Normality of Torus Orbit Closures in G/P

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Abstract. Suppose G is a semisimple algebraic group over $\mathbb{C}$, P is a parabolic subgroup and $T \subset P$ is a maximal torus. It has been asked whether or not all closures of torus orbits in the algebraic homogeneous space G/P are normal. In this paper, we describe all the normal torus orbit closures in G/P, and show, for example, that all torus orbit closures in G/P are normal if G is of type A_n, B_2 or D_4, but in other types, non-normal ones exist. Our description follows from an apparently new result on normality of closures of torus orbits in any smooth projective variety with a torus action.

1 Introduction

In recent years, a number of authors have considered the closure of a so called *generic orbit* of a maximal torus T in the flag variety G/B of a semi-simple Lie group G over $\mathbb{C}$ (e.g. Dabrowski, Dolgachev, Flaschka-Haine, Gelfand-Serganova, Klyachko, and Procesi). These varieties turn out to be nonsingular compactifications of T. In fact, they are the toric varieties for T arising from the Weyl fan in the Lie algebra of T. Consequently all generic T-orbit closures in G/B are isomorphic as T-varieties. For a general algebraic homogeneous space G/P, the correct notion of a generic T-orbit closure is not as easy to formulate. The first definition, given in [5], has the problem that two generic T-orbit closures in a G/P may not be isomorphic [4]. Moreover, as we point out in §7, generic T-orbit closures need not be normal. This means that the results in [5] dealing with the fan of a generic T-orbit closure X really pertain to the normalization of X. (See [4] for more details.)

In [4], Dabrowski refined the notion of a generic torus orbit closure X in G/P. Such X are normal and any two X in a G/P are T-isomorphic. However, the general question of whether arbitrary T-orbit closures in a G/P are normal does not seem to have been treated in the literature. (This was noted by Oda [15], and later it was asked by Dabrowski [4] and Sturmfels [19]). Since the answer is in general no, as mentioned above, the problem is to determine the normal T-orbit closures. That is the purpose of this note.

By considering the closed G-embedding $G/P \hookrightarrow \mathbb{P}(V(\omega))$, $gP \mapsto g[v_\omega]$, where ω is a dominant weight with highest weight module $V(\omega)$ and v_ω is a highest weight vector such that P is the stabilizer of the line $[v_\omega] \in \mathbb{P}(V(\omega))$, we are led to

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the general question of when a T-orbit closure in $\mathbb{P}(V)$ is normal, V being a finite-dimensional vector space over $\mathbb{C}$ and $T \subset \mathrm{GL}(V)$ an algebraic torus. If $v \in V$ is nonzero, the T-orbit closure $X(v) := \overline{T[v]} \subset \mathbb{P}(V)$ is determined by its support $\mathcal{F}(v)$ in the lattice $X(T)$ of characters of T, namely the set of all characters $\chi \in X(T)$ involved in the decomposition $v = \sum v_\chi$ into T-eigenvectors, and many geometric properties of $X(v)$ follow directly from properties of the Newton polytope of v, that is the convex hull $\mathcal{H}(v)$ of $\mathcal{F}(v)$ in $X(T) \otimes \mathbb{R}$ (cf. Proposition 2.2, also [10], [16], [17]). In §3, we show that a T-orbit closure $X(v)$ is normal if and only if at every vertex χ of $\mathcal{H}(v)$, the parallelotope spanned by the primitive parts of the edges of $\mathcal{H}(v)$ touching χ satisfies a certain saturation condition (cf. Corollary 3.7).

The nice fact is that the above saturation condition is easy to express in the G/P setting above, provided the weight ω whose stabilizer is P is sufficiently positive in a sense defined in §6. Every parabolic $P \subset G$ admits sufficiently positive weights, so we are able to classify normal torus orbit closures for all G and P. One consequence is that if $X(v_\omega)$ is a T-orbit closure in G/B such that $\mathcal{H}(v_\omega) \cap X(T) = \mathcal{F}(v_\omega)$, then $X(v_\omega)$ is normal as long as ω is sufficiently positive. The analogous property also holds in G/P, so T-orbit closures in G/P are somewhat special, since for general $X(v) \subset \mathbb{P}(V)$, the property $\mathcal{H}(v_\omega) \cap X(T) = \mathcal{F}(v_\omega)$ does not imply the normality of $X(v)$ (cf. Example 3.3). Applying some recent results of J. Morand [12] on affine torus orbit closures, we conclude that all T-orbit closures $X \subset SL_n(\mathbb{C})/P$ are normal (and similarly when G is of type B_2 or D_4). For all other G, there are non-normal T-orbit closures in G/B, as examples due to D. Wehlau show. The explicit non-normal examples given in §7 occur in G/P where G is of type G_2.

2 General remarks on closures of torus orbits

We will always work over $\mathbb{C}$, although much of the paper is valid for any algebraically closed field. Let T be an algebraic torus, $X(T)$ its lattice of characters and $Y(T)$ its lattice of one parameter subgroups. For $\lambda \in Y(T)$ and $\chi \in X(T)$ the integer $\langle \lambda, \chi \rangle$ defined by $\chi(\lambda(s)) = s^{\langle \lambda, \chi \rangle}$ where $s \in \mathbb{C}^*$ gives a perfect pairing, so, as is well known, $X(T)$ and $Y(T)$ are dual lattices. Let $\mathcal{F} \subset X(T)$ be finite, and define $V = \oplus_{\chi \in \mathcal{F}} V_\chi$, where V_χ is a finite-dimensional T-module of weight χ. For $v = \sum_{\chi \in \mathcal{F}} v_\chi$, put

$$X(v) := \overline{T[v]} \subset \mathbb{P}(V),$$

where $[v]$ denotes the line in V spanned by v.

