z}_0$ and satisfying

$$\chi(h) = \chi(\theta^{-1}(h)) \quad .$$

Denote the set of such characters by $\mathfrak{X}(\theta, \mathbb{Z}_0) = \mathfrak{X}$.

There will be a countable set Y of projective equivalence classes such that

(c) $$\Theta_G(\phi) = \sum_{y \in Y} d(y) \int_{\mathfrak{X}} \operatorname{tr}(\pi_\chi(\phi)\pi_\chi(\theta))\,d\chi$$

π denoting some arbitrary element of y. The meaning of $\pi_\chi(\theta)$ will be explained later. It should also be observed that the numbers $d(y)$ may not be positive and, indeed, may not even be real.

As observed in Shelstad's lectures we proceed now in two steps. We first stabilize the ordinary trace formula for quasi-split groups inductively. This is going to lead us from the formula

$$\sum_M J_M^T(\phi) = \sum_M \Theta_M^T(\phi)$$

to a formula

$$\sum_M SJ_M^T(\phi) = \sum_M S\Theta_M^T(\phi) \quad .$$

If H is a cuspidal endoscopic group for G then, G being quasi-split every associate class $\mathfrak{P}$ for H determines one for G. Namely an element of the class has a Levi factor M and the center of M has a maximal split torus A which can be transferred to a torus A' in G. The centralizer of A' is the Levi factor M' of a parabolic whose class $\mathfrak{P}'$ is the image of $\mathfrak{P}$. We write $\mathfrak{P} = \mathfrak{P}_H$ or $M = M_H$ and $\mathfrak{P}' = \mathfrak{P}_G$ or $M' = M_H$ and write $\mathfrak{P}_H \longrightarrow \mathfrak{P}_G$ or $M_H \longrightarrow M_G$.

We set

$$SJ_{M_G}^T(\phi) = J_{M_G}^T(\phi) - \sum_H{}' \iota(G,H) \sum_{M_H \to M_G} SJ_{M_H}^T(\phi^H) \quad ,$$

the prime indicating that we sum over all cuspidal endoscopic groups (or better data) except G itself and ϕ^H being a function associated to ϕ by transfer of orbital integrals.

In the same way we define

$$S\Theta^T_{M_G}(\phi) = \Theta^T_{M_G}(\phi) - \sum_H{}' \iota(G,H) \sum_{M_H \to M_G} S\Theta^T_{M_H}(\phi^H) \quad .$$

Since all the H are cuspidal they have the same split center as G. Thus (c) will continue to hold for $S\Theta_G(\phi) = S\Theta^T_G(\phi)$ and (a) and (b) will hold for the $SJ^T_M(\phi)$ and the $S\Theta^T_M(\phi)$. Moreover the inductive definition has been so made that

$$\sum_M SJ^T_M(\phi) = \sum S\Theta^T_M(\phi)$$

holds.

We are interested in the distribution $S\Theta_G$ and we would like to know in particular that it is stable. In this case $G = G^*$ is already quasi-split, but $\phi^* = \phi^{G^*}$ is not necessarily equal to ϕ. It need only have the same stable orbital integrals as ϕ, and the assertion that $S\Theta_G$ is stable is the assertion that

$$S\Theta_G(\phi^*) = S\Theta_G(\phi)$$

for all choices of ϕ^*.

The plan of attack, and we will see how it works out in particular cases, is to show that SJ_G, which is a sum of orbital integrals, is in fact a sum of stable orbital integrals and thus that

$$SJ_G(\phi^*) = SJ_G(\phi) \quad .$$

This leaves us with the equality of

(A) $$\sum_{M\neq G} (SJ_M^T(\phi) - SJ_M^T(\phi^*)) - \sum_{M\neq G} (S\Theta_M^T(\phi) - S\Theta_M^T(\phi^*))$$

and

(B) $$S\Theta_G(\phi) - S\Theta_G(\phi^*) \quad .$$

The idea is to add one unramified place v_0 to S, working then with $S' = S\vee\{v_0\}$ rather than S, and to take $\phi_{v_0} = \phi^*_{v_0}$ to be an element of the Hecke algebra $H = H(G, \mathbb{Q}_v)$. We then prove the equality of (A) and (B) by treating them as linear forms on H, substituting finally the identity of the Hecke algebra for ϕ_{v_0} to obtain the identity desired.

This is an argument already used to prove base change for GL(2). The Hecke algebra has an involution $\phi_{v_0} \longrightarrow \tilde{\phi}_{v_0} : g \longrightarrow \bar{\phi}_{v_0}(g^{-1})$. The linear forms (A) and (B) may be represented by measures on the set $\mathfrak{A}$ of homomorphisms λ of the Hecke algebra into $\mathbb{C}$ satisfying

$$\lambda(\phi_{v_0}) = \lambda(\tilde{\phi}_{v_0}) \qquad \text{(?)}$$

for all ϕ_{v_0}.

