Lecture 15

(provisional text)

THE FINE χ-EXPANSION

R. Langlands

1. <u>The operators</u> $M_{P'|P}(s, \lambda)$. As usual M_0 is fixed and we consider only Levi factors $M \in \mathcal{L}(M_0)$ and parabolic subgroups $P \in \mathcal{F}(M_0)$. For such an M the Lie algebra $\mathfrak{a}_M$ is well-defined and so is $\Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$.

Let $s \in \Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$, $P \in \mathcal{P}(M)$, $P' \in \mathcal{P}(M')$. We define the operator $M_{P'|P}(s, \lambda)$ taking ϕ to be the function $M_{P'|P}(s, \lambda)\phi$:

$$g \longrightarrow \int_{\mathbf{N}_{P'} \cap w\mathbf{N}_P w^{-1} \backslash \mathbf{N}_{P'}} \phi(w^{-1}ng)e^{(\lambda+\rho_P)(H_P(w^{-1}ng)-(s\lambda+\rho_{P'})(H_{P'}(g))} dn \quad .$$

Some explanation is in order.

Fix a class χ in $\mathfrak{X}$, thus a pair (ρ_χ, M_χ) given up to association. It is referred to as a cuspidal datum. Two cuspidal data (ρ_χ, M_χ) and $(\rho_{\chi'}, M_{\chi'})$ will be said to be equivalent if after conjugation $M_{\chi'} = M_\chi$ and $\rho_{\chi'} = \alpha \otimes \rho_\chi$, α being a character of $\mathbf{G}$ trivial on $\mathbf{G}^1$. To χ is associated a closed subspace $L^2_\chi(M\backslash\mathbf{M})$ of $L^2_\omega(M\backslash\mathbf{M})$, ω being a certain central character of $\mathbf{M}$ determined by ρ_χ. If ϖ is an irreducible unitary representation of $\mathbf{M}$ let $\mathfrak{H}_{\chi,\varpi}(P)$ be the space of measurable functions ϕ on $\mathbf{G}$ satisfying:

(a) $\phi(ng) = \phi(g)$, $n \in \mathbf{N}_P$;

(b) $\phi(\gamma g) = \phi(\gamma)$ $\quad \gamma \in P$;

(c) $m \longrightarrow \phi(mg)$ is a function in $L^2_\chi(M\backslash\mathbf{M})$ for all $g \in \mathbf{G}$ transforming according to the representation σ;

(d)

$$\|\phi\|^2 = \int_{\mathbf{Z}_M M\backslash\mathbf{M}\times K} |\phi(mk)|^2 dn dk < \infty \quad .$$

The intertwining operator is defined by analytic continuation on K-finite functions and is unitary for $\mathrm{Re}\,\lambda = 0$. Notice that

$$(\phi, \psi) = \int_{\mathbf{Z}_M M\backslash\mathbf{M}\times K} \phi(mk)\overline{\psi}(mk)\,dm dk$$

defines an inner product on $\mathfrak{a}_{\chi,\sigma}(P)$.

The forms ρ_P and $\rho_{P'}$ have the usual meaning and w is a representative of s. Since the Iwasawa decomposition $\mathbf{G} = \mathbf{P}K$ is valid we can define $H_P(g)$.

The operator $M_{P'|P}(s, \lambda)$ is certainly an intertwining operator from $\mathfrak{a}_{\chi,\sigma}(P)$ to $\mathfrak{a}_{\chi,s(\sigma)}(P)$, the representation on the first space being $\rho_{\sigma\otimes\lambda}$ and that on the second being $\rho_{s(\sigma)\otimes s(\lambda)}$. Since χ is fixed in the present lectures we may drop it from the notation.

We shall make use of a number of relations which are either elementary or a part of the theory of Eisenstein series.

(a) If $s \in \Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$, $s' \in \Omega(\mathfrak{a}_{M'}, \mathfrak{a}_{M''})$ then

$$M_{P''|P}(s's, \lambda) = M_{P''|P'}(s', s\lambda) M_{P'|P}(s\lambda) \quad .$$

Of course $P'' \in \mathcal{P}(M'')$.

(b) Suppose L is a Levi subgroup containing both M and M' and s fixes the points of $\mathfrak{a}_L$. Associated to every pair R, Q, $R \in \mathcal{P}^L(M)$, $Q \in \mathcal{P}(L)$ is a unique parabolic subgroup $Q(R) \in \mathcal{P}(M)$ satisfying $Q(R) \subseteq Q$, $Q(R) \cap L = R$. Moreover if $\phi \in \mathfrak{a}_\sigma(Q(R))$ then for each k the function $\phi_k : m \longrightarrow \phi(mk)$ lies in $\mathfrak{a}_\sigma(R)$ and

$$(M_{Q(R')|Q(R)}(s, \lambda)\phi)_k = M_{R'|R}(s, \lambda)\phi_k \quad .$$

Notice that $M_{R'|R}(s, \lambda)$ depends only on the projection of λ on $\mathfrak{a}_M^L$.

(c) Suppose $M' = wMw^{-1}$, $P' = wPw^{-1}$ and w is a representation of s in $\Omega(\mathfrak{a}_M, \mathfrak{a}_{M'})$. Then by the definition

$$M_{P'|P}(s, \lambda)\phi : g \longrightarrow \phi(w^{-1}g)e^{(\lambda+\rho_P)H_P(w^{-1}g)-(s\lambda+\rho_{P'})(H_{P'}(g))} \quad .$$

Now if $g = p'h$ then $w^{-1}g = w^{-1}p'w\; w^{-1}k = pw^{-1}k$. Thus

$$H_P(w^{-1}g) = w^{-1}H_{P'}(g) + H_P(w^{-1}) \quad .$$

Since $\rho_{P'} = s\rho_P$ we conclude that

$$M_{P'|P}(s, \lambda)\phi = s\phi \cdot e^{(\lambda+\rho_P)(T_0-s^{-1}T_0)} \quad ,$$

for as Arthur shows in Lemma 1.1 of the Annals paper there exists a T_0 such that $H_P(w^{-1}) = T_0-s^{-1}T_0$ for all w. We define $s\phi$ by $s\phi : g \longrightarrow \phi(w^{-1}g)$.

(d) Combining (a) and (c) we obtain

$$te^{(s\lambda+\rho_{P'})(T_0-t^{-1}T_0)} M_{P'|P}(s, \lambda) = M_{t(P')|P}(ts, \lambda)$$

$$M_{P'|P}(s,\lambda) = M_{P'|t(P)}(st^{-1}, t\lambda)t\cdot e^{(\lambda+\rho_P)(T_0-s^{-1}T_0)} \quad .$$

2. (G, M) <u>families</u>. For the moment fix M. A (G, M)-family is a set of smooth functions $c_P(\Lambda)$, $\Lambda \in i\mathfrak{a}_M^*$, indexed by the parabolic subgroups in $\mathcal{P}(M)$. These functions are to satisfy a compatibility condition. Recall that each P in $\mathcal{P}(M)$ is associated to a Weyl chamber W_P in $\mathfrak{a}_M$. This chamber is defined as the set of H such that $\alpha(H) > 0$ for all roots α in P. Thus $s(W_P) = W_{w(P)}$ if w represents s. If P and P' are adjacent, that is, if W_P and $W_{P'}$ have a wall in common, then the condition is that $c_P(\lambda) = c_{P'}(\lambda)$ on the hyperplane containing this wall.

A family of points $\{X_P | P \in \mathcal{P}(M)\}$ is said to be A_M-orthogonal if $X_P - X_{P'}$ is perpendicular to the wall separating W_P from $W_{P'}$ whenever P and P' are adjacent. Then the collection of functions $\{e^{\lambda(X_P)}\}$ is a (G, M)-family.

The set A_M-orthogonal of all families is a closed subset of $\prod_{\mathcal{P}(M)} \mathfrak{a}_M$ and if ω is any rapidly decreasing measure on $\mathfrak{a}$ then

$$(1) \qquad c_P : \lambda \longrightarrow \int e^{\lambda(X_P)} d\omega \quad .$$

is a (G, M)-family. It is likely that all compactly supported (G, M) families are of this form. Since these initial lectures on the second American Journal paper have as their sole purpose to discover a modification of the method of Arthur which <u>may</u> work in the twisted case, the rigorous treatment to be given later, I shall assume that the compactly supported families that arise are associated to a measure. Otherwise the combinatorics become unmanageable. (This turned out fortunately to be unwarranted pessimism.)

I now recall some constructions and some facts from the Inventiones and the Annals papers, many of which have already appeared in Lectures 9 and 13. First of all if $Q \supseteq P$ then $i\mathfrak{a}_Q^* \subseteq i\mathfrak{a}_P^*$ and we can project $\lambda \in i\mathfrak{a}_P^*$ onto $i\mathfrak{a}_Q^*$ obtaining λ_Q. If c_P is defined we set

$$c_Q(\lambda) = c_P(\lambda_Q) \quad .$$

Then we define c'_Q by

$$c'_Q(\lambda) = \sum_{R \supset Q} (-1)^{a_Q - a_R} \hat{\theta}_Q^R(\lambda)^{-1} c_R(\lambda) \theta_R(\lambda)^{-1} \quad .$$

Recall that

$$\theta_R^S(\lambda) = \frac{1}{c_R^S} \prod_{\alpha \in \Delta_R^S} \langle \lambda, \alpha \rangle$$

$$\hat{\theta}_Q^R(\lambda) = \frac{1}{\hat{c}_Q^R} \prod_{\varpi \in \hat{\Delta}_Q^R} \langle \lambda, \varpi \rangle \quad .$$

Here c_R^S is the volume of the parallelepiped spanned by Δ_R^S and $\hat{c}_Q^R$ the volume of that spanned by $\hat{\Delta}_Q^R$.

The functions c'_Q, $Q \supseteq P$, $P \in \mathcal{P}(M)$, depend on c_P alone and not on the entire (G, M) family. If $c_P(\lambda) = e^{\lambda(X_P)}$ then c'_Q is the Fourier transform of the function $\Gamma'_Q(\cdot, X_Q)$, where X_Q is the projection of X_P onto $\mathfrak{a}_Q$. Thus

$$c'_Q(\lambda) = \int e^{\lambda(H)} \Gamma'_Q(H, X_Q)\,dH \quad .$$

Recall that $\Gamma'_Q(\cdot, X_Q)$ is a function with support in a ball of radius $e\|X_Q\|$. More generally, if the family is attached to a measure ω then

$$c'_Q(\lambda) = \int_{\mathfrak{a}} \int_{\mathfrak{a}_Q} e^{\lambda(H)} \Gamma'_Q(H, X_Q)\,dH\,d\omega \quad .$$

Observe that X_Q is independent of the choice of $P \subset Q$ used to define it because the collection $\{X_P | P \in \mathcal{P}(M)\}$ is an A_M-orthogonal family.

Arthur also introduces a function $c_M(\lambda)$. It is at first defined by

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1} \quad ,$$

but he then shows that

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c'_P(\lambda) \quad .$$

Thus if

$$\Gamma_M(H, \{X_P\}) = \sum_{P \subset \mathcal{P}(M)} \Gamma'_P(H, X_P)$$

then

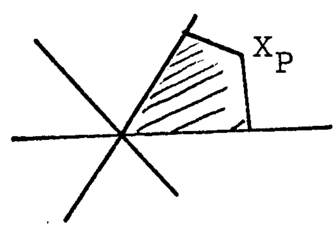
$$c_M(\lambda) = \int_{\mathfrak{a}} \int_{\mathfrak{a}} e^{\lambda(H)} \Gamma_M(H, \{X_P\})\,dH\,d\omega \quad .$$

Since the prime in $\Gamma'_P(H, X_P)$ serves no useful purpose I drop it. The measure ω being rapidly decreasing and the function $\Gamma_M(\cdot, \{X_P\})$ being supported in a ball of radius $c \sup_P \|X_P\|$ the function $c_M(\lambda)$

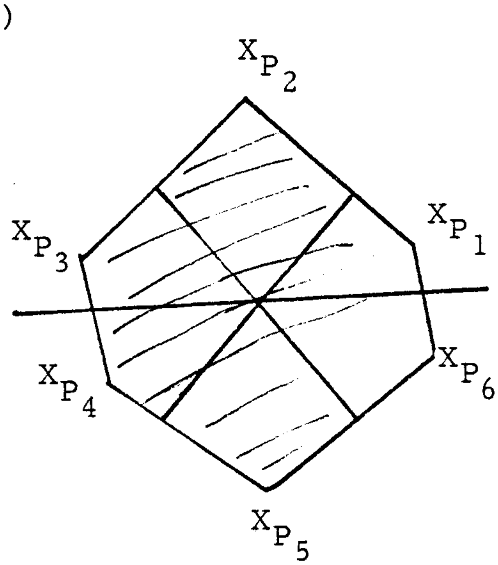
is smooth. Recall that $c_M(0)$ is usually denoted c_M. Of course if L is any Levi factor containing M then the family of functions $\{c_Q(\lambda) | Q \in \mathcal{P}(L)\}$ is also defined, as is $c_L(\lambda)$.

Recall that if each X_P lies in the chamber associated to P then the functions $\Gamma_P(\cdot, X_P)$ and $\Gamma_M(\cdot, \{X_P\})$ are characteristic functions. A typical pair is given by the following diagrams:

$\Gamma_P(\cdot, X_P)$



$\Gamma_M(\cdot, \{X_P\})$



Suppose M is the Levi factor of an ε-stable standard parabolic. Then the ε-roots divide $\mathfrak{a}_M^\varepsilon$ into chambers. Moreover, as we know, every root of G not lying in M has a non-zero restriction to $\mathfrak{a}_M^\varepsilon$. Thus if

W is a chamber in $\mathfrak{a}_M^\varepsilon$ we can let P_W be the group in $\mathcal{P}(M)$ defined by the condition that α is a root of N_{P_W} if and only if α is positive on W. The collection of P_W will be denoted $\mathcal{P}_\varepsilon(M)$. In a similar way, using the faces of the chambers, we define the collection $\mathcal{F}_\varepsilon(M)$ of parabolics.

Suppose more generally that M contains M_0 and that $\mathfrak{a} \subseteq \mathfrak{a}_M$. We say that the pair $(M, \mathfrak{a})$ is ε-special if it is conjugate to $(M', \mathfrak{a}_{M'}^\varepsilon)$, M' being the Levi factor of an ε-stable parabolic. Recall that if w normalizes M_0 and $\mathfrak{a}_{M'}^\varepsilon$ then $w\varepsilon w^{-1}\varepsilon^{-1}$ fixes each point of $\mathfrak{a}_{M'}^\varepsilon$ and thus normalizes the standard parabolic with M' as Levi factor. Consequently w represents an element $\Omega^\varepsilon(\mathfrak{a}_0, \mathfrak{a}_0)$ and maps the sets $\mathcal{P}_\varepsilon(M')$ and $\mathcal{F}_\varepsilon(M')$ onto themselves. So we can transport $\mathcal{P}_\varepsilon(M')$ and $\mathcal{F}_\varepsilon(M')$ from M' to M, thereby obtaining $\mathcal{P}_\varepsilon(M, \mathfrak{a})$ and $\mathcal{F}_\varepsilon(M, \mathfrak{a})$. Once we have these sets we can introduce the notion of a $(G, M, \mathfrak{a})$ family. It consists of a collection of functions c_P on $\mathfrak{a}$, one for each $P \in \mathcal{P}_\varepsilon(M, \mathfrak{a})$ which satisfy the obvious compatibility condition.

LEMMA 1. *If* $\{c_P | P \in \mathcal{P}(M)\}$ *is a* (G, M) *family then the collection* $\{\bar{c}_P | P \in \mathcal{P}_\varepsilon(M, \mathfrak{a})\}$ *is a* $(G, M, \mathfrak{a})$ *family,* $\bar{c}_P$ *being the restriction of* c_P *to* $\mathfrak{a}$.