Define the *support* of v to be $\mathcal{F}(v) = \{\chi \in \mathcal{F} |\ v_\chi \neq 0\}$, and let $\mathcal{H}(v)$ be the convex hull of $\mathcal{F}(v)$ in $X(T) \otimes \mathbb{R}$. For simplicity, we will fix $v \in V$ with $\mathcal{F}(v) = \mathcal{F}$ and denote $\mathcal{H}(v)$ by $\mathcal{H}$.

Lemma 2.1 *For every $[w] \in X(v)$ there is a one parameter subgroup $\lambda \in Y(T)$ with $\lim_{s \to 0} \lambda(s)[v] \in T[w]$.*

Proof: The proof easily reduces to the affine version, which is proved for example in [11, III.2.2 Satz] or [10]. □

The following result giving the orbit configuration of an $X(v)$ is an easy consequence of Lemma 2.1. It can also be deduced as a corollary of a more general result on reductive groups proved by Pauer [16]. There is also a similar but more general constructuction due to Popov and Vinberg [17].

Proposition 2.2 *(a) The fixed point set of the action of T on $X(v)$ is*

$$X(v)^T = \{[v_\chi] \mid \chi \textit{ is a vertex of } \mathcal{H}\}$$

(b) $X(v) = \bigcup_{w \prec v} T[w]$ *where* $w \prec v \Leftrightarrow w = \sum_{\substack{\chi \in \mathcal{F}\cap\mathcal{G} \\ \mathcal{G} \textit{ face of } \mathcal{H}}} v_\chi$

PROOF: (a): Let $[w] \in X(v)^T$, say $w = \sum_{\chi\in\mathcal{E}} w_\chi$ with each $w_\chi \neq 0$. For any λ, $\langle\lambda, \chi\rangle$ must be the same for all $\chi \in \mathcal{E}$. Since the pairing $\langle\ ,\ \rangle$ is perfect, $\mathcal{E} = \{\chi\}$ for some $\chi \in \mathcal{F}$. However, if χ is in the interior of $\mathcal{H}$, then for every $\lambda \in Y(T)$ there are $\chi_1, \chi_2 \in \mathcal{F}$ with $\langle\lambda, \chi_2\rangle > \langle\lambda, \chi\rangle > \langle\lambda, \chi_1\rangle$. So $[v_\chi]$ cannot be expressed as $\lim_{s\to 0} \lambda(s)[v]$ for any $\lambda \in Y(T)$, contradicting Lemma 2.1.

(b): If $\lambda \in Y(T)$, let $M(\lambda) = \{\chi \in \mathcal{F} \mid \langle\lambda, \chi\rangle \leq \langle\lambda, \mathcal{F}\rangle\}$. Then $M(\lambda)$ is of the form $\mathcal{F} \cap \mathcal{G}$ for some face $\mathcal{G}$ of $\mathcal{H}$. Now let $X := \bigcup_{w\prec v} T[w]$. Since $\lim_{s\to 0} \lambda(s)[v] = [\sum_{\chi\in M(\lambda)} v_\chi]$, it is clear from Lemma 2.1 that $X(v) \subset X$. On the other hand, $X \subset X(v)$. For if $\mathcal{G}$ is a face of $\mathcal{H}$, then there exists a $\lambda \in Y(T)$ such that $M(\lambda) = \mathcal{G} \cap \mathcal{H}$, so $w \prec v$ implies $w \in X(v)$. □

In particular, the T-invariant curves in $X(v)$ are in one to one correspondence with the edges of $\mathcal{H}$, and the endpoints of an edge are the two T-fixed points on the curve. We will use this fact in §5.

3 A normality criterion for torus orbit closures

If $X(v)$ is a normal variety, it is a projective toric variety (cf. [10], [6], [14]). On the other hand, non-normal closures of torus orbits have been considered in [4], [18]. We will now give a characterization of the normal $X(v)$, which also allows us to obtain an explicit description of the normalization of $X(v)$. Fix an $\mathcal{F} = \{m_1, \dots, m_s\} \subset X(T)$ which generates $X(T)$. Thus, in the terminology of [10], the natural mapping of T into $X(v)$ is an equivariant affine embedding. We will view T as $\mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{C}^*)$, so the coordinate form of $T[v]$ is

$$\{[t(m_1) : \dots : t(m_s)] \in \mathbb{P}^{s-1}(\mathbb{C}) \mid t \in T\}.$$

Let $\mathbb{Z}_{\geq 0}\mathcal{F} := \sum_{i=1}^{s} \mathbb{Z}_{\geq 0} m_i$ be the semigroup spanned by $\mathcal{F}$. Similarly, let $\mathbb{Q}_{\geq 0}\mathcal{F}$ be the rational cone spanned by $\mathcal{F}$ and $\mathbb{R}_{\geq 0}\mathcal{F}$ the real cone spanned by $\mathcal{F}$ in $X(T) \otimes \mathbb{Q} \subset X(T) \otimes \mathbb{R}$ respectively. Let $\mathcal{M}$ be the lattice spanned by all $m_i - m_j$. Equivalently, for any fixed i, $\mathcal{M} = \mathbb{Z}\{m_j - m_i \mid 1 \leq j \leq s\}$.