It follows from (c), applied to $S\Theta_G$, that the measure attached to (B) is of Lebesgue type and dimension equal to $\dim \mathfrak{X}$, which is often zero, whereas one can expect to prove that (A) is a sum of measures of Lebesgue type and dimension between $\dim \mathfrak{X}+1$ and $\dim \mathfrak{X} + \dim \mathfrak{a}_{M_0}/\mathfrak{a}_G$. The conclusion must be that (A) and (B) are separately 0.

At the moment I only have a clear idea how to do this when G is of rational rank 1 and quasi-split, but this will do for the purposes of this seminar. In this case, as we shall see

$$SJ^T_{M_0}(\phi) = SJ_{M_0}(\phi^T_{M_0})$$

where $\phi^T_{M_0}$ is a function on $\mathbb{M}_0$. In the same way

$$SJ^T_{M_0}(\phi^*) = SJ_{M_0}(\phi^{*T}_{M_0}) \quad .$$

Neither ϕ^T_M nor ϕ^{*T}_M will be smooth in general. However the difference $\phi^T_{M_0} - \phi^{*T}_{M_0}$ will be smooth (cancellation of singularities). So we can apply the trace formula on M_0 to the difference obtaining

$$SJ^T_{M_0}(\phi) - SJ^T_{M_0}(\phi^*) = S\Theta_{M_0}(\phi^T_{M_0} - \phi^{*T}_{M_0}) \quad .$$

Then it will be easy to show that all three linear forms

$$\phi_{v_0} \longrightarrow S\Theta_{M_0}(\phi^T_{M_0} - \phi^{*T}_{M_0})$$

$$\phi_{v_0} \longrightarrow S\Theta^T_{M_0}(\phi)$$

$$\phi_{v_0} \longrightarrow S\Theta^T_{M_0}(\phi^*)$$

are given by measures of Lebesgue type and dimension equal to $\dim \mathfrak{X}+1$.

The stable trace $S\Theta_G(\phi)$ once defined we can at least state what appears to be our final goal. Thus for any group G and any θ we want to show that

$$* \qquad \boxed{\Theta_G(\phi) = \sum_H \iota(G,\theta,H) S\Theta_H(\phi^H)} \quad .$$

The sum is over all cuspidal endoscopic groups for the pair (G, θ).

The proof will of course be about the same. One will show directly, or almost directly, that

$$\text{(C)} \qquad J_G(\phi) = \sum_H \iota(G,\theta,H) SJ_H(\phi^H)$$

and then apply cancellation of singularities and decomposition of measures.

At least one extra difficulty will arise. For example, for the ordinary trace formula the term

$$\mathrm{meas}(G \backslash \mathbb{G}^1)\phi(1)$$

will occur on the left side of (C) and the term

$$\mathrm{meas}(G^* \backslash \mathbb{G}^{*1})\phi^{G^*}(1)$$

will occur on the right, G^* being the quasi-split form of G. The relation between $\phi(1)$ and $\phi^{G^*}(1)$ will be simple, presumably

$$\phi^{G^*}(1) = \phi(1) \quad .$$

Thus to achieve cancellation we will need to show that

$$\mathrm{meas}(G\backslash\mathbb{G}^{1}) = \mathrm{meas}(G^{*}\backslash\mathbb{G}^{*1}) \quad .$$

In some cases this will be known from results on Tamagawa numbers, but it will be preferable to derive it from the trace formula itself, by an elaboration of the measure-theoretic arguments.

Shelstad has explained in her lectures the meaning of the identity * for the twisted trace formula arising from base change for $U(1)$. In this case there are several endoscopic groups, all isomorphic to $U(1)$. Some other cases of the identity are implicit in the literature. If G is the multiplicative group of a quaternion algebra the only cuspidal endoscopic group is $GL(2)$. So the identity is quite simple, and was used in effect in §16 of Jacquet-Langlands. In general if G is the multiplicative group of a division algebra of degree n^2 then there is only one cuspidal endoscopic group, namely $GL(n)$ and weak forms of the identity have been used by Deligne-Kazhdan and Rogawski. For $SL(2)$ there are many cuspidal endoscopic groups, $SL(2)$ itself and all anisotropic tori. For base change for $GL(n)$ there is only one cuspidal endoscopic group and that is $GL(n)$. For $U(3)$ or $SU(3)$ there are, as we have seen, more than one cuspidal endoscopic group. The consequences of this are, as we shall see, quite fascinating.

There are two papers which explore, in a somewhat tentative way but for general groups, the consequences and meaning of * for the ordinary trace formula:

1. J. Arthur, On some problems suggested by the trace formula.

2. R. Kottwitz, Stable trace formula: cuspidal tempered terms.

Our purpose at the moment is however to concentrate on U(3) and SU(3) and to see whether the plan of attack outlined here is feasible or nothing but a pipe dream.