Suppose Q and Q' are in $\mathcal{P}_\varepsilon(M, \mathfrak{a})$ and adjacent. The chambers W_Q^ε, $W_{Q'}^\varepsilon$ in $\mathfrak{a}$ associated to Q and Q' are then separated by a wall defined by an ε-root α. Let $\alpha_1, \ldots, \alpha_r$ be the roots whose restriction to $\mathfrak{a}$ is α. These are the only roots (up to sign) separating W_Q from $W_{Q'}$. Thus, after renumbering, we can find a sequence

$Q_0 = Q, Q_1, \ldots, Q_r = Q'$ such that Q_i is separated from Q_{i-1} by a single wall, that defined by α_i. If $\lambda \in \mathfrak{a}$ and $(\alpha, \lambda) = 0$ then $(\alpha_i, \lambda) = 0$ for all i and

$$\bar{c}_Q(\lambda) = c_{Q_0}(\lambda) = \ldots = c_{Q_r}(\lambda) = \bar{c}_{Q'}(\lambda) \quad .$$

Thus once $\mathfrak{a}$ is specified we can introduce the function $\bar{c}_M$ as well as functions $\bar{c}'_Q$, $Q \in \mathcal{P}_\varepsilon(M, \mathfrak{a})$. If $c_P(\lambda) = e^{\lambda(X_P)}$ and $\bar{X}_P$ is the projection of X_P on $\mathfrak{a}$ then

$$\bar{c}_P(\lambda) = e^{\lambda(X_P)} = e^{\lambda(\bar{X}_P)}, \qquad \lambda \in \mathfrak{a} \quad .$$

Thus if $\{c_P\}$ is associated to the measure ω then

$$\bar{c}_M(\lambda) = \int_{\mathfrak{A}} \int_{\mathfrak{a}} e^{\lambda(H)} \Gamma_{M, \mathfrak{a}}(H, \{\bar{X}_P\}) d\omega$$

where $\Gamma_{M, \mathfrak{a}}(\cdot, \{\bar{X}_P\})$ is the function on $\mathfrak{a}$ associated to the ε-roots and the family $\{\bar{X}_P\}$.

(G, M) - <u>families defined by intertwining operators</u>. Fix a standard P and let $M = M_P$. For brevity I shall denote the representation of $\mathbf{G}$ on the space of functions $g \longrightarrow \phi(g)e^{\lambda(H(g))}$, $\phi \in \mathfrak{A}_{\chi, \sigma}(P)$ by $\rho_{\sigma, \lambda}$. Thus $\rho_{\sigma, \lambda}(h)\phi = \phi'$ means $\phi(gh)e^{\lambda(H(gh))} = \phi'(g)^{\lambda(H(g))}$. There are several (G, M)-families to be introduced. The first is simple to define.

If $Q \in \mathcal{P}(M)$ then $Q = t^{-1}(P_1)$, where t is an element of the Weyl group and P_1 is standard. Let $Y_Q(T)$ be the projection onto $\mathfrak{a}_M$ of $t^{-1}(T-T_0) + T_0$ and set

$$c_Q(\Lambda) = e^{\Lambda(Y_Q(T))}, \qquad \Lambda \in \mathfrak{a}_M .$$

To define the second we choose an $s \in \Omega(\mathfrak{a}_{\varepsilon(P)}, \mathfrak{a}_P)$ and, fixing λ and Λ, we define μ by

$$\Lambda = s\varepsilon\mu - \lambda .$$

Suppose that $s\varepsilon(\sigma)$ and σ are equivalent up to tensoring with a character of $\mathbf{M}^1\backslash\mathbf{M}$. Then we set, suppressing s and σ from the notation and assuming ϕ to be K-finite,

$$d_Q(\Lambda) = \operatorname{tr}(M_{Q|P}(1, \lambda)^{-1} M_{Q|\varepsilon(P)}(s, \varepsilon(\mu))\varepsilon\rho_{\sigma,\lambda}(\phi)) .$$

LEMMA 2. <u>For each</u> λ the collection $\{d_Q\}$ <u>is a</u> (G, M)-<u>family</u>.

Suppose Q' and Q are adjacent. By one of the functional equations

$$M_{Q'|P}(1,\lambda)^{-1}M_{Q'|\varepsilon(P)}(s,\varepsilon(\mu)) = M_{Q|P}(1,\lambda)^{-1}M_{Q'|Q}(1,\lambda)^{-1}M_{Q'|Q}(1,s\varepsilon(\mu))M_{Q|\varepsilon(P)}(s,\varepsilon(\mu))$$

If the wall separating Q and Q' is defined by α and if $(\lambda, \alpha) = 0$ then λ and $s\varepsilon(\mu)$ have the same projection on $\mathbf{C}\alpha$. Thus by the functional equations (b)

$$M_{Q'|Q}(1, s\varepsilon(\mu)) = M_{Q'|Q}(1, \lambda) .$$

The lemma follows.

LEMMA 3. <u>Let</u> $\mathfrak{a}$ <u>be the set of points in</u> $\mathfrak{a}_P$ <u>fixed by</u> $s\varepsilon$. <u>Then</u> $\mathfrak{a}$

is the second term of a unique ε-*special pair* $(L, \mathfrak{a})$ *and* $L \supseteq M$.

The uniqueness of L is clear for it must be the centralizer of $\mathfrak{a}$. So it certainly contains M. To prove its existence we argue by induction on the semi-simple rank of G.

The lemma is clear if $\mathfrak{a}$ is central for the $L = G$. Otherwise let $X \in \mathfrak{a}$ be a point fixed by $s\varepsilon$ on which not all roots vanish. Then $\{\alpha | \alpha(X) \geq 0\}$ defines a proper parabolic subgroup of G which is fixed by $s\varepsilon$. Conjugating M, $\mathfrak{a}$ and $s\varepsilon$ we may suppose that it is standard. Since two standard parabolics which are conjugate are equal, the parabolic subgroup is invariant under both s and ε. In particular s lies in the Weyl group of its Levi factor, to which we then apply the induction assumption.

Since s is fixed we have not included the dependence of $\mathfrak{a}$ on it in the notation. If λ and Λ lie in $\mathfrak{a}$ then μ lies in $\mathfrak{a}$ and $\mu = \Lambda+\lambda$. Therefore

$$M_{Q|P}(1,\lambda)^{-1}M_{Q|\varepsilon(P)}(s,\varepsilon(\mu)) = M_{Q|P}(1,\lambda)^{-1}M_{Q|P}(1,\lambda+\Lambda)M_{P|\varepsilon(P)}(s,\varepsilon(\lambda+\Lambda)) \quad .$$

Recall that to each $Q \in \mathcal{P}(L)$ and to each $R \in \mathcal{P}^L(M)$ there is associated a unique group $Q(R) \in \mathcal{P}(M)$ such that $Q(R) \subset Q$ and $Q(R) \cap M = R$.

LEMMA 4. *If* λ *and* Λ *lie in* $\mathfrak{a}_L$ *then the operators*

$$\mathcal{M}_Q(P,\lambda, \Lambda) = M_{Q(R)|P}(1, \lambda)^{-1}M_{Q(R)|P}(1, \lambda+\Lambda)$$

are independent of R *and for fixed* λ *define an operator-valued* (G, L)-*family*.

The product

$$M_{Q(R')|P}(1, \lambda)^{-1} M_{Q(R')|P}(1, \lambda+\Lambda)$$

is equal to

$$M_{Q(R)|P}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda+\Lambda) M_{Q(R)|P}(1, \lambda+\Lambda) \quad .$$

Since λ and $\lambda+\Lambda$ lie in $\mathfrak{a}_L$, the functional equation (b) yields

$$M_{Q(R')|Q(R)}(1, \lambda)^{-1} M_{Q(R')|Q(R)}(1, \lambda+\Lambda) = 1 \quad .$$

To prove that we have a (G, L)-family we imitate the proof of Lemma 2. It is only necessary to observe that if Q and Q' in $\mathcal{P}(L)$ are adjacent then we can find R and R' in $\mathcal{P}^L(M)$ such that $Q(R)$ and $Q'(R')$ are adjacent. Indeed if $\bar{\alpha} \in \Delta_Q$ defines the wall separating Q and Q' and if $\alpha \in \Delta_P$ restricts to $\bar{\alpha}$ then we may so choose R and R' that $Q(R)$ and $Q'(R')$ are separated by a wall lying in the hyperplane defined by α.

When $\lambda \in \mathfrak{a}$ let $\{{}_\varepsilon\mathcal{M}_Q(P, \lambda, \Lambda)\}$ be the $(G, L, \mathfrak{a})$-family attached to $\{\mathcal{M}_Q(P, \lambda, \Lambda)\}$. Finally set

$${}_\varepsilon\mathcal{M}_Q^T(P, \lambda, \Lambda) = \bar{c}_Q(\Lambda)\,{}_\varepsilon\mathcal{M}_Q(P, \lambda, \Lambda) = e^{\Lambda(Y_Q(T))}\,{}_\varepsilon\mathcal{M}_Q^T(P, \lambda, \Lambda) \quad .$$

In the statement of the following lemma and in its proof P need not be standard.

LEMMA 5. Set ${}_{\varepsilon}T_0 = 1-\varepsilon^{-1}T_0$ and suppose r is so chosen that the ε-special pair $(L, \mathfrak{a})$ associated to $rs\varepsilon(r^{-1})$ is $(M_Q, \mathfrak{a}_Q^{\varepsilon})$ where Q is an ε-invariant standard parabolic. If $\lambda \in \mathfrak{a}$ then

$$M_{P|\varepsilon(P)}(s, \varepsilon(\lambda)) = e^{\langle\lambda, T_{s\varepsilon}\rangle}{}_{\varepsilon}M(P, s) \quad ,$$

where ${}_{\varepsilon}M(P, s)$ is independent of λ and

$$T_{s\varepsilon} = (r^{-1}-1)\,{}_{\varepsilon}T_0 \quad .$$

To prove the lemma we observe that

$$M_{r(P)|\varepsilon(rP)}(rs\varepsilon(r^{-1}), \varepsilon(r\lambda))$$

is equal to

$$M_{r(P)|P}(r, s\varepsilon\lambda)M_{P|\varepsilon(P)}(s, \varepsilon\lambda)M_{\varepsilon(P)|\varepsilon(rP)}(\varepsilon(r^{-1}), \varepsilon(r\lambda)) \quad .$$

Using the functional equations (c) we see that this in turn is equal to

$$e^{(s\varepsilon\lambda+\rho_P)(T_0-r^{-1}T_0)}\,{}_{r}M_{P|\varepsilon(P)}(s\ \varepsilon\lambda)\,\varepsilon(r^{-1})e^{(\varepsilon(r\lambda)+\rho_{\varepsilon(rP)})(T_0-\varepsilon(r)T_0)} \quad .$$

The dependence of the product of the two exponentials on λ is through

$$e^{\lambda(T_0-r^{-1}T_0+r^{-1}\varepsilon^{-1}T_0-\varepsilon^{-1}T_0)} = e^{\langle\lambda, T_{s\varepsilon}\rangle}$$

because $s\varepsilon\lambda = \lambda$.

The upshot is that to prove the lemma we may replace L, $\mathfrak{a}$, P and s by conjugate data. So we take L to be M_Q^{ε} and $\mathfrak{a}$ to be $\mathfrak{a}_L^{\varepsilon}$ for we have seen that this is possible. But then every point in $\mathfrak{a}$ is actually s-invariant and the equality is a consequence of the functional equations (b).

As an aside I observe that if K is ε-invariant, so that $H(\varepsilon(g)) = \varepsilon H(g)$, then

$$\varepsilon^{-1}T_o \equiv T_0 \ (\mathrm{mod}\ \mathfrak{a}_G) \tag{2}$$

and consequently

$$T_{s\varepsilon} = 0 \quad .$$

To verify this we recall (Lemma 1.1 of the Annals paper) that T_0 is uniquely determined modulo $\mathfrak{a}_G$ by the condition

$$H_{P_0}(w^{-1}) = T_0 - s^{-1}T_0$$

for all $s \in \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ and all w in G representing s. However

$$\varepsilon T_0 - \varepsilon s^{-1}\varepsilon^{-1}(\varepsilon T_0) = \varepsilon(T_0 - s^{-1}T_0) = H(\varepsilon(w^{-1})) \quad .$$

Thus εT_0 is another candidate for T_0 and (2) follows.

<u>The fine χ-expansion</u>. The term $J_\chi^T(\phi)$ has been introduced in previous lectures, where it has been shown that it is a polynomial. Our purpose

in this (long) lecture is to prove a formula for it which we now describe.

If $M \in \mathcal{L}(M_0)$ and $\lambda \in \mathfrak{a}_M \otimes \mathbf{C}$ let χ_λ be the character of $\mathbf{M}$ defined by

$$\chi_\lambda(m) = e^{\lambda(H(m))} \quad .$$

If σ and σ' are two representations of $\mathbf{M}$ we write $\sigma' \sim \sigma$ if σ' is equivalent to $\sigma \otimes \chi_\lambda$ for some $\lambda \in \mathfrak{a}_M \otimes \mathbf{C}$. Each class has a distinguished representative, that which is trivial on $\{\exp H \mid H \in \mathfrak{a}_M\}$. We usually work with it.

The formula expresses $J_\chi^T(\phi)$ as a sum over quintuples $(M, L, \mathfrak{a}, \{\sigma\}, s)$ satisfying the following conditions:

(i) $M_0 \subseteq M \subseteq L$ and $\mathfrak{a} \subseteq \mathfrak{a}_M$.

(ii) $\{\sigma\}$ is a class of unitary automorphic representations of $\mathbf{M}$, the equivalence being that just defined, and σ is the distinguished representative.

(iii) $s \in \Omega(\mathfrak{a}_{\varepsilon(M)}, \mathfrak{a}_M)$ and $s\varepsilon(\sigma) \sim \sigma$.

(iv) $(L, \mathfrak{a})$ is ε-special and $\mathfrak{a}$ is the set of fixed points of $s\varepsilon$ in $\mathfrak{a}_M$.

(v) If $P \in \mathcal{P}(M)$ the space $\mathfrak{a}_{\chi,\sigma}(P)$ is not reduced to zero.

We now describe the term corresponding to a given quintuple. Let $\Omega_0 = \Omega(\mathfrak{a}_0, \mathfrak{a}_0)$ and $\Omega_0^M = \Omega^M(\mathfrak{a}_0, \mathfrak{a}_0)$. The linear transformation $s\varepsilon - 1$ is invertible on $\mathfrak{a}_M/\mathfrak{a}$. Let $\Delta = \Delta(s, \varepsilon)$ be the absolute value of its determinant. The term is

$$\frac{|\Omega_0^M|}{|\Omega_0|} \frac{1}{|\mathcal{P}(M)|\Delta} \sum_{P \in \mathcal{P}(M)} \frac{1}{(2\pi)^{a_P}} \int_{i\mathfrak{a}^*} \operatorname{tr}\left({}_\varepsilon M_L^T(P,\lambda) e^{\langle \lambda, T_{s\varepsilon} \rangle} {}_\varepsilon M(P,s)\,\varepsilon\rho_{\sigma,\lambda}(\phi)\right) d\lambda \quad .$$

(Notice that the group **Z** and the character ω of the first lecture have unnoticed become trivial. The general case will have to await the revised lecture notes.) For P which are not standard the integrand is defined by symmetry.

Loose strands. The purpose of this section is to recapitulate results from earlier lectures, but with some minor changes and with a notation convenient for our present purposes. We begin by discussing the coarse χ-expansion more fully than in Lecture 13.