The set of points where $X(v)$ fails to be normal is a closed T-invariant subset of $X(v)$, so normality of $X(v)$ need only be checked on an affine neighborhood of each fixed point, that is at each $[v_m]$, where m is a vertex of $\mathcal{H}$. By [10, I.1 Lemma 1], $X(v)$ is normal at $[v_m]$ if and only if $\mathbb{Z}_{\geq 0}(\mathcal{F} - m)$ is saturated in the lattice $\mathcal{M}$ generated by $\{m_i - m | 1 \leq i \leq s\}$. (Recall that a semigroup S in a lattice L is saturated if whenever $m \in L$ has the property that $km \in S$ for some $k \in \mathbb{N}$, then $m \in S$.) The following lemma is well known and easy to prove.

Lemma 3.1 *Let S be a semigroup in a lattice L which generates L. Then S is saturated in L if and only if*

$$\mathbb{R}_{\geq 0}S \cap L = \mathbb{Q}_{\geq 0}S \cap L = S.$$

Notice that the lattice $\mathcal{M}$ in $X(T)$ defined above is independent of the choice of vertex $m \in \mathcal{F}$. Using the above discussion, we therefore infer:

Corollary 3.2 ([14, 2.4, p. 95/96], [18, 13.10],[10, I.1 Lemma 1]) $X(v)$ *is normal if and only if, for all vertices m of $\mathcal{F}$,*

$$\mathbb{R}_{\geq 0}(\mathcal{F} - m) \cap \mathcal{M} = \mathbb{Q}_{\geq 0}(\mathcal{F} - m) \cap \mathcal{M} = \mathbb{Z}_{\geq 0}(\mathcal{F} - m)$$

The following example, due to Sturmfels, shows there exist non-normal $X(v)$ such that $\mathcal{F} = \mathcal{H} \cap X(T)$.

Example 3.3 ([18, 13.13]) For $\mathcal{F} = \{(0,0,0,1),(0,0,1,1),(1,0,1,1),(0,1,1,1),(1,1,4,1)\}$ the semigroup $\mathbb{Z}_{\geq 0}(\mathcal{F} - (1,1,4,1))$ is isomorphic to the semigroup spanned by $\mathcal{F}' = \{(1,1,4),(1,1,3),(0,1,3),(1,0,3)\}$. Then $x_0 := (1,1,5) = 2(1,1,4) - (1,1,3) \in \mathbb{Z}\mathcal{F}' \cap \mathbb{R}_{\geq 0}\mathcal{F}'$, but $x_0 \notin \mathbb{Z}_{\geq 0}\mathcal{F}'$, and x_0 is not inside the polytope defined by $\mathcal{F}' \cup \{0\}$.

The key to understanding normality of projective T-orbit closures is to consider the parallelotopes spanned by the primitive vectors of the extreme edges of the cones $\mathbb{R}_{\geq 0}(\mathcal{F} - m)$ for the vertices m of $\mathcal{H}$. Let M denote a lattice with a given generating subset $\mathcal{A} = \{a_1, \dots, a_s\} \subset M$. We will call $a_i \in \mathcal{A}$ *primitive* if $\mathbb{R}_{\geq 0}a_i$ is an extreme edge of the cone $\mathbb{R}_{\geq 0}\mathcal{A}$ and a_i is the first vector on $\mathbb{R}_{\geq 0}a_i$ in $\mathcal{A}$. Let $a_1, \dots, a_r$, $r \leq s$ be the primitive vectors of $\mathcal{A}$, and consider the parallelotope $\mathbf{P}(\mathcal{A}) := \{\sum_{i=1}^r t_i a_i \mid 0 \leq t_i \leq 1\}$ generated by the primitive vectors. Put

$$\mathcal{P}_{\mathcal{A}} := \mathbf{P}(\mathcal{A}) \cap M, \qquad S := \mathbb{Z}_{\geq 0}\mathcal{P}_{\mathcal{A}}.$$

Notice that for any $m = \sum_{i=1}^r t_i a_i \in \mathcal{P}_{\mathcal{A}}$, we may assume each $t_i \in \mathbb{Q}$.

We now prove that S is the saturation of $\mathbb{Z}_{\geq 0}\mathcal{A}$. This is in a sense the converse of Gordan's Lemma that the semigroup of integral points in a rational convex polyhedral cone is finitely generated.

Proposition 3.4 *S is saturated, and in fact S is the saturation*

$$\{m \in M \mid km \in \mathbb{Z}_{\geq 0}\mathcal{A} \text{ for some } k \in \mathbb{N}\}$$

of $\mathbb{Z}_{\geq 0}\mathcal{A}$.

Proof: Let $km \in S$ with $m \in M$ and $k \in \mathbb{N}$. Thus $km = \sum_{i=1}^r p_i a_i$ where $p_i \in \mathbb{Z}_{\geq 0}$. It follows that

$$m = \sum_{i=1}^r \frac{p_i}{k} a_i = \sum_{i=1}^r (l_i + t_i) a_i = \sum_{i=1}^r l_i a_i + \sum_{i=1}^r t_i a_i$$

where $l_i \in \mathbb{Z}_{\geq 0}$ and $t_i \in [0,1]$. Hence $m \in S$ since $\sum_{i=1}^r t_i a_i = m - \sum_{i=1}^r k_i a_i \in M \cap \mathbf{P}(\mathcal{A}) = \mathcal{P}_{\mathcal{A}}$, and therefore S is saturated.

