Lecture 4

ABSOLUTE CONVERGENCE OF THE COARSE $\mathcal{O}$-EXPANSION

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4.1. Where the alternating sum is used.

We fix once for all a pair of parabolics $P_1 \subset P_2$. We consider P such that $P_1 \subset P \subset P_2$. We need the

LEMMA 4.1.1. *If* $\sigma_1^2(H(x) - T)F_P^1(x, T) \neq 0$ *then there exist* $\delta \in P_1$ *such that*

(i) $\alpha(H(\delta x) - T) > 0 \qquad \forall \alpha \in \Delta_0^2 - \Delta_0^1$

(ii) $\alpha(H(\delta x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^2$.

First of all, for any $\delta \in P_1$ we have

$$\alpha(H(\delta x) - T) > 0 \qquad \forall \alpha \in \Delta_1^2$$

if $\sigma_1^2(H(\delta x) - T) = \sigma_1^2(H(x) - T) \neq 0$. Now choose $\delta \in P_1$ such that $\delta x \in \mathfrak{S}_P^1(T_0, T)$ then

$$\varpi(H(\delta x) - T) \leq 0 \qquad \forall \varpi \in \hat{\Delta}_0^1 \ ;$$

the Lemma 3.2.2 above yields the assertion (i). We know moreover that

$$\alpha(H(\delta x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^P$$

but Δ_0^P contains Δ_0^1 and since $T - T_0 \in \mathfrak{a}_0^+$ assertion (ii) follows. □

COROLLARY 4.1.2. *If* $P_1 \subset P \subset P_2$

$$\sigma_1^2 F_P^1 = \sigma_1^2 F_2^1 \ . \quad \square$$

Now assume that P is ε-invariant; let M be the unique Levi-component of P containing M_0 and N the unipotent radical. Let $\gamma \in P$, $n \in \mathbb{N}$, $x \in \mathbf{G}$, we need the

LEMMA 4.1.3. Provided T is sufficiently regular, and

$$\sigma_1^2(H(x) - T)F_2^1(x, T)\phi(x^{-1}n^{-1}\gamma\varepsilon(x)) \neq 0$$

$\gamma \in R$ the smallest ε-invariant parabolic containing P_1.

Our assertion is invariant by the transformation $x \longmapsto \delta x$ for $\delta \in P_1$ and hence we may assume that x satisfies the following inequalities, if the above expression is not zero:

(i) $\alpha(H(x) - T) > 0 \qquad \forall \alpha \in \Delta_0^2 - \Delta_0^1$

(ii) $\alpha(H(x) - T_0) > 0 \qquad \forall \alpha \in \Delta_0^2$.

Now write $x = n^* n_* mak$ with $n^* \in \mathbb{N}_2$, $n_* \in \mathbb{N}_0^2 = \mathbb{N}_0 \cap \mathbb{M}_2$, $m \in \mathbb{M}_0^1$, $a \in A_0(\mathbb{R})^o$ and $k \in K$. Since we are free to modify n we may assume $n^* = 1$; since we may change γ to $\delta^{-1}\gamma\varepsilon(\delta)$ with $\delta \in P_0$ we may assume that $n_* m$ remains in a compact set. Now since x verifies the inequalities (i) and (ii) we see that $a^{-1}n_* ma = a^{-1}n_* a.m$ can be assumed to remain in a fixed compact set. Since ϕ is compactly supported all we need to prove is the

LEMMA 4.1.4. Let U be a compact in $\mathbb{M}$, assume that $H(a)$ satisfies the inequalities (i) and (ii) then provided T is sufficiently regular $a^{-1}\gamma\varepsilon(a) \in U$ and $\gamma \in M$ implies $\gamma \in P_1$.

Consider the Bruhat decomposition of γ in M:

$$\gamma = \nu w_s \pi$$

with $\gamma \in N_0^P = N_0 \cap M$, $\pi \in P_0 \cap M$ and $s \in \Omega^M$ the Weyl group of M. Let ϖ be a dominant weight of M. For some integer d there is a rational representation ρ of M, of highest weight $d\varpi$ and with highest weight vector v. We have

$$\|\rho(a^{-1}\gamma\varepsilon(a))v\| = \|\rho(a^{-1}\nu a w_s)v\| . \Lambda(a)$$

with

$$\Lambda(a) = e^{d\left\langle \varpi, \varepsilon.H(a) - s.H(a)\right\rangle} .$$