Recall that this is an expression for $J^T(\phi)$ as a sum,

(3) $$J^T(\phi) = \sum_\chi J_\chi^T(\phi) \quad ,$$

the index of summation running over equivalence classes of pairs of cuspidal data. To obtain it one first expresses the kernel as a sum,

$$K(h, g) = \sum_\chi K_\chi(h, g) \quad .$$

For each standard P and each distinguished σ let $\{\Phi_i\}$ be an orthonormal base of $\mathfrak{A}_{\chi,\sigma}(P)$ and set

$$K_{\chi,P,\sigma}(h, g) = \sum_i E(h, \rho_\sigma(\phi)\Phi_i)\overline{E(g, \Phi_j)} \quad .$$

The collection of unitary automorphic representations of **M** is the union of affine spaces of the form $\{\sigma_0 \otimes \chi_\lambda | \sigma_0$ distinguished, $\lambda \in i\mathfrak{a}_M\}$. Let $d\sigma$ be the measure which on each component is $|d\lambda|$. Then

$$K_\chi(h,\ g) = \sum_P \frac{1}{(2\pi)^{a_\nu} n(P)} \int_{\{\sigma\}} K_{\chi,P,\sigma}(h,\ g)\,d\sigma$$

$$= \sum_P \frac{1}{(2\pi)^{a_P} n(P)} \int_{\{\sigma\}} \sum_i E(h,\ \rho_\sigma(\phi)\Phi_i)\overline{E(g,\ \Phi_j)}\,d\sigma \quad .$$

Moreover

$$(4) \qquad \sum_\chi \sum_P \frac{1}{(2\pi)^{a_P} n(P)} \int_\sigma \sum_i \left|E(h,\ \rho_\sigma(\phi)\Phi_i)\overline{E}(g,\ \Phi_j)\right| d\sigma < \infty \quad .$$

The integer a_P is equal to the dimension of the split component of P and $n(P)$ is the number of parabolic subgroups with the same Levi factor as P. The absolute convergence of (4) was proven in Lecture 10.

We can also introduce

$$K_\chi^\varepsilon(h,\ g) = K_\chi(h,\ \varepsilon(g))$$

and, when P is ε-invariant,

$$K_{P,\chi}^\varepsilon(h,\ g) = K_{P,\chi}(h,\ \varepsilon(g)) \quad .$$

Of course

$$K_P^\varepsilon(h,\ g) = \sum_\chi K_{P,\chi}^\varepsilon(h,\ g) \quad .$$

The basic identity, viz. the equality of

$$\sum_{P_0 \subset P} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \sum_{\delta \in P \backslash G} K_{P,\chi}^\varepsilon(\delta g,\ \delta g)\,{}_\varepsilon\hat{\tau}_P(H(\delta g) - T)$$

and

$$\sum_{P_0 \subset P_1 \subset P_2} \sum_{\delta \in P_1 \backslash G} {}_\varepsilon\sigma_1^2(H(\delta g)-T)(\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon - a_G^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}(\delta g, \delta g)$$

remains valid.

The expansion (3) is obtained from the χ-expansion of the left or the right side of the basic identity by integration over $G\backslash \mathbf{G}_\varepsilon^1$. It is however necessary to verify absolute convergence in order to justify the interchange of summation and integration. For this we use the right side. The left side is used only for the purpose of showing that $J_\chi^T(\phi)$ is a polynomial in T, the argument imitating that in §2 of the Annals paper.

To prove the convergence of the coarse χ-expansion we show that for each pair of standard parabolic subgroups $P_1 \subset P_2$ the sum

$$\sum_\chi \int_{P_1 \backslash \mathbf{G}_\varepsilon^1} {}_\varepsilon\sigma_1^2(H(g)-T) \Big| \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(g, g) \Big| dg < \infty \quad .$$

This is a stronger assertion than that treated in Lectures 7 and 8 but the proof proceeds along similar lines. We indicate the necessary modifications including those entailed by the replacement of σ_1^2 by ${}_\varepsilon\sigma_1^2$. The critical observation is Lemma 2.3 of the Compositio paper.

The first step is to find a substitute for the argument on pp. 3-5 in order to replace (we are taking $\omega \equiv 1$)

$$(5) \qquad \sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(h, g)$$

by

$$(6) \qquad (-1)^{a_Q^\varepsilon} \sum_{\gamma \in F_\varepsilon(P_1,P_2)} \Lambda^{T,P_1} K^\varepsilon_{P_1,\chi}(h, \gamma\varepsilon(g))$$

if there is an ε-invariant parabolic Q between P_1 and P_2 and by zero otherwise.

Lemma 2.3 allows one first to extend Lemma 7.6 in Lecture 7 to $K_{P_1,\chi}$. Then following §2 of the Compositio paper we deduce the equality of (5) with (6) or with 0 from the corresponding equality for the original kernels.

Variants of Lemma 2.3 can be obtained from a simple observation, which was drawn to my attention by Clozel. The kernel $K_{P,\chi}$ is the kernel of an operator on $L^2(\mathbf{N}_P P \backslash \mathbf{G})$ and is equal to

$$\prod_\chi^1 K_P(h, g) = \prod_\chi^2 K_P(h, g)$$

where the superscript indicates whether we operate on the first or the second variable and $\prod_\chi$ is the projection on the space attached to χ. The operator $\prod_\chi$ acts of course on a function f according to

$$(\prod_\chi f)_g = \prod_\chi(\mathbf{M}^1) f_g$$

where $\prod_\chi(\mathbf{M}^1)$ is an operator on functions on $\mathbf{M}^1$ and $f_g(m) = f(mg)$, $m \in \mathbf{M}^1$.

Since

$$\Lambda^{T,P_1} K_{P_1,\chi}(h, g) = \Lambda^{T,P_1} \prod_\chi^2 K_{P_1}(h, g)$$

$$= \prod_\chi^2 \Lambda^{T,P_1} K_{P_1}(h, g)$$

we conclude that when

$$\Lambda^{T,P_1} K_{P_1,\chi}(h, mg) \not\equiv 0$$

as a function of $m \in \mathbf{M}^1$ then

$$\Lambda^{T,P_1} K_{P_1}(h, mg) \not\equiv 0 \quad .$$

The modified Lemmas 7.2, 7.4, and 7.6 follow immediately and the changes in the proof of Lemma 7.1 are minimal, for $\Lambda^{T,P_1} \prod_\chi^2 K_{P_1}(h, g) \neq 0$ implies that $K_{P_1}(h, mg) \neq 0$ for some $m \in \mathbf{M}^1$.

The proof of Lemma 7.3 for ${}_\varepsilon\sigma_1^2$ is the same as its proof for σ_1^2 and the modified Lemma 7.5 is implied by Lemma 4.4 of the first Duke Jour. paper and was proved by Clozel.

The coarse χ-expansion is the expansion

$$J^T(\phi) = \sum_\chi J_\chi^T(\phi)$$

and $J_\chi^T(\phi)$ is the sum over pairs $P_1 \subset P_2$ of standard parabolic subgroups of

$$\int_{P_1 \backslash \mathbf{G}_\varepsilon^1} {}_\varepsilon\sigma_1^2(H(g)-T)\{\sum_{P_1 \subset P \subset P_2} (-1)^{a_P^\varepsilon} \Lambda^{T,P_1} K_{P,\chi}^\varepsilon(g, g)\}dg \quad .$$

Following the arguments of Lectures 8, 10, and 11 one shows that this may be written as

$$(7)\quad (-1)^{a_Q^\varepsilon} \sum_{\gamma \in P_1 \backslash F_\varepsilon(P_1,P_2)/\varepsilon^{-1}(P_1)} \int_{P_1 \cap \varepsilon^{-1}\gamma^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} {}_\varepsilon\sigma_1^2(H(g)-T)\Lambda^{T,P_1} K_{P_1,\chi}(g,\gamma\varepsilon(g))dg \quad .$$

It was shown in Lectures 10 and 11 that each term is zero unless there is a unique ε-invariant parabolic Q between P_1 and P_2. However more can be squeezed out of the arguments given there, namely that even when the ε-invariant parabolic between P_1 and P_2 is unique the only contribution which perhaps does not vanish is that attached to the class of $\gamma = 1$. Since this leads to an indispensable simplification, we give the necessary supplementary argument.

The element γ may of course be taken to normalize M_0. Let it represent the element s in the Weyl group. We may assume that $s\varepsilon(\alpha) > 0$ for $\alpha \in \Delta_0^1$, for we are free to modify s on the right by an element of $\Omega^{\varepsilon^{-1}(M_1)}$. Recall that there is a unique standard parabolic subgroup $P_{s\varepsilon}$ such that

$$P_{s\varepsilon} \cap M_1 = \varepsilon^{-1}s^{-1}(P_1) \cap M_1$$

and that if $H = H(g)$ the following conditions must be satisfied if the term of (7) corresponding to γ is not to vanish:

(i) $\varpi(H-T) \leq 0$, $\varpi \in \hat{\Delta}_{s\varepsilon}^1$.

(ii) ${}_\varepsilon\sigma_1^2(H-T) \neq 0$.

(iii) $\varpi(H-s\varepsilon H) \leq C$, $\varpi \in \hat{\Delta}_1^Q$. (For this it is necessary to apply the original arguments within Q.)

The conditions (i) and (ii) allow us to write the projection of $H-T$ on $\mathfrak{a}_s^Q$ as

$$-\sum_{\alpha \in \Delta^1_{s\varepsilon}} c_\alpha \alpha + \sum_{\varpi \in \Delta_1^Q} c_\varpi \varpi \quad ,$$

with all coefficients non-negative. Thus $\varpi_0(H-s\varepsilon H)$ is equal to

$$(8) \qquad \varpi_0(T-s\varepsilon T) + \sum_{\alpha \in \Delta^1_{s\varepsilon}} c_\alpha \varpi_0(s\varepsilon(\alpha)) + \sum_{\varpi \in \hat{\Delta}_1^Q} c_\varpi \varpi_0(\varpi - s\varepsilon\varpi)$$

if $\varpi_0 \in \hat{\Delta}_1^Q$. Notice that $\varpi_0(\alpha) = 0$ if $\alpha \in \Delta^1_{s\varepsilon}$.

The expression (8) must be bounded by a constant independent of T and H. The space $\mathfrak{a}_0^{s\varepsilon}$ is spanned by roots of $M_{s\varepsilon}$. Thus $s\varepsilon(\mathfrak{a}_0^{s\varepsilon})$ is orthogonal to $\mathfrak{a}_1$ and if $\alpha \in \Delta^1_{s\varepsilon}$ is the image of $\alpha' \in \Delta^1_0$ then $\varpi_0(s\varepsilon(\alpha)) = \varpi_0(s\varepsilon(\alpha'))$. Consequently $\varpi_0(s\varepsilon(\alpha)) \geq 0$. We conclude that

$$\varpi_0(T-s\varepsilon T) + \sum c_\varpi \varpi_0(\varpi - s\varepsilon\varpi) \leq C \quad .$$

Let

$$X = \sum_{\varpi_0 \in \hat{\Delta}_1^Q} c_{\varpi_0} \varpi_0 \quad .$$

Multiplying by c_{ϖ_0} and summing we conclude that

$$(X,\ T-s\varepsilon T) + (X,\ X) - (X,\ s\varepsilon X) \leq C\|X\| \quad .$$

Now $X \in \mathfrak{a}_1^{Q_+}$. Thus if we assume (and we shall) that T is ε-invariant then

$$(X, T-s\varepsilon T) = (X, T-sT) \geq 0 \quad .$$

Moreover there is a constant $\delta > 0$ such that

$$(X, X) - (X, s\varepsilon X) \geq \delta \|X\|^2$$

on $\mathfrak{a}_1^{Q_+}$. To see this we have only to verify that

$$\min_{\{X \in \mathfrak{a}_1^{Q_+} \mid \|X\|=1\}} ((X, X) - (X, s\varepsilon X)) > 0 \quad .$$

The minimum is certainly not negative. If it is 0 then for some $Y \in \mathfrak{a}_1^{Q_+}$

$$Y = s\varepsilon Y \quad .$$

The set of roots α in Δ_0^Q such that $(\alpha, Y) \geq 0$ define a standard parabolic subgroup between P_1 and Q which is properly smaller than Q. Since it is invariant under $s\varepsilon$ it is invariant under both ε and s. This contradicts the definition of Q.

We conclude that

$$\|X\|^2 \leq c^1 \|X\|$$

and thus that $\|X\|$ is bounded. We obtain finally inequalities

$$\varpi_0(T-sT) \le C'' , \qquad \varpi_0 \in \hat{\Delta}_1^Q .$$

These inequalities can be violated for T sufficiently regular unless $s \in \Omega^{M_1}$. This leads to the desired conclusion.

So we are to consider

$$(-1)^{a_Q^\varepsilon} \int_{P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} \varepsilon^{\sigma_1^2(H(g)-T)} \Lambda^{T,P_1} K_{P_1,\chi}(g, \varepsilon(g))dg .$$

It was shown in Lectures 10 and 11 that this could be expanded as

$$(-1)^{a_Q^\varepsilon} \sum_P \frac{1}{(2\pi)^{a_P} n_1(P)} \int_{P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{G}_\varepsilon^1} \varepsilon^{\sigma_1^2(H(g)-T)} \int_{\Sigma(P)} \Lambda^{T,P_1} K_{P_1\chi,P,\sigma}(g,\varepsilon(g))d\sigma dg ,$$

$\Sigma(P)$ being the set of possible $\{\sigma\}$. Recall that

$$K_{P_1,\chi,P,\sigma}(g, \varepsilon(g)) = \sum_j E_{P_1}(g, \rho_\sigma(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(g), \Phi_j)} .$$

Thus we have an integral over the space parametrized by P, σ, j, and g. Some care is necessary because there is an element of conditional convergence, which we recall explicitly. The group $i\mathfrak{a}_1$ acts on Σ and we can clearly decompose $\Sigma(P)$ as a product $\Sigma_1(P) \times i\mathfrak{a}_1$, the connected components of Σ_1 being affine spaces over $i\mathfrak{a}_P^1$ (The attempt to distinguish between spaces and their duals becomes too much of a burden on the notation and I abandon it).

If $\sigma_1 \in \Sigma_1$, $\lambda_1 \in i\mathfrak{a}_1$ denote $\rho_{\sigma_1 \otimes \chi_{\lambda_1}}$ by $\rho_{\sigma_1,\lambda_1}$. The integral is obtained by iterating two other integrals, each of which is absolutely convergent although their iteration may not be. The first is

$$(9)\qquad \frac{1}{(2\pi)^{a_P} n_1(P)} \int_{i\mathfrak{a}_1} \sum_j \Lambda^{T,P_1} E_{P_1}(g, \rho_{\sigma_1,\lambda_2}(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(g), \Phi_j)}\,|d\lambda| \quad .$$

The second is over

$$\{(P, \sigma)\,|\,P \supseteq P_0,\ \sigma_1 \in \Sigma_1(P)\} \times P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{P}^1 \times K \times \mathfrak{a}^1_{1,\varepsilon}$$

where ${}_{\varepsilon}\mathfrak{a}^1_1$ is the set of $H \in \mathfrak{a}_1$ such that $d\chi(H) = 0$ for every ε-invariant character χ of G defined over $\mathbf{Q}$. The integrand for the second is the product of (9) with ${}_{\varepsilon}\sigma_1^2(H-T)e^{-2\rho_{P_1}(H)}$, g being $p(\exp H)k$. For this we do not need to assume that ϕ is K-finite. However for the first part of the proof of the fine χ-expansion we do, the assumption and the theory of Eisenstein series assuring us that the set of (P, σ, j), σ distinguished, which yield a non-zero contribution for a given χ is finite, so that the sum over these parameters presents no analytical problems. Thus until we explicitly return to the general case ϕ will be K-finite.

The integrand of (9) is clearly an entire function of λ_1. It is shown in Lectures 10 and 11 that the contour can be deformed to $\mathrm{Re}\,\lambda_1 = -\Lambda$, $\Lambda \in \mathfrak{a}_1$ arbitrary, without changing the value of the integral, which remains absolutely convergent, the parameter implicit in $E_{P_1}(\varepsilon(g), \Phi_j)$ being $-\overline{\lambda}_1$. We choose Λ such that $(\Lambda, \alpha) \gg 0$ for all $\alpha \in \Delta_1$.

Then, and this will be shown in Lectures 10 and 11, the double integral

$$(10)\int_{P_1\cap\varepsilon^{-1}(P_1)\backslash P_1^1\times K}\int_{\operatorname{Re}\lambda_1=-\Lambda}\Lambda^{T,P_1}E_{P_1}(pak,\rho_{\sigma_1,\lambda_1}(\phi)\Phi_j)\overline{E_{P_1}(\varepsilon(pak),\Phi_j)}\,|d\lambda_2|\,dp\,dk,\quad a=\exp H\ ,$$

is absolutely convergent. Given the properties of the truncation operator this follows from an estimate

$$(11)\qquad \sum_{\gamma\in P_1\cap\varepsilon^{-1}(P_1)\backslash P_1}|E_{P_1}(\varepsilon(\gamma pak),\ \Phi_j)|\le C(a)\,|m|^N\ .$$

Here $p = nm$, N is some fixed real number depending on Λ but not on λ_1 with $\operatorname{Re}\lambda_1 = -\Lambda$, and m lies in a Siegel domain of $\mathbf{M}_1$.