Now assume $m \in M$ and $lm = \sum_{i=1}^{s} k_i a_i \in \mathbb{Z}_{\geq 0}\mathcal{A}$ for some $l \in \mathbb{N}$. Since each $a_j \in \mathbb{Q}_{\geq 0}\{a_1, \ldots, a_r\}$, we may write $lm = \sum_{i=1}^{r} q_i a_i$ with $q_i \in \mathbb{Q}_{\geq 0}$. Hence $Nm \in S$ for some $N \in \mathbb{N}$, and this implies $m \in S$ since S is saturated. Thus S contains the saturation of $\mathbb{Z}_{\geq 0}\mathcal{A}$. Conversely, for any $m \in S$, the remark before the proposition implies $km \in \mathbb{Z}_{\geq 0}\{a_1, \ldots, a_r\} \subset \mathbb{Z}_{\geq 0}\mathcal{A}$ for some $k \in \mathbb{N}$, so S is contained in the saturation. □

Corollary 3.5 *$\mathbb{Z}_{\geq 0}\mathcal{A}$ is saturated if and only if $\mathcal{P}_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}\mathcal{A}$.*

The generating set $\mathcal{A} = \{a_1, \ldots, a_s\} \subset M$ defines an affine T-orbit

$$\mathcal{O} = \{(t(a_1), \ldots, t(a_s)) \mid t \in T\} \subset \mathbb{C}^s.$$

Then $Z(\mathcal{A}) := \overline{\mathcal{O}}$ is an affine T-orbit closure. Now $Z(\mathcal{A})$ is normal if and only if $\mathbb{Z}_{\geq 0}\mathcal{A}$ is saturated [18, 13.5], so using the proposition one gets:

Corollary 3.6 *$Z(\mathcal{A})$ is normal if and only if $\mathcal{P}_{\mathcal{A}} \subset \mathbb{Z}_{\geq 0}\mathcal{A}$.*

Returning to the projective T-orbit closure $X(v) = \overline{T[v]}$ defined by $\mathcal{F}$, we obtain analogously the

Corollary 3.7 *$X(v)$ is normal if and only if for all vertices $m \in \mathcal{H}$,*

$$\mathbf{P}(\mathcal{F} - m) \cap \mathcal{M} \subset \mathbb{Z}_{\geq 0}(\mathcal{F} - m).$$

A different criterion for normality of an $X(v)$ was given by Jurkiewicz [9]. We can now give a criterion for normality pertaining to Example 3.3 which will be strengthened in the G/P setting (§6).

Proposition 3.8 *Suppose that for each vertex $m \in \mathcal{H}$, $\mathbf{P}(\mathcal{F} - m) \subset \mathcal{H} - m$. Suppose also that $\mathcal{H} \cap X(T) = \mathcal{F}$. Then $X(v)$ is normal.*

PROOF: Fix a vertex m. Then $\mathbf{P}(\mathcal{F} - m) \cap \mathcal{M} \subset (\mathcal{H} - m) \cap \mathcal{M}$, which easily implies $\mathbf{P}(\mathcal{F} - m) \cap \mathcal{M} \subset \mathcal{F} - m$. Hence the result follows from the last corollary. □

Remark 3.9 Similarly, one has that $X(v)$ is nonsingular if and only if for all vertices $m \in \mathcal{H}$, the edges of $\mathbf{P}(\mathcal{F} - m)$ at 0 are contained in a basis of $\mathcal{M}$. This is proved in a manner similar to [10], and we will omit the details.

4 The fan associated to a projective T-orbit closure

As usual let $\mathcal{F} \subset X(T)$ be a finite subset and define $\mathcal{F}_\lambda := \{\chi \in \mathcal{F} \mid \langle \lambda, \chi \rangle$ is minimal$\}$ for $\lambda \in Y(T)$. For $\lambda_1, \lambda_2 \in Y(T)$ we define $\lambda_1 \sim \lambda_2$ if $\mathcal{F}_{\lambda_1} = \mathcal{F}_{\lambda_2}$. Clearly, $\sim$ is an equivalence relation. It can also be characterized as follows: $\lambda_1 \sim \lambda_2$ if and only if $\lambda_1(0)[v] = \lambda_2(0)[v]$ where $v = \sum_{\chi \in \mathcal{F}} v_\chi$, $v_\chi \neq 0$ for all $\chi \in \mathcal{F}$.