But $\|\rho(a^{-1}\nu a w_s)v\|$ is bounded from below by a constant times $\|\rho(w_s)v\|$; so if $a^{-1}\gamma\varepsilon(a)$ remains in the compact U there exists a real c independent of T such that

$$\Lambda(a) \leq e^c .$$

Now if $\varpi = \varepsilon.\varpi$ we simply have

$$\left\langle \varpi, H(a) - s.H(a)\right\rangle \leq c .$$

We can write

$$H(a) = \sum_{\varpi_\alpha \in \hat{\Delta}_0^P} \lambda_\alpha \varpi_\alpha^\vee + H_P$$

where $H_P \in \mathfrak{a}_P$ the intersection of the kernels of the $\alpha \in \Delta_0^P$, and where the λ_α are subjected to the inequalities

$$\lambda_\alpha > \alpha(T_0) \qquad \forall \alpha \in \Delta_0^P$$

$$\lambda_\alpha > \alpha(T) \qquad \forall \alpha \in \Delta_0^P - \Delta_0^1 .$$

Hence if T is sufficiently large this implies

$$\left\langle \varpi, \varpi_\alpha - s\varpi_\alpha \right\rangle = 0$$

for any $\alpha \in \Delta_0^P - \Delta_0^1$ and any ε-invariant ω. Now assume $\varpi = \sum_{r=0}^{\ell-1} \varepsilon^r . \varpi_0$; since $\varpi_\alpha - s\varpi_\alpha$ is a sum of positive roots we also have

$$\left\langle \varpi_0 , \varpi_\alpha - s\varpi_\alpha \right\rangle = 0$$

for any $\varpi_0 \in \hat{\Delta}_0^P$, and hence $\varpi_\alpha - s\varpi_\alpha = 0$ for all $\alpha \in \Delta_0^P - \Delta_0^1$. This implies $s \in \Omega^1$ the Weyl group of M_1, and hence $\gamma \in P_1$. □□

Recall that R is the minimal ε-invariant parabolic containing P_1. As usual let M_R be the Levi-component containing M_0 and N_R the unipotent radical. Corollary 4.1.2 and Lemma 4.1.3 show that

$$H_1^2(x)_{\mathfrak{o}}^T = F_2^1(x, T)\sigma_1^2(H(x) - T)\omega(x)\varphi_{\mathfrak{o}}(x)$$

where $\varphi_{\mathfrak{o}}(x)$ is the sum over $\gamma \in M_R \cap \mathfrak{o}$ of

$$\sum_{\substack{\varepsilon(P)=P \\ P_1 \subset P \subset P_2}} (-1)^{a_P^\varepsilon} \int_{(N)} \sum_{\eta \in N_R} \phi(x^{-1}n^{-1}\eta\gamma\varepsilon(x))dn .$$

Notice that this expression is non-zero only if $R \subset P_2$.

The exponential mapping is an isomorphism of N_R onto its Lie algebra n_R. Let ψ be a non-trivial additive character of $\mathbf{Q}\backslash\mathbb{A}$. Using the Poisson summation formula we get that

$$\sum_{\eta \in N_R} \phi(x^{-1} n \eta \gamma \varepsilon(x)$$

equals

$$\sum_{Y \in n_R^*} \int_{n_R(\mathbb{A})} \phi(x^{-1} n \ \exp(X) \gamma \varepsilon(x)) \psi(\langle X, Y \rangle) dX$$

where n_R^* is the dual of n_R (as a $\mathbf{Q}$-vector space).

By integration over (N) the contributions of the Y that are non-trivial on $N(\mathbb{A})$ vanish, and we are left with a sum over $n_{R,P}^*$, the subspace of n_R^* orthogonal to n_P the Lie algebra of N.

To take care of the alternating sum over P we need the

LEMMA 4.1.5. <u>Let</u> Q <u>and</u> R <u>be two invariant parabolics then</u>

$$\sum_{\left\{P \,\middle|\, \begin{subarray}{c} R \subset P \subset Q \\ \varepsilon(P) = P \end{subarray}\right\}} (-1)^{a_P^\varepsilon} = \begin{cases} 0 & \text{if } Q \neq R \\ 1 & \text{if } Q = R \end{cases}$$

An ε-invariant parabolic P between R and Q is defined by an ε-invariant subset S of Δ_R^Q. The number of orbits of E in S is $a_R^\varepsilon - a_P^\varepsilon$. The lemma is an immediate consequence of the binomial formula for $(1-1)^d$. □