The parameter in the Eisenstein series is λ_1. All we need do is estimate

$$(12)\qquad \sum_{\gamma\in P_1\backslash G}|E_{P_1}(\gamma g,\ \Phi_j)|$$

on a Siegel domain of $\mathbf{G}$, for taking $g = \varepsilon(pak)$ we majorize (10). That (12) is bounded by $C|g|^N$ follows from the elements of the theory of Eisenstein series (see the remarks following Lemma 4.1 of my notes on the subject).

Apart from a finite sum over P, distinguished σ, and j and a constant

$$(-1)^{a_Q^\varepsilon}\frac{1}{(2\pi)^{a_P}n_1(P)}$$

we have to consider

$$(13)\quad \int_{\varepsilon\mathfrak{a}_1^1} \varepsilon\sigma_1^2(H-T)e^{-2\rho_{P_1}(H)} \int_{\operatorname{Re}\lambda=-\Lambda} \int_{P_1\cap\varepsilon^{-1}(P_1)\backslash \mathbf{P}_1^1\times K} a(pak,\lambda) = A^T(\phi)$$

where

$$a(pak,\lambda) = \Lambda^{T,P_1} E_{P_1}(pak, \rho_{\sigma,\lambda}(\phi)\Phi_j\overline{(E_{P_1}(\varepsilon(pak), \Phi_j)}\quad ,$$

$a = \exp H$, and $_\varepsilon\mathfrak{a}_1^1 \subseteq \mathfrak{a}_1$ is the intersection of the kernels of the ε-invariant rational characters of G.

In contrast to the integrals appearing in the ordinary trace formula, (13) does not seem to admit a useful explicit expression even when Φ_j is a cusp form. So we derive an approximate formula for it, anticipating the needs of the arguments in the two Amer. Jour. papers. Recall that they involve substituting $\phi_\gamma = \phi_H$ for ϕ where $\gamma = \gamma_H$ is the distribution

$$\gamma = \frac{1}{\Omega} \sum_{s\in\Omega} \gamma_{s^{-1}H}$$

and ϕ_γ is obtained from ϕ by applying the multiplier associated to γ. Then $A^T(\phi_H)$ is a function of T and H, $H \in \mathfrak{h}$ (Arthur works with a subspace $\mathfrak{h}^1 \subseteq \mathfrak{h}$, but with our formulations $\mathfrak{h}$ is better).

In order to simplify the formulation at various places, we formalize the inequality (5.1) of Amer. Jour. I into a definition. Fix an integer $d_0 \geq 0$. If $\psi^T(H)$ is a function of T and H we write

$$\psi^T(H) \sim 0$$

if there are positive constants ε and C and for every invariant differential operator D on $\mathfrak{h}$ a constant c_D such that

$$|D\psi^T(H)| \leq c_D e^{-\varepsilon\|T\|}\,(1+\|T\|)^{d_0}$$

whenever $d(T) > C(1+\|H\|)$. Recall that

$$d(T) = \min_{\{\alpha, Q \mid \alpha \in \Delta_Q, Q \supseteq P_0\}} \alpha(T) \quad .$$

Set $\mu = -\varepsilon^{-1}(\lambda)$ and define Ψ_j by

$$\Phi_j(\varepsilon(g)) = \Psi_j(g) \quad .$$

It is a function in $\mathfrak{a}_{\varepsilon^{-1}(\chi),\varepsilon^{-1}(\sigma)}(\varepsilon^{-1}(P))$. Thus if $P' \subseteq \varepsilon^{-1}(P_1) \cap P_1$ and $s \in \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'})$ we may build the associated Eisenstein series $E_{P_1}(g, M_{P'|\varepsilon^{-1}(P)}(s,\bar{\mu})\Psi_j)$. Set

$$(14)\ b(pak,\lambda) = \sum_{P'}\sum_s \Lambda^{T,P_1} E_{P_1}(pak, \rho_{\sigma,\lambda}(\phi)\Phi_j)\bar{E}_{P_1}(pak, M_{P'|\varepsilon^{-1}(P)}(s,\bar{\mu})\Psi_j) \quad .$$

The sum over P' is a sum over associate classes within $\varepsilon^{-1}(P_1) \cap P_1$. Thus we take only one representative from each class, several classes appearing only because they become associate in $\varepsilon^{-1}(P_1)$. The variable s runs over

$$\Omega^{\varepsilon^{-1}(P_1)\cap P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P'}) \setminus \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'}) \quad .$$

Finally set

$$B^T(\phi) = \int_{\varepsilon \mathfrak{a}_1^1} \varepsilon \sigma_1^2(H-T) e^{-2\rho_{P_1}(H)} \int_{\operatorname{Re}\lambda=0} \int_{M_1\backslash \mathbf{M}_1^1\times K} b(mak, \lambda) \quad .$$

We shall show that the iterated integral converges and that

$$A^T(\phi_H) - B^T(\phi_H) \sim 0 \quad .$$

We begin by studying

$$\int_{P_1 \cap \varepsilon^{-1}(P_1)\backslash \mathbf{P}_1^1\times K} a(pak, \lambda)$$

when Φ_j is a cusp form. This is best regarded as a triple integral, over

$$(M_1\backslash \mathbf{M}_1^1 \times K) \times (M_1 \cap \varepsilon^{-1}(P_1)\backslash M_1) \times N_1\backslash \mathbf{N}_1 \quad ,$$

and we begin with the integral over $N_1\backslash \mathbf{N}_1$.

Since

$$\Lambda^{T,P_1} E_{P_1}(ng, \rho_{\sigma,\lambda}(\phi)\Phi_j) = \Lambda^{T,P_1} E_{P_1}(g, \rho_{\sigma,\lambda}(\phi)\Phi_j)$$

we are led to consider

$$\int_{N_1\backslash \mathbf{N}_1} E_{P_1}(\varepsilon(ng), \Phi_j)dn = \int_{N_1\backslash \mathbf{N}_1} E_{\varepsilon^{-1}(P_1)}(ng, \Psi_j)dn \quad .$$

I claim that it is equal to

$$(15) \qquad \sum_{P'} \sum_{s} E_{P_1 \cap \varepsilon^{-1}(P_1)}(g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad ,$$

the range of summation being the same as in (14).

It is enough to verify this equality in the domain of absolute convergence of the Eisenstein series. The proof will be easier to follow if for a few brief moments we change the notation, letting $\varepsilon^{-1}(M_1)$ be G, $P_1 \cap \varepsilon^{-1}(M_1)$ be Q, and $\varepsilon^{-1}(P)$ be P. Our integral is then

(16) $$\int_{N_Q \backslash \mathbf{N}_Q} E_G(ng, \Psi_j)\,dn \quad .$$

Let

$$F(g, \Psi_j) = \Psi_j(g) e^{(\mu+\rho_P)(H_P(g))} \quad ,$$

so that (16) is equal to

(17) $$\sum_{\gamma \in P \backslash G/Q} \int_{N_Q \backslash \mathbf{N}_Q} \sum_{\delta \in Q \cap \gamma^{-1} P \gamma \backslash Q} F(\gamma \delta n g, \Psi_j)\,dn \quad .$$

Each γ may be chosen to lie in the normalizer of $\mathfrak{a}_0$ and thus to represent an element s^{-1} of the Weyl group $\Omega(\mathfrak{a}_0, \mathfrak{a}_0)$. We have sufficient freedom to suppose that $s\alpha > 0$ for $\alpha \in \Delta_0^{\varepsilon^{-1}(P)}$. The group

$$\gamma \delta Q \delta^{-1} \gamma^{-1} \cap M_P = \gamma Q \gamma^{-1} \cap M_P$$

is then a standard parabolic subgroup of M_P with unipotent radical $\gamma N_Q \gamma^{-1} \cap M_P$. Since Ψ_j is a cusp form the term of (17) associated to γ is 0 unless $\gamma N_Q \gamma^{-1} \cap M_P = 1$ and thus unless $\gamma M_Q \gamma^{-1} \supset M_P$.

We now assume this and in addition that $s^{-1}\alpha > 0$ for $\alpha \in \Delta_0^Q$

which implies that $s\alpha > 0$ for $\alpha \in \Delta_0^P$. The group

$$P' = N_Q(\gamma^{-1}P\gamma \cap M_Q) = N_Q(\gamma^{-1}N_P\gamma \cap M_Q)\cdot\gamma^{-1}M_P\gamma$$

is a parabolic subgroup associate to P and $s \in \Omega(\mathfrak{a}_P, \mathfrak{a}_{P'})$.

The term associated to γ is equal to

$$\sum_{\delta \in Q \cap \gamma^{-1}P\gamma\backslash Q/N_Q} \int_{N_Q \cap \delta^{-1}\gamma^{-1}P\gamma\delta\backslash \mathbf{N}_Q} F(\gamma\delta n g, \Psi_j)\,dn$$

or

$$\sum_{\delta \in Q \cap \gamma^{-1}P\gamma\backslash Q/N_Q} \int_{N_Q \cap \gamma^{-1}P\gamma\backslash \mathbf{N}_Q} F(\gamma n\delta g, \Psi_j)\,dn \quad .$$

The domain of integration is $N_Q \cap \gamma^{-1}N_P\gamma\backslash \mathbf{N}_Q$ and may be replaced by $\mathbf{N}_Q \cap \gamma^{-1}\mathbf{N}_P\gamma\backslash\mathbf{N}_Q$. Since $N_{P'} = N_Q(\gamma^{-1}N_P\gamma \cap M_Q)$ and $\gamma^{-1}N_P\gamma \cap M_Q \subseteq \gamma^{-1}N_P\gamma$ the domain of integration may in fact be taken to be $\mathbf{N}_{P'} \cap \gamma^{-1}\mathbf{N}_P\gamma\backslash\mathbf{N}_{P'}$. Hence the integration yields

$$F(\delta g, M_{P'|P}(s, \overline{\mu})\Psi_j)$$

The range of summation is

$$(M_Q \cap \gamma^{-1}P\gamma)N_Q\backslash Q = P'\backslash Q \quad .$$

So we obtain

$$\sum_{P'\backslash Q} F(\delta g, M_{P'|P}(s, \overline{\mu})\Psi_j) \quad .$$

Summing over γ and reverting to our original notation we obtain (15).

The next step is to replace g by γg in (15) and to sum over $\gamma \in M_1 \cap \varepsilon^{-1}(P_1)\backslash M_1$. Interchanging the order of summation we obtain

$$(17)\ \sum_{P'} \sum_{s} \sum_{\gamma \in M_1 \cap \varepsilon^{-1}(P_1)\backslash M_1} E_{P_1 \cap \varepsilon^{-1}(P_1)}(\gamma g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad .$$

To justify the interchange we must show that the inner sum converges absolutely. If so, it yields

$$E_{P_1}(g, M_{P'|\varepsilon^{-1}(P)}(s, \overline{\mu})\Psi_j) \quad .$$

For absolute convergence we need

$$\mathrm{Re}(\alpha, s\mu_1) = (\alpha, s\varepsilon^{-1}(\Lambda)) \gg 0$$

for all $\alpha \in \Delta^{P_1}_{P_1 \cap \varepsilon^{-1}(P_1)}$ or for $\alpha \in \Delta_0^{P_1} - \Delta_0^{P_1 \cap \varepsilon^{-1}(P_1)}$, which is of course $\Delta_0^{P_1} - \Delta_0^{\varepsilon^{-1}(P_1)}$, a subset of $\Delta_0 - \Delta_0^{\varepsilon^{-1}(P_1)}$. Here $\mu_1 = -\varepsilon^{-1}(\lambda_1)$ where λ_1 is the projection of λ on $\mathfrak{a}_1$. Since $s\varepsilon^{-1}(\Lambda) = \varepsilon^{-1}(\Lambda)$,

$$(\alpha, s\varepsilon^{-1}(\Lambda)) = (\varepsilon(\alpha), \Lambda)$$

and $\varepsilon(\alpha) \in \Delta_0 - \Delta_0^1$. Hence $(\varepsilon(\alpha), \Lambda) \gg 0$ by assumption.

We are left with the evaluation of

$$\sum_{P'} \sum_{t} \int_{M_1\backslash \mathbf{M}_1^1 \times K} \Lambda^{T,P_1} E_{P_1}(mak, \rho_{\sigma,\lambda}(\phi)\Phi_j) E_{P_1}(mak, M_{P'|\varepsilon^{-1}(P)}(t, \overline{\mu})\Psi_j)\, dmdk \quad ,$$

in which, for convenience later, the variable of summation s has been replaced by t. These integrals are evaluated by the inner product formula of the Compositio paper, which has (in effect) been proved in Lecture 12. The integral corresponding to P' and t is equal to $e^{2\rho_{P_1}(H)}$ times

$$(18)\quad \sum_{P''}\sum_{s_1,s_2} \frac{e^{(s_1\lambda+s_2t\mu)(T^1+H)}}{\theta^1_{P''}(s_1\lambda+s_2t\mu)}(M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j,\ M_{P''|\varepsilon^{-1}(P)}(s_2t,\overline{\mu})\Psi_j) \quad .$$

Here $P_0 \subseteq P'' \subseteq P_1$, $s_1 \in \Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P''})$, $s_2 \in \Omega^{P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$. The projection of T on $\mathfrak{a}^1_0$ is denoted T^1.

In general the terms of (18) are not individually defined on the domain of integration, $\mathrm{Re}\,\lambda = \mathrm{Re}\,\lambda_1 = -\Lambda$, because of the zeros of the denominator, which we now examine more closely. Apart from a constant $\theta^1_{P''}(s_1\lambda+s_2t\mu)$ is equal to

$$\prod_{\alpha\in\Delta^1_{P''}} \alpha(s_1\lambda+s_2t\mu) = \prod_{\alpha\in\Delta^1_{P''}} \alpha(s_1\lambda-s_2t\varepsilon^{-1}(\lambda)) \quad .$$

Moreover

$$\mathrm{Re}\ \alpha(s_1\lambda-s_2t\varepsilon^{-1}(\lambda)) = \alpha(s_2t\varepsilon^{-1}(\Lambda)) = \beta(\Lambda) \quad ,$$

with $\beta = \varepsilon(t^{-1}s_2^{-1}(\alpha))$, and β either is identically 0 on $\mathfrak{a}_1$ or vanishes nowhere on $\mathfrak{a}_1^+$. If it vanishes identically and s_α, s_β are the reflections corresponding to α and β then

$$s_\alpha s_2 t\varepsilon^{-1} = s_2 t\varepsilon^{-1}(s_\beta)\varepsilon^{-1} \quad .$$

If we sum (18) over P' and t we obtain sums over P', P'', and s_2, whose ranges of summation are to be specified, of

$$(19)\ \sum_{s_1,t} \frac{e^{s_1\lambda+s_2t\mu(T^1+H)}}{\mathcal{O}^1_{P''}(s_1\lambda+s_2t\mu)} (M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j, M_{P''|\varepsilon^{-1}(P)}(s_2t,\overline{\mu})\Psi_j)) \quad ,$$

s_1 running over $\Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P''})$ and t over $\Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P'})$. The remarks above allow us to apply the usual arguments and to conclude that the zeros of the denominators do not contribute to the singularities of (19). The sum is over P'' which we associate to P in P_1, and for each P'' over a set of representatives P' for the associate classes in $P_1 \cap \varepsilon^{-1}(P_1)$ which lie in the associate class of P'' in P_1. Once P'' and P' are fixed,

$$s_2 \in \Omega^{P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})/\Omega^{\varepsilon^{-1}(P_1)\cap P_1}(\mathfrak{a}_{P'}, \mathfrak{a}_{P'}) \quad .$$

We are taking $(\Lambda, \alpha) \gg 0$, $\alpha \in \Delta_1$. Thus the numerators of (19) are well-behaved functions and we can consider

$$(20) \int_{\mathrm{Re}\,\lambda_1=-\Lambda} \sum_{s_1,t} \frac{e^{(s_1\lambda+s_2t\mu)(T^1+H)}}{\theta^1_{P''}(s_1\lambda+s_2t\mu)} (M_{P''|P}(s_1,\lambda)\rho_{\sigma,\lambda}(\phi)\Phi_j, M_{P''|\varepsilon^{-1}(P)}(s_2t,\mu)\Psi_j))\,|d\lambda_1|$$

I claim that this is zero for T sufficiently regular and ${}_\varepsilon\sigma_1^2(H-T) \neq 0$ unless $s_2 \in \Omega^{\varepsilon^{-1}(P_1)\cap P_1}$. The reader will note that sufficiently regular means $d(T) \geq C(1 + \|H_0\|)$ when $\phi = \phi_{H_0}$.