Lemma 4.1 *For $\lambda \in Y(T)$ the equivalence class $[\lambda] := \{\mu \in Y(T) \mid \mu \sim \lambda\}$ is a saturated semigroup in $Y(T)$.*

PROOF: For $\mu_1, \mu_2 \in [\lambda]$ it is easy to show that $\mathcal{F}_{\mu_1+\mu_2} = \mathcal{F}_\lambda$; hence $\mu_1+\mu_2 \in [\lambda]$. If $\mu \in Y(T)$ with $k\mu \in [\lambda]$ for some $k \in \mathbb{N}$ then $\mathcal{F}_\lambda = \mathcal{F}_{k\mu} = \mathcal{F}_\mu$, thus $\mu \in [\lambda]$ as well. □

Let $\sigma_{[\lambda]} := \overline{\mathbb{R}_{\geq 0}[\lambda]}$ (closure in $\mathbb{R}$-topology) and define the fan associated to $X(v)$ (which is induced by $\mathcal{F}$) by

$$\Delta_{\mathcal{F}} \;=\; \bigcup_{\lambda \in Y(T)} \sigma_{[\lambda]}. \tag{1}$$

The set of all $\mathcal{F}_\lambda$ consists of the faces of $\mathcal{F}$. So by Proposition 2.2 (b) there is a $1-1$-correspondence between the T-orbits in $X(v)$ and the set of all cones $\sigma_{[\lambda]}$ of $\Delta_{\mathcal{F}}$. Now look at the (normal) toric variety

$$X(\Delta_{\mathcal{F}}) = \bigcup_{\sigma_{[\lambda]} \in \Delta_{\mathcal{F}}} X_{[\lambda]}$$

corresponding to the fan $\Delta_{\mathcal{F}}$, where $X_{[\lambda]} := \mathrm{Spec}(\mathbb{C}[\sigma_{[\lambda]}{}^\vee \cap \mathcal{M}])$. Recall that $\sigma_{[\lambda]}{}^\vee$ denotes the dual cone in $X(T) \otimes \mathbb{R}$ and $\mathcal{M} \subset X(T)$ is generated by $\{\chi - \nu \mid \chi, \nu \in \mathcal{F}\}$.

For each vertex χ of $\mathcal{F}$ there is a $\lambda \in Y(T)$ with $\mathcal{F}_\lambda = \{\chi\}$. By definition $\langle \lambda, \mathcal{F} - \chi \rangle \geq 0$, so $\mathcal{F} - \chi \subset \sigma_{[\lambda]}{}^\vee \cap \mathcal{M}$ and therefore $\mathbb{C}[\mathbb{Z}_{\geq 0}(\mathcal{F} - \chi)] \subset \mathbb{C}[\sigma_{[\lambda]}{}^\vee \cap \mathcal{M}]$. The last inclusion induces a dominant morphism of affine varieties $X_{[\lambda]} \to U_\chi$ where $U_\chi \subset X(v)$ is the affine open set containing the T-fixed point $[v_\chi]$. These pieces lead to a dominant morphism

$$\eta : X(\Delta_{\mathcal{F}}) \longrightarrow X(v).$$

In fact, $X(\Delta_{\mathcal{F}})$ together with η is the normalization of $X(v)$ since $\sigma_{[\lambda]}{}^\vee \cap \mathcal{M}$ is the saturation of $\mathbb{Z}_{\geq 0}(\mathcal{F} - \chi)$. By [13, III.8, Theorem 4], the normalization of a projective variety is projective. Therefore:

Proposition 4.2 *Let $\mathcal{F} \subset X(T)$ be a finite subset and $X(v)$ the corresponding projective T-orbit closure in $\mathbb{P}(V)$. Then the normalization of $X(v)$ is the projective toric variety $X(\Delta_{\mathcal{F}})$, where $\Delta_{\mathcal{F}}$ is the fan defined by (1).*

5 Torus orbit closures in G/P

Let G be a simply connected semisimple algebraic group. As usual $B \subset G$ will denote a Borel subgroup, T a maximal torus in B, $U \subset B$ the maximal unipotent subgroup and U^- the maximal unipotent subgroup opposite to U. Let $\Phi \subset X(T)$ denote the set of roots of G with respect to T, Φ^+ the roots which determine U, $\Phi^- = l - \Phi^+$ the roots which determine U^- (the positive and negative roots respectively) and $\Delta \subset \Phi^+$ the set of simple roots. $W = \mathrm{Nor}_G(T)/T$ will denote the Weyl group of G with respect to T, and $r_\alpha \in W$ will denote the reflection associated to $\alpha \in \Phi$.

It is well known that for any closed subgroup $P \supset B$, i.e., standard parabolic subgroup, there is a unique subset $\Delta_P \subset \Delta$ such that $P = BW_PB$, where $W_P \subset W$ denotes the subgroup generated by the r_α, $\alpha \in \Delta_P$ [1, 14.18]. Let $\Phi_P \subset \Phi$ denote the set of roots which are not linear combinations of roots in Δ_P. If U_α ($\alpha \in \Phi$) denotes the (unipotent) root subgroup of G, then for any ordering

$\alpha_1, \ldots, \alpha_k$ of Φ_P^-, multiplication defines a T-equivariant isomorphism in the sense of varieties from $U_{\alpha_1} \times \ldots \times U_{\alpha_k}$ onto a closed unipotent subgroup $U_P^- \subset U^-$ with the property that $U_P^- P/P$ is a T-stable open affine cell in G/P. Finally, let W^P denote the set of minimal length coset representatives for W/W_P. Thus every element in $w \in W$ has a unique expression $w = w^P w_P$, where $w^P \in W^P$ and $w_P \in W_P$ [8, 1.10].

Let $\omega \in X(T)$ be a dominant weight, and let $V(\omega)$ be the simple G-module of highest weight ω. Define $X_\omega(T)$ to be the set of $\mu \in X(T)$ in the convex hull of $W\omega$ which have the form $\mu = \omega - \sum_i k_i \beta_i$, where $k_i \geq 0$ and $\beta_i \in \Delta$ for each i.