Let $\tilde{n}_{1,2}$ be the set of elements in n_R^* that belong to one and only one $n_{R,P}^*$ for $P_1 \subset R \subset P \subset P_2$ and $\varepsilon(P) = P$. Using the previous lemma we see that

$$\varphi_{\sigma}(x) = \sum_{\gamma \in M_R \cap \sigma} \sum_{Y \in \tilde{n}_{1,2}} \hat{\phi}(x, Y, \gamma)$$

where

$$\hat{\phi}(x, Y, z) = \int_{n_R(\mathbb{A})} \phi(x^{-1}\exp(X) z\varepsilon(x))\psi(\langle X, Y \rangle) dX \quad .$$

Let Q be the maximal ε-invariant parabolic contained in P_2. Let $p \in \mathbb{P}_1$; since $P_1 \subset R$ normalizes N_R we have

$$\hat{\phi}(p, Y, \gamma) = \hat{\phi}(1, Ad^*(p)Y, p^{-1}\gamma\varepsilon(p))\delta_R(p)$$

where $\delta_R(p)$ is the absolute value of the determinant of $Ad(p)$ on $n_R(\mathbb{A})$. Now let $p \in \mathbb{P}_1 \cap \mathbb{G}^1$. We may write $p = n^* n_* m a$ with $n^* \in \mathbb{N}_2$, $n_* \in \mathbb{N}_1^2$, $m \in \mathbb{M}_1^1$ and $a \in A_1(\mathbb{R})^0$. Since N_2 and $\varepsilon(N_2)$ are in N_Q and $Y \in \tilde{n}_{1,2}$ then $\hat{\phi}$ is independent of n^*. Choose a compact set $\omega_1 \subset \mathbb{N}_1^2$ such that $N_1^2\omega_1 = \mathbb{N}_1^2$. There exists a compact set $\omega_2 \subset \mathbb{M}_1^1$ such that if m is such that $F_2^1(ma, T) = 1$

then $m \in M_1 \cdot \omega_2$; moreover if $\sigma_1^2(H(a) - T) = 1$ we have $\alpha(H(a)) > 0$ for all $\alpha \in \Delta_1^2$; this implies that $a^{-1}\omega_1\omega_2 a = a^{-1}\omega_1 a\omega_2$ remains in a fixed compact set $\omega_3 \subset \mathbf{N}_1^2\mathbf{M}_1^1$.

From this we conclude that the integral over $P_1\backslash\mathbf{G}^1$ of

$$\sum_{\sigma \in \mathcal{O}} |H_1^2(x)^T|$$

is bounded by

$$\int_{p \in \omega_3 K} \left(\int_{A_1(\mathbb{R})^o \cap \mathbb{G}^1} \Xi\,(ap)\delta_1(a)^{-1} da \right) dp$$

where

$$\Xi(x) = F_2^1(x, T)\sigma_1^2(H(x) - T) \sum_{\gamma \in M_R} \sum_{Y \in \tilde{n}_{1,2}} |\hat{\phi}\,(x, Y, \gamma)| \quad .$$

4.2. Final estimates.

Let $\mathfrak{a}_1$ be the Lie algebra of $A_1(\mathbf{R})^o$ and $\mathfrak{a}_1^\varepsilon$ be the set of ε-fixed vectors in $\mathfrak{a}_1$. Let $\mathfrak{b}_1$ be the orthogonal complement of $\mathfrak{a}_1^\varepsilon$ in $\mathfrak{a}_1$. Since ε is of finite order on $\mathfrak{a}_0$ we may and shall assume that the scalar product on $\mathfrak{a}_0$ is ε-invariant. Let π_1 be the orthogonal projection from $\mathfrak{a}_0$ onto $\mathfrak{a}_1$. We have the

LEMMA 4.2.1. Let $H \in \mathfrak{a}_1$; the projection on $\mathfrak{a}_1$ of $\varepsilon(H) - H$ is an injective map from $\mathfrak{b}_1$ into $\mathfrak{a}_1$.

Assume $\pi_1(\varepsilon(H) - H) = 0$, since $H \in \mathfrak{a}_1$ we have $\pi_1(\varepsilon(H)) = H$;

but ε preserves the scalar products and hence $\varepsilon(H) = H$. $\square$

At the end of the preceding section we introduced a function Ξ on $\mathbf{G}$. Assume $\Xi(ap) \neq 0$ with $p \in \omega_3 K$ and $a \in A_1(\mathbf{R})^o$. Since we assume

$$\hat{\phi}(ap, Y, \gamma) = \hat{\phi}(p, Ad^*(a)Y, a^{-1}\gamma\varepsilon(a))\delta_R(a)$$

is not zero, this implies that $a^{-1}\gamma\varepsilon(a)$ remains in a compact set $\omega_4 \subset \mathbf{M}_R$ independent of $p \in \omega_3 K$ and $Y \in n_R^*$ (but depending on the support of ϕ). We moreover assume that $F_2^1(ap, T)\sigma_1^2(H(a) - T) \neq 0$, and then assumptions of Lemma 4.1.4 are fulfilled. This implies that for sufficiently regular T, $\gamma \in P_1 \cap M_R$.