Recall that Q is the smallest ε-invariant parabolic containing P_1. Let $H = T_1+X+Y$ with $Y \in \mathfrak{a}_Q$ and with $X \in \mathfrak{a}_1^Q$. Then $T = T_1+T^1$ and $\alpha(X) > 0$ for all $\alpha \in \Delta_1^Q$. We deform the contour to $\mathrm{Re}\,\lambda_1 = -\Lambda-tX$. Then

$$e^{s_1\lambda+s_2t\mu(T^1+H)}$$

is multiplied by

$$e^{-t(X-s_2t\varepsilon^{-1}(X),T)-t(X-s_2t\varepsilon^{-1}(X),X)} \quad .$$

Now, as we saw above,

$$(X-s_2t\varepsilon^{-1}(X),X) \geq \delta\|X\|^2$$

and

$$(X-s_2t\varepsilon^{-1}(X),T) = (X-\varepsilon s_2t\varepsilon^{-1}(X),T) \geq 0 \quad .$$

Since we can estimate

$$(M_{P''|P}(s_1, \lambda)\rho_{\sigma,\lambda}(\phi_{H_0})\Phi_j, M_{P''|\varepsilon^{-1}(P)}(s_2t, \mu)\Psi_j)$$

when $\mathrm{Re}\,\lambda = -\Lambda-tX$ by

$$e^{c\|X\|\|H_0\|}f(\mathrm{Im}\,\lambda) \quad ,$$

with f integrable, we see that the integral vanishes unless $\|X\| \leq c\|H_0\|$.

For $\|X\| \leq c\,\|H_0\|$ we take $\alpha \in \Delta_1^Q$ and deform the contour to $\mathrm{Re}\,\lambda_1 = -\Lambda-t\varpi_\alpha$. If $\varepsilon s_2t^{-1}\varepsilon^{-1}(\varpi_\alpha) \neq \varpi_\alpha$ then

$$(\varpi_\alpha - \varepsilon s_2 t^{-1}(\varpi_\alpha),\ T) \geq c\|T\| \ .$$

If $s_2 \notin \Omega^{\varepsilon^{-1}(P_1)}$ we can choose α such that $\varepsilon s_2 t^{-1}\varepsilon^{-1}(\varpi_\alpha) \neq \varpi_\alpha$ and then we obtain vanishing for $\|T\| \geq c(1+\|H_0\|)$. Note that the value of the constant c changes from line to line.

If $s_2 \in \Omega^{\varepsilon^{-1}(P) \cap P_1}$ then it may be taken to be 1. Then the integral (20) has a very useful property. Neither $M_{P''|P}(s_1, \lambda)$ nor $M_{P''|\varepsilon^{-1}(P)}(t, \mu)$ depends on λ_1 but only on the projection of λ onto $\mathfrak{a}_P^1$. Thus they have no singularities to obstruct the deformation of the contour, which may therefore be taken to be defined by any Λ with $(\alpha, \Lambda) < 0$ for all $\alpha \in \Delta_1$, or even Δ_1^Q. We may not however allow the (α, Λ) to become zero, for the zeros of the denominators could then cause trouble.

To obviate this we choose δ such that none of the functions

$$M_{P''|\varepsilon^{-1}(P)}(s_2 t, \mu)\Psi_j$$

which appear in (19) have singularities in the region $\|\mathrm{Re}\,\lambda\| < \delta$, $(\mathrm{Re}\,\lambda, \alpha) \leq 0$, $\alpha \in \Delta_1^Q$, even if $s_2 \notin \Omega^{\varepsilon^{-1}(P_1) \cap P_1}$. Then we choose a Λ with $\|\Lambda\| < \delta$. Having deformed the contour we take once again the sum over all P', P'' and s_2, thereby introducing an error which must be estimated. This done the zeros of the denominator no longer cause any trouble; so we can deform to $\Lambda = 0$. Putting back the factor $e^{2\rho_{P_1}(H)}$, then integrating over $\mathfrak{a}_P^1$, and finally multiplying by $\varepsilon\sigma_1^2(H-T)e^{-2\rho_{P_1}(H)}$ (as we have in effect already done) and integrating over ${}_{\varepsilon}\mathfrak{a}_1^1$ we obtain $B^T(\phi)$.

The error is a sum of integrals

$$\int_{{}_{\varepsilon}\mathfrak{a}_1^1} {}_{\varepsilon}\sigma_1^2(H-T) \int_{i\mathfrak{a}_P^1} C(\lambda, H)\,|d\lambda|\,dH \quad ,$$

where $C(\lambda, H)$ is given by (20) with $s_2 \notin \Omega^{\varepsilon^{-1}(P_1) \cap P_1}$ but with $\|\Lambda\| < \delta$. The estimations, which establish incidentally that the integrals defining $B^T(\phi)$ converge, mimic the earlier proof of vanishing.

Let $H = T_1+X+Y$ as before then the same arguments establish that for $\|X\| \geq c\|H_0\|$ we have

$$\|C(\lambda, \ H)\| \leq c_1 e^{-c\|X\|} \quad .$$

Notice that the ε-form of Lemma 7.3 implies that $\|Y\| \leq c\|X\|$ if ${}_{\varepsilon}\sigma_1^2(X+Y) \neq 0$ (c is a highly <u>variable</u> constant). On the other hand if $d(T) > C(1+\|H_0\|)$ and $\|X\| \leq c\|H_0\|$ with $C \gg c$ then

$$C(\lambda, \ H) \leq c_2 e^{-c\|T\|} \ .$$

Since

$$\int_{\{H=X+Y \mid \|X\| \geq c\|H_0\|\}} {}_{\varepsilon}\sigma_1^2(X+Y) e^{-c\|X\|} \leq c_3 e^{-c\|H_0\|}$$

the asserted estimates follow easily. (The reader will have observed that the arguments are often sketchy. This is partly because they will ultimately be included in the notes of the earlier lectures.)

To show that

$$A^T(\phi_H) - B^T(\phi_H) \sim 0$$

even when Φ_j is not a cusp form we have to use techniques from the second Duke Journal paper. There will not be time to discuss this paper; so we merely sketch the argument envisaged, referring for a careful exposition to the revised lecture notes.

An Eisenstein series on the group P_1 associated to P may be built up with residues of Eisenstein series associated to cusp forms on groups Q contained in P. Recall that taking a residue involves nothing more than a contour integration over a small cycle surrounding the point at which the residue is wanted. These cycles lie in $\mathfrak{a}_Q^P \otimes \mathbf{C}$ and the parameter which is important for the transition from $A^T(\phi)$ to $B^T(\phi)$ was $\lambda_1 \in \mathfrak{a}_1$. So they do not interfere with each other.

Hence we are able, in imitation of Lemma 3.1 of the Duke Journal paper, to show that all operators commute with the formation of residues, thereby deducing the general statements from those for cusp forms. For example, this is certainly so of the integrations over $N_1 \backslash \mathbf{N}_1$ and $M_1 \backslash \mathbf{M}_1^1 \times K$ and of the summation over $M_1 \cap \varepsilon^{-1}(P_1) \backslash M_1$ that appeared in the treatment of $A^T(\phi)$. So we will obtain formulas like (18) but by no means so simple. Nonetheless these terms whose apparent singularities prevent us from deforming the contour back to $\operatorname{Re} \lambda_1 = -\Lambda$, $(\Lambda, \alpha) > 0$, $\|\Lambda\| < \delta$ can still be shown by the previous arguments to be zero. So we can deform the contour and then restore these terms and estimate the error introduced as before.

More loose strands. To obtain the fine χ-expansion for the twisted case we imitate the arguments in the Amer. Jour. papers but, once again for lack of time, we can only sketch the modifications.

We know that $J_{\chi}^{T}(\phi)$ is a polynomial (presumably for $d(T) \geq c(1+\|H\|)$ if ϕ is replaced by ϕ_H). It is given by

$$(21)\quad \sum_{P} \frac{1}{(2\pi)^{a_P}} \sum_{\sigma} \sum_{P\subset P_1 \subset P_2}{}' \frac{(-1)^{a_Q^{\varepsilon}}}{n_1(P)} \int_{\varepsilon \mathfrak{a}_1^1} \varepsilon \sigma_1^2(X-T) e^{-2\rho_{P_1}(X)} \int_{\operatorname{Re}\lambda=0} \Psi_{\sigma}^{T}(X,\lambda,\phi)\, d\lambda\, dX$$

plus an error term, $E^T(\phi)$. The error term satisfies

$$E^T(\phi_H) \sim 0 \quad .$$

The sum over P, σ, which is effectively finite provided ϕ is K-finite runs over $P \supseteq P_0$ and distinguished σ for which $\mathfrak{a}_{\chi,\sigma}(P) \neq 0$. The sum over $P_1 \subset P_2$ runs over pairs of standard parabolics which are separated by a unique ε-invariant standard parabolic Q. The expression $\Psi_{\sigma}^{T}(X, \lambda, \phi)$ which implicitly depends on P, is equal to

$$\sum_{P'} \sum_{s} \sum_{j} \int_{M_1\backslash \mathbf{M}^1 \times K} \Lambda^{T,P_1} E_{P_1}(mak, \rho_{\sigma,\lambda}(\phi)\Phi_j) \overline{E}_{P_1}(mak, M_{P'|\varepsilon^{-1}(P)}(s,\overline{\mu})\Psi_j)\, dm\, dk \quad .$$

If we let $P^T(H)$ be the polynomial in T which equals $J_{\chi}^{T}(\phi_H)$ for $d(T) > c(1+\|H\|)$ and if we let $\psi^T(H)$ be the value of (21) when $\phi = \phi_H$ we still have

$$P^T(H) - \psi^T(H) \sim 0$$

and

$$\psi^T(H) = \sum_\Gamma \psi_\Gamma^T(H) e^{X_\Gamma(H)} .$$

Thus Prop. 5.1 of the first Amer. Jour. paper allows us to write

$$P^T(H) - \sum_\Gamma P_\Gamma^T(H) e^{X_\Gamma(H)} ,$$

the $P_\Gamma^T(H)$ being polynomials in T, and various estimates obtaining.

At this point we can take over the argument of Amer. Jour. I almost literally. It allows us first of all to consider not (21) itself, but (21) with $\Psi_\sigma^T(X, \lambda, \phi)$ replaced by

$$\Psi_\sigma^T(X, \lambda, \phi) B_\sigma(\lambda) ,$$

where B is a Weyl group invariant function on the Schwartz space of $i\mathfrak{f}$. Then §7 of the paper allows us to replace $\Psi_\sigma^T(X, \lambda, \phi)$ by a much more convenient expression, namely $e^{2\rho_{P_1}(X)}$ times

$$(22)\quad \sum_{P'} \sum_{s,t} \frac{e^{(s\lambda+t\mu)(T^1+X)}}{\theta_{P'}^1(s\lambda+t\mu)} \operatorname{tr}(\varepsilon M_{P'|\varepsilon^{-1}(P)}(t, \varepsilon^{-1}(\lambda))^{-1} M_{P'|P}(s, \lambda) \rho_{\sigma,\lambda}(\phi)) .$$

Here P' runs over standard parabolic subgroups of P_1 associate to P and $s \in \Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_{P'})$. On the other hand t runs over all elements of $\Omega^Q(\mathfrak{a}_{\varepsilon^{-1}(P), P'})$ which can be expressed as a product $t_1 t_2$ with

$t_1 \in \Omega^{P_1}(\mathfrak{a}_{P''}, \mathfrak{a}_{P'})$, $t_2 \in \Omega^{\varepsilon^{-1}(P_1)}(\mathfrak{a}_{\varepsilon^{-1}(P)}, \mathfrak{a}_{P''})$.

After making these two substitutions we obtain a function of T which depends on B. All we need do is to find a polynomial $P^T(B)$ to which it is approximately equal for $d(T) > \delta\|T\| \gg 0$ and to see what happens to $P^T(B)$ as $B \longrightarrow 1$ in an appropriate sense. The primary purpose of this lecture was to deal with the first step, mimicking the second Amer. Jour. paper. We need only consider compactly supported B.

Symmetrization. In (21), with the substitutions indicated, we may sum over all pairs $P_1 \subset P_2$ provided we remove $(-1)^{a_Q^\varepsilon}$ from before the integral and insert

$$\sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^\varepsilon}$$

after it, the sum on Q now being taken over all ε-invariant parabolics between P_1 and P_2.

There are also a number of simple modifications of (22) to be made. First of all

$$\varepsilon M_{P'|\varepsilon^{-1}(P)}(t, \varepsilon^{-1}(\lambda))^{-1} M_{P'|P}(s,\lambda) = M_{\varepsilon(P')|P}(\varepsilon(t),\lambda)^{-1} M_{\varepsilon(P')|\varepsilon(P)}(\varepsilon(s),\varepsilon(\lambda))\varepsilon \quad ,$$

and by the functional equation the right side is equal to

$$M_{\varepsilon(t^{-1}(P'))|P}(1,\lambda)^{-1} \varepsilon(t)^{-1} \varepsilon(t) M_{\varepsilon(t^{-1}(P'))|\varepsilon(P)}(\varepsilon(t^{-1}s),\varepsilon(\lambda))$$

times

$$e^{\left\langle -\lambda+\rho_{\varepsilon(t^{-1}(P'))}+\varepsilon t^{-1}s\lambda-\rho_{\varepsilon(t^{-1}(P'))},T_0-\varepsilon(t^{-1})T_0\right\rangle_\varepsilon} .$$

So we change the notation, letting $\varepsilon(t^{-1}(P'))$ become P', $\varepsilon(t)$ become t, and $\varepsilon(t^{-1}s)$ become s, thereby simplifying the product to

$$M_{P'|P}(1, \lambda)^{-1}M_{P'|\varepsilon(P)}(s, \varepsilon(\lambda))e^{\left\langle s\varepsilon\lambda-\lambda,T_0-t^{-1}T_0\right\rangle} .$$

The new s lies in $\Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{\varepsilon(P)})$ and the sole condition on it is that it be expressible as a product

$$s = t^{-1}s_1$$

with $s_1 \in \Omega^{\varepsilon(P_1)}(\mathfrak{a}_{\varepsilon(P)}, \mathfrak{a}_{t(P')})$. Observe also that t is determined by the condition that t applied to the new P' be standard.

With the new notation the denominator is replaced by

$$\theta_{P'}^{t^{-1}(\varepsilon(P_1))}(s\varepsilon\lambda-\lambda) .$$

Moreover we can combine the two exponential factors that appear in the numerator to obtain

$$e^{\left\langle s\varepsilon\lambda-\lambda,t^{-1}\varepsilon(X-T_1)\right\rangle+\left\langle s\varepsilon\lambda-\lambda,Y_{P'}(T)\right\rangle}$$

where

$$Y_{P'}(T) = t^{-1}(\varepsilon(T)) + T_0 - t^{-1}T_0 .$$

We are thus concerned with

$$(23)\quad \sum_P \frac{1}{(2\pi)^{a_P}} \sum_\sigma \sum_{P_1} \int_{\varepsilon \mathfrak{a}_1^1} \sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2(X-T) \frac{1}{n_1(P)} \int_{\mathrm{Re}\,\lambda=0} \Omega_\sigma^T(X,\lambda)\, d\lambda\, dX$$

where $\Omega_\sigma^T(X, \lambda)$ is the sum over the indicated P' and s of

$$(24)\quad \frac{e^{\langle s\varepsilon\lambda-\lambda,\, Y_{P'}(T)+t^{-1}\varepsilon(X-T_1)\rangle}}{\theta_{P'}^{\varepsilon(P_1)}(s\varepsilon\lambda-\lambda)} \operatorname{tr}(M_{P'|P}(1,\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_\sigma(\lambda) \quad .$$

Recall that $n_1(P)$ is the number of parabolic subgroups of P_1 with a given Levi factor in common with P. The next step is to replace $n_1(P)$ by $n(P)$, defined in the same way but with G replacing P_1.