Lemma 5.1 *The set of weights of T on $V(\omega)$ is $X_\omega(T)$, and the set of extreme weights in $X_\omega(T)$ is the full Weyl group orbit $W\omega$.*

For a proof, consult [2]. Now fix a highest weight vector $v_\omega \in V(\omega)$. The stabilizer of $[v_\omega]$ is a standard parabolic P such that the stabilizer of ω is W_P. Conversely, every standard parabolic P is obtained in this way for an appropriate (but not unique) choice of ω. For simplicity, we will say that ω and P are *associated.*

For the rest of this section, we will assume P is a standard parabolic and ω is an associated dominant weight. Then G/P imbeds in $\mathbb{P}(V(\omega))$ as the orbit $G[v_\omega]$. It is a basic fact that the image $G[v_\omega]$ is Zariski closed in $\mathbb{P}(V(\omega))$, and moreover, $G[v_\omega]^T = \{[n_w v_\omega] \mid w \in W^P\}$, where n_w is a representative of w ([8]). Consequently, closures of T-orbits in G/P are projective. If X is the closure of a T-orbit in G/P and $w \in W$, then $n_w X = n_w TX$, so we may define wX to be $n_w X$. Note that wX is also a T-orbit closure, and $(wX)^T = w(X^T)$.

In the sequel, we will denote the coset of G/P containing $g \in G$ by $g \cdot P$. If $w \in W$, then $w \cdot P$ will denote $n_w \cdot P$. Since $T \subset P$, this definition is unambiguous. The identity coset will be expressed as $e \cdot P$.

Lemma 5.2 *Let P be a standard parabolic and let $w \in W^P$. Then the closure X of a T-orbit in G/P containing $w \cdot P$ is of the form $w\overline{Tu \cdot P}$ for some $u \in U_P^-$ (depending on w). Moreover, any orbit closure of this form contains $w \cdot P$.*

PROOF: Let X be a T-orbit closure such that $w \cdot P \in X$. Then $w^{-1}X$ is the closure of a T-orbit containing $e \cdot P$, say $w^{-1}X = \overline{Tg \cdot P}$. By Lemma 2.1, there exists a $\lambda \in Y(T)$ such that $\lim_{s \to 0} \lambda(s) g \cdot P = e \cdot P$. Since $U_P^- \cdot P$ is a T-stable open set in G/P containing $e \cdot P$, it follows that $\lambda(s) g \cdot P \in U_P^- \cdot P$ if $s \neq 0$. Therefore we may put $u = \lambda(s_0) gp$ for a suitable $p \in P$ and $s_0 \in \mathbb{C}^*$. The proof of the second statement is similar. □

In view of the previous lemma, we will only need to consider orbit closures of the form $\overline{Tu \cdot P} \simeq X(uv_\omega) = \overline{T[uv_\omega]}$, where $u \in U_P^-$. Notice that the convex polytope of $wX(uv_\omega)$ is $w\mathcal{H}(uv_\omega)$, where the action of W on $X(T) \otimes \mathbb{R}$ is induced in the usual way from its natural action on T.

In order to describe a torus orbit closure $X = X(uv_\omega) \subset \mathbb{P}(V_\omega)$, one needs to find the support $\mathcal{F}_u(\omega) := \mathcal{F}(uv_\omega)$ (see §2). On the other hand, to just describe the T-orbits contained in X, it suffices, by Proposition 2.2 to find the fixed point set $X(uv_\omega)^T$, since determining the set of T-fixed points is equivalent to determining the set of vertices of the associated Newton polytope $\mathcal{H}_u(\omega) :=$

$\mathcal{H}(uv_\omega)$ and hence the set of all its faces, since a convex polytope is the hull of its extreme points. The problem, by Lemma 5.1, therefore is to determine the set of $w \in W^P$ for which $w\omega \in \mathcal{H}_u(\omega)$.

To find the T-fixed points in $X = X(uv_\omega)$, it will suffice, by Lemma 5.4 to describe the T-invariant curves in X. On the one hand, these curves correspond to the extreme edges in $\mathcal{H}_u(\omega)$. On the other hand, the T-invariant curves in G/P have the following general description (cf. [3, Theorem F]).

Lemma 5.3 *Every T-invariant curve in G/P has the form $C(x,\alpha) := \overline{U_\alpha x \cdot P}$ for some $x \in W^P$ and $\alpha \in \Phi$ with the property that $x^{-1}(\alpha) \in \Phi_P^-$. Moreover, $C(x,\alpha)^T = \{x, r_\alpha x\}$.*