Now for such a γ we have

$$a^{-1}\gamma\varepsilon(a) = \gamma_1 . a^{-1}\eta a . a^{-1}\varepsilon(a)$$

for some $\gamma_1 \in M_1$ and $\eta \in N_1 \cap M_R$. Since $a^{-1}\gamma\varepsilon(a)$ remains in a compact set of $\mathbf{M}_R \cap \mathbf{P}_1$, the projection on $A_1(\mathbf{R})^o$ of $a^{-1}\varepsilon(a)$ must also remain in a compact set. But if $a = a_1 b$ where $a_1 = \exp H_1$ with $H_1 \in \mathfrak{a}_1^\varepsilon$ and $b = \exp H_2$ with $H_2 \in \mathfrak{b}_1$ we have

$$a^{-1}\varepsilon(a) = b^{-1}\varepsilon(b) = \exp(\varepsilon(H_2) - H_2) \quad .$$

Using Lemma 4.2.1 we conclude that b has to remain in a compact set ω_5. Moreover since $\mathfrak{a}_1^\varepsilon \subset \mathfrak{a}_R$ we have

$$a^{-1}\gamma\varepsilon(a) = \gamma_1 . b^{-1}\eta b . b^{-1}\varepsilon(b) \in \omega_4$$

with $b \in \omega_5$. We conclude that the set of γ_1 and η_1 (and hence of γ) that may occur is finite, and in the definition of Ξ the sum over $\gamma \in M_R$ may be restricted to a sum over a finite set E. For $Y \in n_R^*(\mathbb{A})$ we define

$$\theta(Y) = \int_{p \in \omega_5 . \omega_3 . K} \sum_{\gamma \in E} |\hat{\phi}(p, Y, \gamma)| dp \quad .$$

Since $\delta_R(a) = \delta_1(a)$ on A_R all we need to prove is that, given a compact set $\omega_6 \subset \mathfrak{a}_0$, the integral

$$\int_{\mathfrak{z}^\varepsilon \setminus \mathfrak{a}_1^\varepsilon} \sigma_1^2(H-X) \sum_{Y \in \tilde{n}_{1,2}} \theta(Ad^*(\exp H)Y)\, dH$$

is convergent, with an upper bound independent of $X \in \omega_6$. (Here $\mathfrak{z}^\varepsilon$ is the ε-fixed part of the Lie algebra of the split part of the center of G.)

The space $\mathfrak{a}_1^\varepsilon = \mathfrak{a}_R^\varepsilon$ can be further decomposed into a sum

$$\mathfrak{a}_1^\varepsilon = (\mathfrak{a}_R^Q)^\varepsilon \oplus \mathfrak{a}_Q^\varepsilon$$

where Q is the maximal ε-invariant parabolic contained in P_2. Let $H = H_1 + H_2$ be the associated decomposition of $H \in \mathfrak{a}_R^\varepsilon$; we have the

LEMMA 4.2.2. <u>Assume that</u> $H \in \mathfrak{a}_1^\varepsilon$ <u>and</u> $X \in \omega_6$ <u>are such that</u> $\sigma_1^2(H-X) = 1$. <u>Then there exist a constant</u> c <u>independent of</u> X <u>such that</u>

$$\|H_2\| \leq c(1 + \|H_1\|) \quad .$$

Any $\alpha \in \Delta_R - \Delta_R^Q$ is the restriction of some $\alpha' \in \Delta_1 - \Delta_1^2$ and hence we have

$$\alpha(H_2) = \alpha(H-X) - \alpha(H_1) + \alpha(X) < -\alpha(H_1) + \alpha(X) < -\alpha(H_1) + c_1$$

for some constant c_1. For any $\varpi \in \hat{\Delta}_2$ we have

$$\varpi(H_2) = \varpi(H) > \varpi(X)$$

since $\mathfrak{a}_R^Q$ is orthogonal to $\varpi \in \hat{\Delta}_2 \subset \hat{\Delta}_Q$. Now H and H_2 are ε-invariants, and then the same inequalities hold if we replace ϖ by $\varepsilon^r\varpi$ and X by $\varepsilon^{-r}X$. But any $\varpi_1 \in \hat{\Delta}_Q$ is of the form $\varepsilon^r\varpi$ for some integer r and some $\varpi \in \hat{\Delta}_2$ and hence