Suppose $r \in \Omega(\mathfrak{a}_{P''}, P)$. We replace in (23) and (24) the variable λ by $r\lambda$ and σ by $r\sigma$. The expression (24) becomes the product of

$$(25)\quad \frac{e^{\langle r^{-1}s\varepsilon(r)\varepsilon\lambda-\lambda,\ Y_{P'''}(T) - T_0 + r^{-1}(T_0) + r^{-1}t^{-1}\varepsilon(X-T_1)\rangle}}{\theta_{P'''}^{r^{-1}t^{-1}\varepsilon(P_1)}(r^{-1}s\varepsilon(r)\varepsilon\lambda-\lambda)} \quad ,$$

with $P''' = r^{-1}(P')$, and

$$(26)\quad \operatorname{tr} M_{P'|P}(1,\ r\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\ \varepsilon(r\lambda))\varepsilon\rho_{r\sigma,r\lambda}(\phi))B_{r\sigma}(r\lambda) \quad .$$

Since B is invariant under the Weyl group,

$$B_{r\sigma}(r\lambda) = B_\sigma(\lambda) \quad .$$

Moreover the functional equations allow us to rewrite

$$M_{P'|P}(1,\ r\lambda)^{-1}M_{P'|\varepsilon(P)}(s,\ \varepsilon(r\lambda))$$

as the product of

$$M_{P|P''}(r,\lambda)M_{P'|P''}(r,\lambda)^{-1}M_{P'|\varepsilon(P'')}(s\varepsilon(r),\varepsilon(\lambda))M_{\varepsilon(P)|\varepsilon(P''')}(\varepsilon(r),\varepsilon(\lambda))^{-1} \quad .$$

We also have

$$M_{\varepsilon(P)|\varepsilon(P'')}(\varepsilon(r),\ \varepsilon(\lambda))^{-1}\varepsilon = \varepsilon M_{P|P''}(r,\lambda)^{-1}$$

and

$$M_{P|P''}(r,\ \lambda)^{-1}\rho_{r\sigma,r\lambda}(\phi) = \rho_{\sigma,\lambda}(\phi)M_{P|P''}(r,\ \lambda)^{-1} \quad .$$

Thus (26) is equal to

$$\mathrm{tr}(M_{P'|P''}(r,\ \lambda)^{-1}M_{P'|\varepsilon(P'')}(s\varepsilon(r),\ \varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda) \quad .$$

Making use of the functional equations as before we see that this is equal to the product of

$$\mathrm{tr}(M_{r^{-1}(P')|P''}(1,\ \lambda)^{-1}M_{r^{-1}(P')|\varepsilon(P'')}(r^{-1}s\varepsilon(r),\ \varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_{\sigma}(\lambda)$$

and

$$e^{\langle r^{-1}s\varepsilon(r)\varepsilon(\lambda)-\lambda,T_0-r^{-1}(T_0)\rangle} \quad .$$

Notice that this exponential cancels part of the numerator of (25).

Now we have to try to simplify the results. The sum in (23) is over all standard parabolics P, P_1 containing P. The invariance with respect to r just established allows us, for a given P_1, to replace P by a fixed element in its associate class provided we replace $n_1(P)$ by $|\Omega^{P_1}(\mathfrak{a}_P, \mathfrak{a}_P)|$. However we can then use the invariance once again to sum over all P'' and all r provided we replace $n_1(P)$ by $n(P)$. So we change the notation, denoting P'' by P, P''' by P', tr by t, and $r^{-1}s\varepsilon(r)$ by s.

We obtain

$$\sum_P \sum_s \frac{1}{(2\pi)^{a_P} n(P)} \sum_\sigma \sum_{P_1} \int_{\varepsilon\mathfrak{a}_1^1} \sum_{P_1 \subset Q \subset P_2} (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2(X - Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P'} \omega_{P'}^T(X, \lambda)$$

where $\omega_{P'}^T(X, \lambda)$ is equal to

$$\frac{e^{\left\langle s\varepsilon\lambda - \lambda, Y_{P'}(T) + (X - Y_1(T)) \right\rangle}}{\theta_{P'}^{P_1}(s\varepsilon\lambda - \lambda)} \operatorname{tr}(M_{P'|P}(1,\lambda)^{-1} M_{P'|\varepsilon(P)}(s, \varepsilon(\lambda)) \varepsilon\rho_{\sigma,\lambda}(\phi)) B_\sigma(\lambda) .$$

There are further changes in the notation to explain. The group formerly labelled $r^{-1}t^{-1}\varepsilon(P_1)$ is now labelled P_1, and the condition on P_1 is that it contain P'. Moreover P_2 is the former $r^{-1}t^{-1}\varepsilon(P_2)$. The new t is determined by the condition that $t(P')$, and thus $t(P_1)$ and $t(P_2)$ be standard. The function ${}_\varepsilon\sigma_1^2$ is then defined by transport of structure and $Y_1(T)$ is the projection of $t^{-1}T$ on $\mathfrak{a}_1$. So are the groups Q, but they can clearly be defined intrinsically. If $(L, \mathfrak{a})$ is the ε-special pair attached to s then the groups Q

are the elements of $\mathcal{F}_\varepsilon(L, \mathfrak{a})$ which contain P_1.

We can unburden ourselves of P_2 if we recall from Lecture 9 that

$$\sum_{\{Q,P_2|P_1 \subset Q \subset P_2\}} (-1)^{a_Q^\varepsilon} {}_\varepsilon\sigma_1^2 = \sum_{P_1 \subset Q} (-1)^{a_Q^\varepsilon} \tau_1^Q {}_\varepsilon\hat{\tau}_Q \quad .$$

The sum over P, s, and σ can now be forgotten as can $\frac{1}{(2\pi)^{a_P}} \cdot \frac{1}{n(P)}$ for they appear in the statement of the fine χ-expansion, the transition from P to M being effected as in the second Amer. Jour. paper.

Thus we are reduced to considering

$$(27) \quad \sum_{P_1} \int_{\varepsilon\mathfrak{a}_1^1} \sum_{Q \supset P_1} (-1)^{a_Q^\varepsilon} \tau_1^Q(X-Y_1(T))\, {}_\varepsilon\hat{\tau}_Q(X-Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P'} \omega_{P'}^T(X, \lambda) \quad ,$$

in which we have still to be precise about which P' and P_1 occur.

Choose the unique standard P'' and the unique $t \in \Omega(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$ such that $t(P') = P''$. If we review the calculations that led to this point, we see that

$$ts\varepsilon(t^{-1}) = s_1 s_2 \varepsilon(s_1^{-1})$$

with $s_1 \in \Omega^{t(P_1)}(\mathfrak{a}_{P''}, {}_{t(P')})$, $s_2 \in \Omega^{t(P_1)}(\mathfrak{a}_{P''}, \mathfrak{a}_{\varepsilon(P'')})$, where $P'' \in \mathcal{F}(M_0)$. We see indeed that the necessary and sufficient condition that P' and P_1 occur is that $ts\varepsilon(t^{-1})$ have this form. In particular if one P' occurs then, as we should expect, all parabolics associate to it in P_1 occur. Thus for a given P_1 either no $P' \subseteq P_1$ occur or

P' runs over $\mathcal{P}^{M_1}(M_P)$.

<u>Combinatorics</u>. We take P_1 with $\mathfrak{a}_{P_1} \subseteq \mathfrak{a}_P$ and consider

$$(28)\quad \int_{\varepsilon\mathfrak{a}_1^1} \sum_{\{Q \in \mathcal{F}_\varepsilon(L,\mathfrak{a}) | Q \supseteq P_1\}} (-1)^{a_Q^\varepsilon} \tau_1^Q(X-Y_1(T))_\varepsilon \hat{\tau}_Q(X-Y_1(T)) \int_{i\mathfrak{a}_P} \sum_{P' \in \mathcal{P}^{P_1}(M_P)} \omega_{P'}^T(X,\lambda$$

without asking whether it actually occurs in (27). We shall see that if it does not occur then it approaches 0 as T approaches ∞, and therefore may be added to the sum (27), in whose behavior we are interested only for large T.

Let $(L, \mathfrak{a})$ be the ε-special pair determined by s. We introduce new coordinates on $i\mathfrak{a}_P$, replacing λ by the pair (ν, Λ), with ν the projection of λ on $i\mathfrak{a}$ and $\Lambda = s\varepsilon\lambda-\lambda$. It is this change of coordinates which introduces the factor $\frac{1}{\Delta}$ into the statement of the fine χ-expansion.

We may write $Y_{P'}(T) + X-Y_1(T)$ as $X + Y_{P'}^1(T)$, where $\{Y_{P'}^1(T) | P' \in \mathcal{P}^1(M_P)\}$ is an $A_{M_P}^1$-orthogonal family. $Y_{P'}^1(T)$ is given by

$$(r^{-1}T)^1 + T_0 - r^{-1}T_0 \quad ,$$

where $(r^{-1}T)^1$ is the projection of $r^{-1}T$ on $\mathfrak{a}_0^1$, r being any element of the Weyl group such that $r(P^1)$ is standard.

The inner integral in (28) is equal to

$$(29)\qquad \int e^{\langle \Lambda, X\rangle} \operatorname{tr}(M_M(\lambda) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi)) B_\sigma(\lambda)\, |d\Lambda d\lambda|$$

with

$$M_M(\lambda) = \sum_{P'} \frac{M_{P'|P}(1,\lambda)^{-1} M_{P'|P}(1,\lambda+\Lambda)}{\theta^1_{P'}(\Lambda)} e^{\langle \Lambda, Y^1_{P'}(T)\rangle}$$

and $M = M_P$. We have used the identity

$$M_{P'|P}(1,\lambda)^{-1} M_{P'|\varepsilon(P)}(s,\varepsilon(\lambda)) = M_{P'|P}(1,\lambda)^{-1} M_{P'|P}(1,s\varepsilon(\lambda)) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda)) \quad .$$

We denote the value of (28) by $f(X)$, the function f being defined on $\mathfrak{a}_1$ or on $\mathfrak{a}_1/\mathfrak{a} \cap \mathfrak{a}_1$ because Λ is orthogonal to $\mathfrak{a}$. We remark that if D is any differential operator with constant coefficients on $i\,\mathfrak{a}_1$ then

(30) $$|D(\mathrm{tr}(M_M(\lambda) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi))B_\sigma(\lambda))| \le c_D(1+\|T\|)^{d_0}$$

where d_0 is independent of D. I omit the verification, which is easy with the help of the expansion $\sum c^S_M d^1_{\mathfrak{S}}$ to be developed below. The essential observation is that $\langle \Lambda, Y^1_{P'}(T)\rangle = 0$ for $\Lambda \in i\,\mathfrak{a}_1$.

We deduce from (30) that if $X = U+V, V \in i\,\mathfrak{a}$, U orthogonal to $i\,\mathfrak{a}$ then for any n we have

(31) $$|f(X)| \le c_n(1+\|U\|)^{-n}(1+\|T\|)^{d_0} \quad .$$

LEMMA 6. <u>The expression</u> (28) <u>goes to</u> 0 <u>as</u> T <u>approaches</u> ∞ <u>unless</u> $\mathfrak{a}_1$ <u>is invariant under</u> $s\varepsilon$ <u>and</u> M_1 <u>is the centralizer of</u> $\mathfrak{a}_1^{s\varepsilon}$.

To prove this lemma it is useful to write

$$\sum_{\substack{Q \supset P_1 \\ Q \in \mathfrak{F}_\varepsilon(L, \mathfrak{a})}} (-1)^{a_Q^\varepsilon} \tau_1^Q(X - Y_1(T)) \, {}_\varepsilon\hat{\tau}_Q(X - Y_1(T))$$

once again as

$$\sum_{P_2 \supset Q \supset P_1}' (-1)^{a_Q^\varepsilon} \, {}_\varepsilon\sigma_1^2(X - Y_1(T))$$

the sum running over those P_2 such that there is exactly one element of $\mathfrak{F}_\varepsilon(L, \mathfrak{a})$ between P_1 and P_2.

It is clear that we may reduce ourselves to the case that P_1 is standard. We shall show that if P_1 is standard and (28) does not approach 0 as T approaches ∞ then P_1 is ε-invariant and $s \in \Omega^{P_1}$. A consequence will be that if (28) does not approach 0 then it actually occurs in (27).

It will approach 0 if there is a positive constant c such that

$$(32) \qquad \|U\| \geq c\|V\| \;, \; \|U\| \geq c\|T\|$$

whenever ${}_\varepsilon\sigma_1^2(X - Y_1(T)) \neq 0$ for some P_2 such that there is a unique ε-invariant parabolic between P_1 and P_2.

Recall that $d(T) \geq c\|T\|$. Consequently $\|T_1\| \geq c\|T\|$, $T_1 = Y_1(T)$ being the projection of T on $\mathfrak{a}_1$ (unless $P_1 = G$, the trivial case).

If (32) doesn't hold then we can find a sequence $\{(X_n, T_n)\}$ such that

$$\frac{U_n}{\|U_n\| + \|V_n\| + \|T_n\|} \longrightarrow 0 \quad .$$

Taking the limit we find a non-zero pair (X, T) with $X \in \mathfrak{a} \cap \mathfrak{a}_1$ and with

$$\alpha(X-T_1) \geq 0, \ \alpha \in \Delta_1^2 , \quad \alpha(X-T_1) \leq 0 \quad \alpha \in \Delta_1 - \Delta_1^2 ,$$
$$_{\varepsilon}\varpi_{\alpha}(X-T_1) \geq 0 , \quad \alpha \in {}_{\varepsilon}\Delta_0 - {}_{\varepsilon}\Delta_0^Q \ .$$

Let $X = X_1^Q + X_Q$, $X_Q \in \mathfrak{a}_Q$, $X_1^Q \in \mathfrak{a}_1^Q$, $T_1 = T_1^Q + T_Q$. Since $X_1^Q \in \mathfrak{a}_1^Q \cap \mathfrak{a}$ and since Q is the smallest ε-invariant parabolic containing P_1 we conclude that $X_1^Q = 0$. Then $X_1^Q \in \mathfrak{a}^{Q^+}$

$$\alpha(-T_1^Q) \geq 0 , \qquad \alpha \in \Delta_1^Q \ .$$

If $P_1 \neq Q$, so that Δ_1^Q is not empty, and if $\alpha \in \Delta_1^Q$ then

$$\alpha(T_1^Q) = \alpha(T_1) \geq \alpha(T) \geq c\|T\| \ .$$

We deduce that $T = 0$ if $P_1 \neq Q$. On the other hand the proof of Lemma 7.3 of Lecture 7 shows that

$$\|X_1^Q - T_1^Q\| \geq c\|X_Q - T_Q\| , \qquad\qquad c > 0 \ .$$

Thus $X_1^Q = T = 0$ implies that $X_Q = X = 0$. We conclude that $P_1 = Q$.

The upshort is that in (27) we are free to sum over all P_1 with $\mathfrak{a}_{P_1} \subset \mathfrak{a}_P$ or only over those P_1 such $M_{P_1} = M_Q$ for some $Q \in \mathcal{F}_{\varepsilon}(L, \mathfrak{a})$. We take the former, larger set.