PROOF: Note first that $C(x,\alpha) = xU_{x^{-1}(\alpha)} \cdot P$ is a curve in G/P as long as $x^{-1}(\alpha) \in \Phi_P^-$. It is T-invariant since T normalizes each U_α. Now suppose $u \cdot P$ is on a T-invariant curve for some $u \in U_P^-$. For each $\alpha \in \Phi$, choose a one parameter group $x_\alpha : \mathbb{C} \to U_\alpha$ such that $tx_\alpha(s)t^{-1} = x_\alpha(\alpha(t)s)$ for all $t \in T$. Write $u = x_{\beta_1}(s_1)\cdots x_{\beta_r}(s_r)$ with $\beta_1, \dots, \beta_r \in \Phi_P^-$ and each $s_i \neq 0$. Then if $t \in T$, $tu \cdot P = x_{\beta_1}(\beta_1(t)s_1)\cdots x_{\beta_r}(\beta_r(t)s_r) \cdot P$. Since T is a maximal torus, $Tu \cdot P$ cannot be one-dimensional unless $r = 1$, so $Tu \cdot P = C(e, \beta_1)$ as asserted. It is not hard to show that all other T-invariant curves in G/P are W^P-translates $wC(e,\beta_1)$. We leave the details of this to the reader. Clearly, if $\alpha \in \Phi_P^-$, $C(e,\alpha)^T = \{e, r_\alpha\}$, so $xC(e,\alpha)^T = x\{e, r_\alpha\} = \{x, xr_\alpha\} = \{x, r_{x(\alpha)}x\}$, which completes the proof since $x^{-1}(x(r_\alpha)) \in \Phi_P^-$. □

The next fact is a special case of a property of T-invariant subvarieties in any G/P proved in [3]. Since there is an easy self contained proof for T-orbit closures, we state it in that case only.

Lemma 5.4 *If $X \subset G/P$ is the closure of a T-orbit, then there are at least $\dim X$ T-invariant curves in X through each T-fixed point of X.*

PROOF: This is a consequence of the fact that each vertex of $\mathcal{H}_u(\omega)$ lies on at least $\dim \mathcal{H}_u(\omega)$ extreme edges, and $\dim \mathcal{H}_u(\omega) = \dim X(uv_\omega)$. □

We next describe the T-invariant curves in an orbit closure $X = X(uv_\omega)$ at $e \cdot P$ in terms of the edges of $\mathcal{H}_u(\omega)$ at ω. After that, we give a more precise description of these edges.

Lemma 5.5 *Let ω be a dominant weight, let P be the associated parabolic, and let $u \in U_P^-$. Then the edges of $\mathcal{H}_u(\omega)$ are parallel to roots. If an edge $\mathcal{E}$ is parallel to $\alpha \in \Phi$, then the endpoints of $\mathcal{E}$ have the form $x\omega, y\omega$ where $x, y \in W^P$ with $r_\alpha x = y$ and the T-invariant curve corresponding to $\mathcal{E}$ is a $C(x,\alpha)$ with $x^{-1}(\alpha) \in \Phi_P^-$.*

PROOF: By Proposition 2.2, an edge of $\mathcal{H}_u(\omega)$ having endpoints $x\omega, y\omega$ corresponds uniquely to a T-invariant curve in X with T-fixed points $x \cdot P$ and $y \cdot P$. Thus $y = r_\alpha x$ where $x^{-1}(\alpha) \in \Phi_P^-$ since $X \subset G/P$. □

One can say more about the edges by looking more closely at $u \in U_P^-$. An expression $u = u_{\alpha_{n_1}} \dots u_{\alpha_{n_r}}$ according to the above decomposition of U_P^- will be called *simple* provided no $u_{\alpha_{n_j}} = e$. To simplify notation, we will write $u = u_{\beta_1}u_{\beta_2}\cdots u_{\beta_r}$ where $\beta_i = \alpha_{n_i}$.

Proposition 5.6 *Assume $u = u_{\beta_1}u_{\beta_2}\cdots u_{\beta_r} \in U_P^-$ is a simple expression as above. Then every edge of $\mathcal{H}_u(\omega)$ at ω is parallel to an extreme root in the cone $\mathbb{R}_{\geq 0}\{\beta_1,\ldots,\beta_r\}$, and conversely, every extreme root in the cone determines an edge.*

PROOF: Suppose $\gamma \in \Phi_P^-$ corresponds to an edge at ω as in Lemma 5.5. Then $C(e,\gamma) \subset X$, so by Lemma 2.1, there exists a $\lambda \in Y(T)$ such that $\lim_{\zeta\to 0}\lambda(\zeta)u[v_\omega] = x_\gamma(s)[v_\omega]$ for some $s \neq 0$. If $\langle\lambda,\beta_i\rangle < 0$ for some i, then $\lim_{\zeta\to 0}\lambda(\zeta)u[v_\omega] \notin U_P^-[v_\omega]$, so each $\langle\lambda,\beta_i\rangle \geq 0$. Therefore $\lim_{\zeta\to 0}\lambda(\zeta)u[v_\omega] = x_{\beta_{m_1}}(s_1)\cdots x_{\beta_{m_p}}(s_p)[v_\omega]$ for the maximal subsequence $1 \leq m_i \leq r$ for which $\langle\lambda,\beta_i\rangle = 0$. By uniqueness, $p = 1$ and $\gamma = \beta_{m_1}$.