$$\varpi_1(H_2) > c_2$$

for any $\varpi_1 \in \hat{\Delta}_Q$ and some constant c_2. □

COROLLARY 4.2.3. <u>If</u> $H = H_1 + H_2$ <u>as above, the set of</u> $H_2 \in \mathfrak{a}_Q^\varepsilon$ <u>such that</u> $\sigma_1^2(H_1+H_2-X) = 1$ <u>for some</u> $X \in \omega_6$ <u>has a volume bounded by a polynomial in</u> $\|H_1\|$. □

Let V be the cone in $\mathfrak{z}^\varepsilon \backslash (\mathfrak{a}_R^Q)^\varepsilon$ defined by $\alpha(H) > 0$ for all $\alpha \in \Delta_R^Q$. If $a_2 = \exp H_2$ with $H_2 \in \mathfrak{a}_Q^\varepsilon$ then

$$\mathrm{Ad}(a_2)^* Y = Y$$

for $Y \in \tilde{n}_{1,2} \subset n_{R,Q}^*$. Then all that is left to prove is that

$$\|Y\| \geq \sup_{\lambda \in \Lambda} \|Y_\lambda\|$$

then if $n(Y)$ is the number of λ such that $Y_\lambda \neq 0$ we have

$$\|Y\|^{n(Y)} \geq \prod_{\substack{\lambda \in \Lambda \\ Y_\lambda \neq 0}} \|Y_\lambda\| .$$

Let L be a lattice in $n^*_{R,Q} \otimes \mathbb{R}$, then there is a constant c_1 such that if $Y \in L - \{0\}$

$$\|Y\|^n \geq c_1 \|Y\|^{n(Y)}$$

with n the cardinal of Λ. Then for $Y \in \tilde{n}_{1,2} \cap L$ and $H \in V$ we have

$$\|Ad^*(\exp H)Y\|^n \geq c_1 e^{c_2\|H\|} \prod_{\substack{\lambda \in \Lambda \\ Y_\lambda \neq 0}} \|Y_\lambda\| ,$$

for some strictly positive constant c_2. Since the function θ is obtained by integration over p in a compact set of the absolute value of a Schwartz-Bruhat function on $n^*_{R,Q} \otimes \mathbf{A}$ depending smoothly on p, the convergence is now an easy exercise left to the reader. □

$$\int_V \|H_1\|^r \sum_{Y \in \tilde{n}_{1,2}} \theta(\mathrm{Ad}^*(H_1)Y)\,dH_1$$

is finite for any positive real number r. To prove this we must recall the definition of $\tilde{n}_{1,2}$: it is the subset of the $Y \in n^*_{R,Q}$ that belong to one and only one $n^*_{R,P}$ with $R \subset P \subset Q$ and P ε-invariant; in other words

$$\tilde{n}_{1,2} = n^*_{R,Q} - \bigcup_{\substack{R \subset P \subsetneq Q \\ \varepsilon(P)=P}} n^*_{R,P} \ .$$

The space $n^*_{R,Q}$ can be decomposed into root subspaces under the action of $(\mathfrak{a}^Q_R)^{\varepsilon}$:

$$n^*_{R,Q} = \bigoplus_{\lambda \in \Lambda} n^*_{\lambda} \ .$$

The set Λ is in natural bijection with the orbits of E in the roots of $\mathfrak{a}^Q_R$ in $n^*_{R,Q}$. Let ϖ_α be a weight in $\hat{\Delta}^Q_R$ and define

$$\varpi_{\bar{\alpha}} = \frac{1}{\ell} \sum_{r=0}^{\ell-1} \varepsilon^r \varpi_\alpha$$

where $\bar{\alpha}$ represents the orbit of α under E. The $\varpi_{\bar{\alpha}}$ are a basis of $(\mathfrak{a}^Q_R)^{\varepsilon}$. An element $Y = \sum_{\lambda \in \Lambda} Y_\lambda$ in $n^*_{R,Q}$ is in $\tilde{n}_{1,2}$ if and only if for any $\bar{\alpha}$ there exist $\lambda \in \Lambda$ such that $Y_\lambda \neq 0$ and $\langle \lambda, \varpi_{\bar{\alpha}} \rangle \neq 0$ (and hence strictly positive).

Choose a norm on $n^*_{R,Q} \otimes \mathbb{R}$ such that