We set

Replace $d_{p_1}(\lambda, \Lambda)$, λ, Λ varying freely

by $d_{p_1}(\lambda, \Lambda) A(\Lambda)$, $A(\Lambda)$ compact support

Since $\Lambda = s_\epsilon \lambda - \lambda$, $B_\sigma(\lambda) \neq 0$, Λ in effect is

compactly supported

$$c^1_{P'}(\Lambda) = e^{\langle \Lambda, Y^1_{P'}(T)\rangle}, \qquad \Lambda \in \mathfrak{a}_{P'}$$

But can multiply with fn of compact support $A(\Lambda)$

and we set

$\Lambda = s\varepsilon\lambda - \lambda$ ⇒ compact support No!

$$d_{P'}(\lambda,\Lambda) = \mathrm{tr}\{M_{P'|P}(1,\lambda)^{-1}M_{P'|P}(1,\lambda+\Lambda)M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\varepsilon\rho_{\sigma,\lambda}(\phi)\}B_\sigma(\lambda) \quad \cdot \hat{A}(\Lambda)$$

Moreover for convenience in the following discussion we denote the variable X appearing in (29) by H_1. Then by Lemma 6.3 of the Annals paper the expression (29) is equal to

$$\int e^{\langle \Lambda, H_1\rangle} \sum_{S \in \mathcal{F}^1(M)} c^S_M(\Lambda)\, d^{P_1}_S(\lambda, \Lambda)\, d\Lambda\, d\nu \quad .$$

Recall that if $S \in \mathcal{F}^1(M)$ then we attach to S the point $Y^1_S(T)$, obtained by projecting any $Y^1_{P'}(T)$ on $\mathfrak{a}_S$, $P' \subset S$, and the collection $\mathcal{Y}^S_M$

$$\mathcal{Y}^S_M = \{Y^1_{S(R)}(T) - Y^1_S(T) \mid R \subset S,\ R \in \mathcal{P}^1(M)\} \quad .$$

~~Each point of $\mathcal{Y}^S_M$ projects to $Y^1_S(T)$~~. If $\Gamma^S_M(\cdot, \mathcal{Y}^S_M)$ is the characteristic function of the convex hull of the points in $\mathcal{Y}^S_M$ then

$$c^S_M(\Lambda) = e^{\langle \Lambda, Y^1_S(T)\rangle} \int_{\mathfrak{a}^S_M} e^{\Lambda(H^S_M)}\, \Gamma^S_M(H^S_M, \mathcal{Y}^S_M)\, dH^S_M \quad .$$

The notation for the function $d^{P_1}_S$ is not good. For example d^G_S is the function formerly denoted d'_S. In any case for each fixed λ we express $d_{P'}(\lambda, \Lambda)$ as a Fourier transform

$$d_{P'}(\lambda,\ \Lambda) = \int_{i\mathfrak{a}_M} \hat{d}_{P'}(\lambda,\ U) e^{\langle \Lambda, U\rangle} dU \quad .$$

Then, as was observed in Lecture 13,

$$d_S^{P_1}(\lambda, \Lambda) = \int_{i\mathfrak{a}_M} dU \int_{\mathfrak{a}_S^1} dH_S^1\ \hat{d}_{P'}(\lambda, U) e^{\langle \Lambda, H_S^1\rangle + \langle \Lambda, U_1\rangle}\ \Gamma_S^1(H_S^1, U_S^1)$$

if $P' \subseteq S$. Since we are dealing with a (G, M) family this may also be written

$$\int_{\mathfrak{a}_S} dU_S \int_{\mathfrak{a}_S^1} dH_S^1 \hat{d}_S(\lambda, U_S) e^{\langle \Lambda, H_S^1\rangle + \langle \Lambda, U_1\rangle}\ \Gamma_S^1(H_S^1, U_S^1)$$

with

$$\hat{d}_S(\lambda,\ U_S) = \int_{\mathfrak{a}_M^S} \hat{d}_{P'}(\lambda,\ U_S + V) dV\ , \qquad P' \subseteq S \quad .$$

Putting this all together we see that (28) is equal to

$$(33) \int_{\varepsilon\mathfrak{a}_M^1} \sum_S \int_{\mathfrak{a}_S} \sum_{Q \supset P_1 \supset S} (-1)^{a_Q^\varepsilon} {}_\varepsilon\hat{\tau}_Q(H_1 - Y_1(T) - U_1) \tau_{P_1}^Q(H_1 - Y_1(T) - U_1) \Gamma_S^1(H_S^1 - Y_S^1(T), U_S^1) \Gamma_M^S(H_M^S, Y_M^S) \phi_S(H, U_S) dH \quad ,$$

where $H = H_1 + H_S^1 + H_M^S$ and the inner sum is over Q. The function $\phi_S(H,\ U_S)$ is given by

$$\phi_S(H,\ U_S) = \int d\Lambda d\nu\, e^{\langle \Lambda, H\rangle} \hat{d}_S(\lambda,\ U_S) \quad .$$

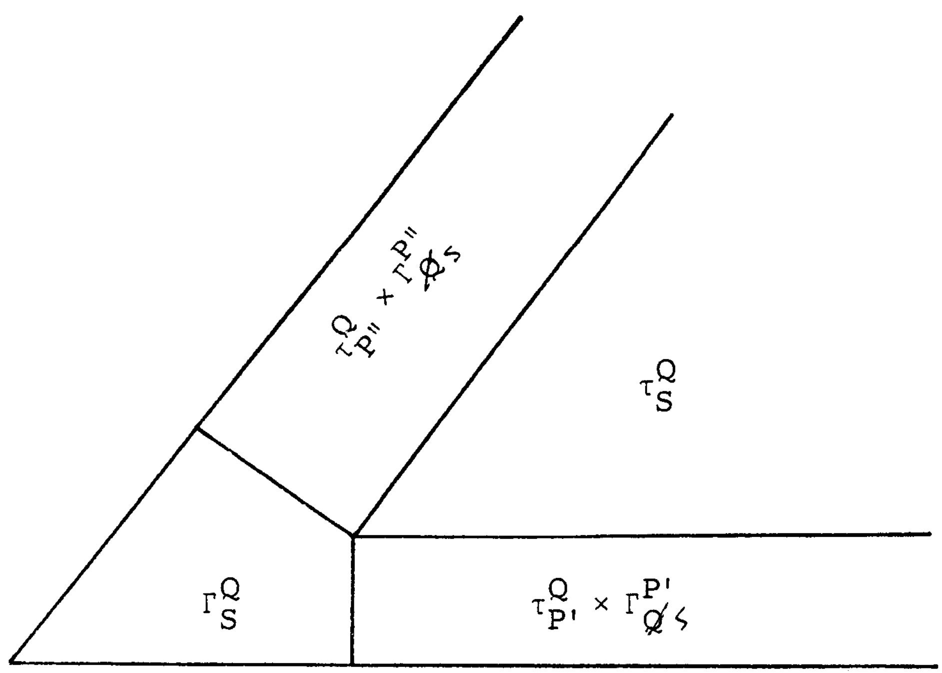
Recall that $\lambda = \lambda(\Lambda,\ \nu)$, where $\Lambda = s\varepsilon\lambda - \lambda$ and ν is invariant under $s\varepsilon$.

In (33) the space ${}_{\varepsilon}\mathfrak{a}^1_M$ is the set of all X in $\mathfrak{a}_M$ such that $d\chi(X) = 0$ for all ε-invariant rational characters of G (The notation is not ambiguous, but almost so, for there is a danger of confounding ${}_{\varepsilon}\mathfrak{a}^1_M$ with $\mathfrak{a}^1_M$). The point is that the domain of integration is independent of P_1, so that we can take the sum of (27) under the integral sign obtaining (33) again but with the summation extending not only over Q and S but also over P_1.

So we can simplify it, because

$$\sum_{P_1} \tau^Q_{P_1}(H_1 - Y_1(T) - U_1)\Gamma^1_S(H^1_S - Y^1_S(T),\ U^1_S) = \tau^Q_S(H_S - Y_S(T)) \quad .$$

This formula is implicit in §2 of the Annals paper, and corresponds to the following diagram:



Thus the sum over P_1 of the integrals (33) becomes the sum over S of

$$(34)\ \int_{\mathfrak{a}_S} \int_{\varepsilon\mathfrak{a}^1_M} \sum_{Q \supset S} (-1)^{a^{\varepsilon}_Q} {}_{\varepsilon}\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau^Q_S(H^Q_S - Y^Q_S(T))\Gamma^S_M(H^S_M, Y^S_M)\phi_S(H, U_S)\,dH \ .$$

LEMMA 7. <u>The integral</u> (34) <u>converges</u>. <u>It approaches</u> 0 <u>as</u> T <u>approaches infinity unless</u> $\mathfrak{a}_S \subseteq \mathfrak{a}_L$.

For the purposes of this lemma it is best not to work with (34) but to return to (33), removing the sum over S and replacing

$$\sum_Q (-1)^{a^{\varepsilon}_Q} {}_{\varepsilon}\hat{\tau}_Q(H_1 - Y_1(T) - U_1)(\tau^Q_{P_1}(H_1 - Y_1(T) - U_1)$$

by

$$\sum_{\mathcal{P}_\nu}{}' \ {}_{\varepsilon}\sigma^2_1(H_1 - Y_1(T) - U_1) \quad .$$

The function $\hat{d}_S$ is a Schwartz function of Λ, ν, U. Thus ϕ_S is a Schwartz function on $\mathfrak{a}_M/\mathfrak{a} \times \mathfrak{a}_S$. For convergence we need to show that if

$$H_S = X + V$$

with $V \in \mathfrak{a}$, X orthogonal to $\mathfrak{a}$ then there is a positive constant c such that

$$\|X\| + \|U_S\| \geq c\ \|V\|$$

on the support of the integrand, T being for the moment fixed. If this were not so then the usual argument shows the existence of a non-zero $V \in \mathfrak{a} \cap \mathfrak{a}_1$ which takes in the closure of the support of ${}_{\varepsilon}\sigma_1^2$. This contradicts the proof of Lemma 7.3 of Lecture 7.

We use a similar argument to show that the integral approaches 0 if $\mathfrak{a}_S$ is not contained in $\mathfrak{a}_L$. For this we write

$$H = H_M = X+V \quad ,$$

$V \in \mathfrak{a}$, X orthogonal to $\mathfrak{a}$ and show that for some $c > 0$

$$\|X\| + \|U_S\| \geq c\|V\| \ , \quad \|X\| + \|U_S\| \geq c\|T\|$$

on the support of the integrand. Because of Lemma 6, or rather because of its proof, we can simplify the situation somewhat. We can suppose that $P_1 = Q$ is ε-invariant and standard and that s acts trivially on $\mathfrak{a}_Q$. Thus $\mathfrak{a}_L \supset \mathfrak{a}_Q$, $\mathfrak{a} \supseteq \mathfrak{a}_Q^{\varepsilon}$. We may also suppose that S is standard.

If we cannot find the constant c then we can find a non-zero pair (V, T) such that $V-T$ is in the closure of the support of ${}_{\varepsilon}\sigma_1^2$ and such that

$$(35) \qquad \begin{aligned} &\Gamma_S^1(V_S^1-T_S^1),\ 0) \neq 0 \\ &\Gamma_M^S(V_M^S,\ \bar{y}_M^S) \neq 0 \quad , \end{aligned}$$

where $\bar{y}_M^S$ is defined like y_M^S but with $T_0 = 0$. Moreover $V \in \mathfrak{a}$. Thus $V^1 \in \mathfrak{a}$, for $P^1 = Q$ is invariant under s and ε. However the first of the inequalities (35) implies that

(36) $$Y_S^1 = T_S^1 \quad .$$

Since $d(T) \geq c\|T\|$ we have $T_S^1 = 0$ only if $T = 0$. If $T = 0$ the second inequality implies that $V_M^S = 0$. However the proof of Lemma 7.3 shows that $V_1 - T_1 = 0$. Thus T cannot be 0 without V being 0. We conclude that $T_S^1 \neq 0$. The inequalities (35) actually imply that V is in the convex hull of the collection

$$\{r^{-1}(T_M) \mid r \in \Omega^S(M)\} \quad .$$

If α is a root of S which does not vanish on $\mathfrak{a}_S$ then there is a positive constant c such that

$$\alpha(r^{-1}(T_M)) \geq c\|T\|$$

for all $r \in \Omega^S(M)$. We conclude that $\alpha(V) \geq c\|T\| \neq 0$. However if $\mathfrak{a}_S$ is not contained in $\mathfrak{a}_L$ we can find a root α which vanishes on $\mathfrak{a}_L$ and thus on V but not on $\mathfrak{a}_S$: Changing its sign if necessary we obtain a root in S and then a contradiction.

We continue to work with the modified (33) assuming that $\mathfrak{a}_S \subseteq \mathfrak{a}_L$. We show that we may substitute $\Gamma_L^S(H_L^S, y_L^S)$ for $\Gamma_M^S(H_M^S, y_M^S)$. For lack of time we simply quote two lemmas from the second Amer. Jour. paer. The first states that

$$\Gamma_L^S(H_L^S, Y_L^S) - \Gamma_M^S(H_M^S, Y_M^S) \geq 0 \quad .$$

The second states that where the difference is not 0,

$$\|H_M^S - H_L^S\| \geq c\|T\| \quad ,$$

c being as usual a positive constant. Since $H_M^S - H_L^S$ is the projection of X on $\mathfrak{a}_M^L$ the conclusion follows readily. See diagram at end of lecture.

Final combinatorics. We are reduced to considering

$$(37) \int_{\varepsilon\mathfrak{a}_M^1} \sum_{\mathfrak{a}_S \subseteq \mathfrak{a}_L} \int_{\mathfrak{a}_S} \sum_{Q \supset S} (-1)^{a_Q^\varepsilon} \,{}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, Y_L^S)\phi_S(H, U_S) \quad .$$

We first treat

$$(38) \sum_{\mathfrak{a}_S \subseteq \mathfrak{a}_L} \int_{\mathfrak{a}_S} \sum_{Q \supset S} (-1)^{a_Q^\varepsilon} \,{}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, Y_L^S)\phi_S(H, U_S) \quad .$$

We may interchange the order of summation and integration. We write $U_S = U_Q^{s\varepsilon} + V$ with $U_Q^{s\varepsilon}$ in $\mathfrak{a}_Q^{s\varepsilon}$ and V orthogonal to $\mathfrak{a}_Q^{s\varepsilon}$ and integrate first with respect to V.

$$(39) \int_{s\varepsilon\mathfrak{a}_S^Q} \phi_S(H, U_Q^\varepsilon + V)\,dV = \int d\Lambda d\nu\, e^{\langle \Lambda, H\rangle} \int_{s\varepsilon\mathfrak{a}_S^Q} \hat{d}_S(\lambda, U_Q^\varepsilon + V)\,dV$$

where ${}_{s\varepsilon}\mathfrak{a}_S^Q$ is the orthogonal complement of $\mathfrak{a}_Q^{s\varepsilon}$ in $\mathfrak{a}_S$. If $P' \in P^S(M)$ the right side is equal to

$$\int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \int_{s\varepsilon \mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda, U_Q^{\varepsilon}+V)\,dV \;.$$

Impossible notation! $s\varepsilon\mathfrak{a}_M^Q = s\varepsilon\mathfrak{a}_S^Q + \mathfrak{a}_M^S$

However

$$\int_{s\varepsilon\mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda, U_Q^{\varepsilon}+V) = \int_{s\varepsilon\mathfrak{a}_Q^Q} dV_1 \int_{\mathfrak{a}_M^Q} dV_2 \;\; \hat{d}_{P'}(\lambda, U_Q^{\varepsilon}+V_1+V_2)$$

and

$$\int_{\mathfrak{a}_M^Q} \hat{d}_{P'}(\lambda, U_Q+V_2)\,dV_2$$

is the Fourier transform of the restriction of $d_{P'}$ to $\mathfrak{a}_Q$, and therefore the same for all $P' \subseteq Q$. Thus (39) is equal to

$$\int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \int_{s\varepsilon\mathfrak{a}_Q^Q} \hat{d}_Q(\lambda, U_Q^{\varepsilon}+V)\,dV \quad ,$$

and, in particular, is independent of S.

Set

$$\psi_Q(H, U_Q) = \int d\Lambda d\nu \; e^{\langle \Lambda, H\rangle} \hat{d}_Q(\lambda, U_Q) \quad .$$

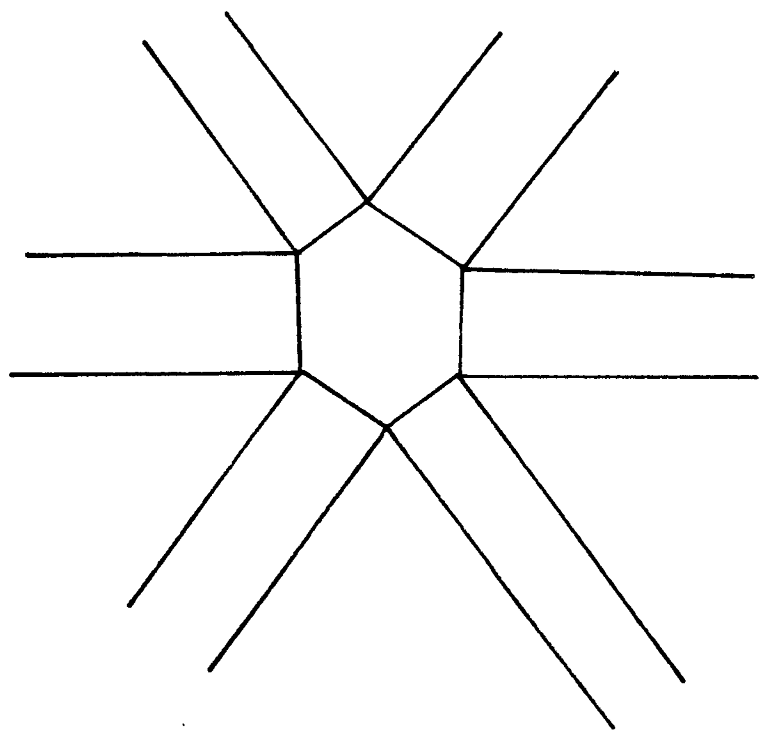
The sum (38) is equal to

$$(40) \qquad \sum_Q (-1)^{a_Q^{\varepsilon}} \int_{\mathfrak{a}_Q} {}_{\varepsilon}\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q)\,\psi_Q(H, U_Q)$$

because

$$\sum_{S \in \mathcal{P}^Q(L)} \tau_S^Q(H_S^Q - Y_S^Q(T))\Gamma_L^S(H_L^S, Y_L^S) \equiv 1 \quad .$$

This identity, which I do not prove formally, is another form of Lemma 6.3 of the Annals paper and corresponds to the following diagram:



The outer integral in (37) can be taken first over ${}_{s\varepsilon}\mathfrak{a}_M^M$ and then over $\mathfrak{a}_M^{s\varepsilon} = \mathfrak{a}$. To integrate (40) over ${}_{s\varepsilon}\mathfrak{a}_M^M$ we have to take

$$\int_{{}_{s\varepsilon}\mathfrak{a}_M^M} \psi_Q(H, U_Q) \quad .$$

Since ${}_{s\varepsilon}\mathfrak{a}_M^M$ is the domain in which $-i\Lambda$ varies this integration yields

$$\int_{i\mathfrak{a}} \hat{d}_Q(\lambda, U_Q)\, d\lambda \quad .$$

We next integrate this over $_{s\varepsilon}\mathfrak{a}_Q^Q$ taking

$$\int_{s\varepsilon\mathfrak{a}_Q^Q} dV \int_{i\mathfrak{a}} \hat{d}_Q(\lambda, U_Q^{s\varepsilon}+V)\,d\lambda$$

and observing that

$$\int_{s\varepsilon\mathfrak{a}_Q^Q} \hat{d}_Q(\lambda, U_Q^{s\varepsilon}+V)\,dV = \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon}) \quad ,$$

where we now regard Q as an element of $\mathcal{F}_\varepsilon(L, \mathfrak{a})$ and where $\bar{d}_Q$ is defined by the $(G, L, \mathfrak{a})$-family attached to $\{d_{P'}\}$.

Summing up the results so far, we see that (37) is equal to

$$\int_{\mathfrak{a}} \left\{ \sum_Q \int_{\mathfrak{a}_Q^{s\varepsilon}} (-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q^{s\varepsilon}) \left\{ \int_{i\mathfrak{a}} \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon})\,d\lambda \right\} dU_Q^{s\varepsilon} \right\} dH \quad .$$

(The notational difficulties and inconsistencies are growing more and more severe.) Let L' run over the Levi factors of groups in $\mathcal{F}_\varepsilon(L, \mathfrak{a})$.

Using the results of §2 of the Annals paper we expand

$$(-1)^{a_Q^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q(H_Q - Y_Q(T) - U_Q^{s\varepsilon})$$

as

$$\sum_{R \supset Q} (-1)^{a_Q^\varepsilon - a_R^\varepsilon}\, {}_\varepsilon\hat{\tau}_Q^R(H_Q - Y_Q(T))\, {}_\varepsilon\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon}) \quad ,$$

it being understood that the sum is over $R \in \mathcal{F}_\varepsilon(L, \mathfrak{a})$.

We postpone consideration of

$$\int_{\mathfrak{a}_Q^{s\varepsilon}} {}_{\varepsilon}\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon}) \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon}) \, dU_Q^{s\varepsilon}$$

observing for the moment only that it depends on R alone and not on Q because

$$\int \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon} + V) \, dV$$

does, the integration being taken over the orthogonal complement of $\mathfrak{a}_R^{s\varepsilon}$ on $\mathfrak{a}_Q^{s\varepsilon}$.

This observation allows us to sum

$$\sum_{Q \subseteq R} (-1)^{a_Q^{\varepsilon} - a_R^{\varepsilon}} {}_{\varepsilon}\hat{\tau}_Q^R(H_Q - Y_Q(T))$$

obtaining (cf. Lemma 5.3.5 of Lecture 5)

$${}_{\varepsilon}\Gamma_L^R(H_L^R, {}_{\varepsilon}y_L^R) \quad .$$

Since this is a function with compact support we see that (37) is equal to the integral over $i\mathfrak{a}$ of the sum over R of the product of

$$\int_{\mathfrak{a}_L^R} {}_{\varepsilon}\Gamma_L^R(H_L^R, {}_{\varepsilon}y_L^R) = \bar{c}_L^R(T)$$ Poor notation

and

$$\int_{\mathfrak{a}_R^G} \int_{\mathfrak{a}_Q^{s\varepsilon}} {}_{\varepsilon}\Gamma_R^G(H_Q - Y_Q(T), U_Q^{s\varepsilon}) \hat{\bar{d}}_Q(\lambda, U_Q^{s\varepsilon}) \, dU_Q^{s\varepsilon} dH_Q$$

which equals

Since Λ is not equal to 0 the multiplication by A(Λ) has no effect.

$$\bar{d}_R^G(\lambda) \quad .$$

We obtain finally

$$\int_{i\mathfrak{a}} \sum_R \bar{c}_L^R(T)\bar{d}_R^G(\lambda)\,d\lambda \quad ,$$

a polynomial in T. Using Lemma 6.3 of the Annals paper to collapse the sum and examining the definitions we see that this is equal to

$$\int_{i\mathfrak{a}} \operatorname{tr}({}_\varepsilon M_L(P,\lambda) M_{P|\varepsilon(P)}(s,\varepsilon(\lambda))\,\varepsilon\rho_{\sigma,\lambda}(\phi))B_\sigma(\lambda)\,|d\lambda| \quad .$$

All that is left is to rid ourselves of the $B_\sigma(\lambda)$ and for this we must consider the normalization of intertwining operators.

Normalization of intertwining operators. In the second Amer. Jour. paper Arthur assumes a normalization of the local intertwining operators with the properties to be stated below. I now want to point out, with minimal explanation, that such a normalization can be easily deduced from results already in the literature.

References 1. J. Arthur, On the invariant distributions attached to weighted orbital integrals (preprint).

2. K. F. Lai, Tamagawa number of reductive algebraic groups, Comp. Math.

3. A. Silberger (i) Introduction to harmonic analysis on reductive p-adic groups P.U.P. (ii) Special representations of reductive p-adic groups are not integrable, Annals. (iii) On Harish-Chandra μ-functions for p-adic groups, Transactions.

I observe that we do not need the formula of Th. 1.6 of paper [3.ii], which, as Shahidi has pointed out to me, is not correct. The source of error is perhaps the assertion preceding Lemma 1.2.

If G is a group over a local field with standard parabolic P_0 then for any $P \supseteq M_0$ and any unitary representation σ of $M = M_P$ we can introduce as usual the induced representations $\rho_{\sigma,\lambda}$ on the space $\mathfrak{A}_\sigma(P)$. Here λ lies in the complex dual of $\mathfrak{a}_M$. We also introduce the intertwining operators $M_{Q|P}(1,\lambda)$, $M_{Q|P}(\sigma,\lambda) = M_{Q|P}(1,\sigma,\lambda)$ which send $\phi \in \mathfrak{A}_\sigma(P)$ to $\phi' \in \mathfrak{A}_\sigma(Q)$ with

$$\phi'(g) = \int_{N_Q \cap N_P \backslash N_P} \phi(ng) e^{(\lambda+\rho_P)(H_P(ng))-(\lambda+\rho_Q)(H_Q(g))} dn \quad .$$

The global operators are tensor products of these local operators.

We need decompositions

$$M_{Q|P}(\sigma,\lambda) = n_{Q|P}(\sigma,\lambda) N_{Q|P}(\sigma,\lambda) \quad ,$$

where $n_{Q|P}(\sigma,\lambda)$ is a scalar and both functions on the right are meromorphic in λ, $\lambda \in \mathfrak{a}_M^* \otimes \mathbf{C}$. The following conditions are to be satisfied.

(i) $$N_{R|P}(\sigma,\lambda) = N_{R|Q}(\sigma,\lambda) N_{Q|P}(\sigma,\lambda)$$

(ii) $$N_{Q|P}(\sigma,\lambda)^* = N_{Q|P}(\sigma,-\bar{\lambda})$$

(iii) $$(N_{S(R')|S(R)}(\sigma,\lambda)\phi)_k = N_{R'|R}\phi_k \quad .$$

(iv) If σ and G are unramified and ϕ is fixed by a hyperspecial K, then

$$N_{Q|P}(\sigma, \lambda)\,\phi$$

is independent of λ.

(v) If σ is tempered then $n_{Q|P}(\sigma, \lambda)$ has neither zeros nor poles in the positive chamber attached to P.

(vi) If the local field is non-archimedean then $N_{Q|P}(\sigma, \lambda)$ is a rational function of $\{q^{-\alpha(\lambda)} \mid \alpha \in \Delta_P\}$.

In the paper [1] Arthur has established the existence of such a normalization for real groups. Much of his argument is also applicable to p-adic groups and shows that it is enough to verify the existence of $n_{Q|P}$ and $N_{Q|P}$ when σ is tempered, P is maximal, and $Q = \overline{P}$ is opposite to P.

In this case, by [3]

$$M_{\overline{P}|P}(\sigma, -\overline{\lambda})^{*} M_{\overline{P}|P}(\sigma, \lambda) = c\,\mu(\sigma;\lambda)$$

where c is a positive constant and μ is the function appearing in Harish-Chandra's Plancherel formula. Again by [3], this function is a rational function of $z = q^{-\alpha(\lambda)}$ (α is now the unique simple root)

$$\mu(\sigma,\lambda) = U(\sigma, z) \quad .$$

All we need do is to decompose $U(\sigma, z)$ as

$$U(\sigma, z) = V_P(\sigma, z)\overline{V}_P(\sigma, \overline{z}^{-1})$$

where $V_P(\sigma, z)$ is a rational function with neither zero pole in $|z| < 1$ and with $V_P(\sigma, z) = \overline{V}_{\overline{P}}(\sigma, \overline{z})$, for we then set

$$n_{\overline{P}|P}(\sigma, \lambda) = \sqrt{c}\, V_P(\sigma, q^{-\alpha(\lambda)})$$

and

$$N_{\overline{P}|P}(\sigma, \lambda) = \frac{M_{\overline{P}|P}(\sigma, \lambda)}{n_{\overline{P}|P}(\sigma, \lambda)} .$$

Since

$$M^*_{\overline{P}|P}(\sigma, \lambda) = M_{P|\overline{P}}(\sigma, -\overline{\lambda})$$

and

$$\overline{n}_{\overline{P}|P}(\sigma, \lambda) = n_{P|\overline{P}}(\sigma, -\overline{\lambda})$$

the condition (ii) is fulfilled. Observe that $\overline{\alpha} = -\alpha$, that is, replacing by $\overline{P}$ entails replacing α by $-\alpha$.

To verify (i) we need only check that

$$N_{P|\overline{P}}(\sigma, \lambda) N_{\overline{P}|P}(\sigma, \lambda) = 1 .$$

By (ii) the left side is

$$N_{\overline{P}|P}(\sigma, -\overline{\lambda})^* N_{\overline{P}|P}(\sigma, \lambda)$$

which equals

$$\frac{c\,\mu(\sigma,\lambda)}{\overline{V}_p(\sigma,\overline{z}^{-1})V_p(\sigma,z)} = 1 \quad .$$

That $V_p(\sigma,z)$ can be so chosen that (iv) is satisfied follows from the calculations in [2].

To prove the existence of $V_p(\sigma,z)$ I use an argument of Shahidi. It exploits the following two properties of $U(\sigma,z)$, both consequences of the fact that $U(\sigma,z)$ is real and positive for $|z| = 1$:

(i) $U(\sigma,z) = \overline{U}(\sigma,\overline{z}^{-1})$.

(ii) Any zero of $U(\sigma,z)$ on $|z| = 1$ is of even multiplicity.

It follows from (i) that if α is a root of $U(\sigma,z) = 0$ then $\overline{\alpha}^{-1}$ is also. The same assertion is valid for poles. Thus we may write

$$U(\sigma,z) = a\,\frac{\prod_{i=1}^{r}(1-\alpha_i z)(1-\overline{\alpha}_i^{-1}z)}{\prod_{i=1}^{r}(1-\beta_i z)(1-\overline{\beta}_i^{-1}z)}$$

where $|\alpha_i| \le 1$, $|\beta_i| \le 1$, $1 \le i \le r$, and

$$a\,\frac{\prod \alpha_i}{\prod \beta_i} > 0 \quad .$$

We let

$$b\overline{b}\,\frac{\prod \overline{\alpha}_i}{\prod \overline{\beta}_i} = a$$

and set

$$V_P(\sigma, z) = b \frac{\prod_{i=1}^{r} (1-\alpha_i z)}{\prod_{i=1}^{r} (1-\beta_i z)} .$$

Then

$$\bar{V}_P(\sigma, \bar{z}^{-1}) = \frac{\prod \bar{\alpha}_i}{\prod \bar{\beta}_i} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i^{-1} z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i^{-1} z)}$$

and

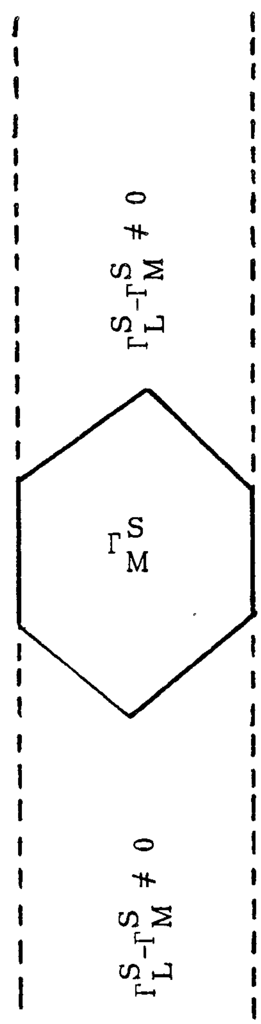
$$U(\sigma, z) = V_P(\sigma, z)\bar{V}_P(\sigma, \bar{z}^{-1}) .$$

If we replace P by $\bar{P}$ then $\mu(\sigma, \lambda)$ is not changed but z is replaced by z^{-1} and $U(\sigma, z)$ by $U(\sigma, z^{-1})$, which equals

$$\bar{a} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i z)(1-\alpha_i^{-1} z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i z)(1-\beta_i^{-1} z)} .$$

So we may take

$$V_{\bar{P}}(\sigma, z) = \bar{b} \frac{\prod_{i=1}^{r} (1-\bar{\alpha}_i z)}{\prod_{i=1}^{r} (1-\bar{\beta}_i z)} .$$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