Conversely, assume β_j is on an extreme edge of $\mathbb{R}_{\geq 0}\{\beta_1,\ldots,\beta_r\}$. Recall that this means there exists a $\lambda \in Y(T)$ such that $\langle\lambda,\beta_i\rangle > 0$ if $i \neq j$ and $\langle\lambda,\beta_j\rangle = 0$. Now choose $s_i \in \mathbb{C}^*$ such that $x_{\beta_i}(s_i) = u_{\beta_i}$ for each i.Then, for all $\zeta \in \mathbb{C}$,

$$\lambda(\zeta)\cdot u[v_\omega] = \lambda(\zeta)u\lambda(\zeta)^{-1}[\lambda(\zeta)v_\omega] = \prod_{1\leq i\leq l} x_{\beta_i}(\zeta^{\langle\lambda,\beta_i\rangle}s_i)[v_\omega].$$

Therefore, $\lim_{\zeta\to 0}\lambda(\zeta)\cdot u[v_\omega] = x_{\beta_j}(s_j)[v_\omega]$. It follows that $x_{\beta_j}(s_j)[v_\omega] \in X$, so the T-invariant curve $C(e,\beta_j) \subset X$. So by Lemma 5.5, the extreme β_j determine edges of $\mathcal{H}_u(\omega)$ at ω. □

Consequently, if $\mathcal{E}_1,\ldots,\mathcal{E}_k$ are the edges of $\mathcal{H}_u(\omega) - \omega$ at 0, then the first vectors along the $\mathcal{E}_i$ in $X_\omega(T) - \omega$ are exactly the extreme roots in the cone $\mathbb{R}_{\geq 0}\{\beta_1,\ldots,\beta_r\}$.

The above proof generalizes to describe more of the orbit closure X.

Proposition 5.7 *Let ω be a dominant weight associated to the parabolic P, and let $u = u_{\beta_1}u_{\beta_2}\cdots u_{\beta_r} \in U_P^-$ be a simple expression. Then every T-orbit closure $Y \subset X(uv_\omega)$ containing P has the form $Y = X(u'v_\omega)$, where u' is a subexpression of u. That is, $u' = u_{\gamma_1}\cdots u_{\gamma_p}$, where $\gamma_i = \beta_{m_i}$ for some strictly increasing subsequence $1 \leq m_i \leq r$.*

PROOF: Let $y \in Y$ be an element of the open T-orbit closure of Y. By Lemma 2.1 there exists a $\lambda \in Y(T)$ such that $\lim_{\zeta\to 0}\lambda(\zeta)u[v_\omega] \in Ty$. As in the proof above, $\lim_{\zeta\to 0}\lambda(\zeta)u[v_\omega] = x_{\beta_{m_1}}(s_1)\cdots x_{\beta_{m_p}}(s_p)[v_\omega]$ for the maximal subsequence $1 \leq m_i \leq r$ for which $\langle\lambda,\beta_i\rangle = 0$. □

We now show that $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega)$ depends only on a simple expression of $u \in U_P^-$ and not on the associated dominant weight ω.

Proposition 5.8 *Suppose $u \in U_P^-$ has a simple expression $u = u_{\beta_1}\ldots u_{\beta_r}$ as above. Then $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega) = \mathbb{Z}_{\geq 0}\{\beta_1,\ldots,\beta_r\}$. In particular, $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega)$ is independent of the dominant weight ω associated to P.*

PROOF: By a repeated application of the fact that $u_\alpha v_\omega \in \sum_{k\geq 0}\mathbb{C}v_{\omega+k\alpha}$ for each $\alpha < 0$, the fact that weights occur in strings through ω, and the fact that $uv_\omega = u_{\beta_1}(u_{\beta_2}\cdots(u_{\beta_r}v_\omega))\cdots)$, we obviously obtain that

$$\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega) \subset \mathbb{Z}_{\geq 0}\{\beta_1,\ldots,\beta_r\}.$$

For the other inclusion, it suffices to notice that for each β_j, uv_ω has a non-vanishing component of weight $\omega + \beta_j$, so $\beta_j \in \mathcal{F}_u(\omega) - \omega$. □

The above discussion still does not settle the problem of determining the support of uv_ω. An added complication is that the T-isomorphism type of an $X(uv_\omega) \subset G/P$ for a $u \in U_P^-$ with simple expression $u = u_{\alpha_1} \dots u_{\alpha_r}$ may depend on the ordering of the roots $\{\alpha_1, \dots, \alpha_r\}$.

Example 5.9 Let $G = B_2$ and let α and β be the (negative) short and long simple roots respectively (see §7 for definitions). Consider $X_1 = X(u_\alpha u_{\alpha+\beta} v_\omega)$ and $X_2 = X(u_{\alpha+\beta} u_\alpha v_\omega)$ in G/B. Looking at Table I (§7), we find that $|X_1^T| = 6$ while $|X_2^T| = 4$. Thus X_1 and X_2 are not T-isomorphic.

Finally we state a result about the set of simple expressions of a $u \in U^-$.

Proposition 5.10 *Let $u \in U^-$ have two simple expressions $u = u_{\beta_1} \dots u_{\beta_l} = u_{\gamma_1} \dots u_{\gamma_m}$. Then*

$$\mathbb{Z}_{\geq 0}\{\beta_1, \dots, \beta_l\} = \mathbb{Z}_{\geq 0}\{\gamma_1, \dots, \gamma_m\},$$

and if $\beta_i = \gamma_j$ are two equal extreme edges, then $u_{\beta_i} = u_{\gamma_j}$.

PROOF: The first statement follows directly from 5.8. The second statement follows from the degeneration argument of 5.6. □

6 Normal T-orbit closures in G/P

In this section, we will give a classification of the normal torus orbit closures X in any G/P. By Lemma 5.2, it suffices to consider X of the form $\overline{Tu \cdot P}$, where $u \in U_P^-$. When we are considering a fixed embedding of G/P in $\mathbb{P}(V(\omega))$, we have the identification $X = X(uv_\omega)$. As in §3, let $\mathcal{M}$ denote the lattice $\mathbb{Z}\{\chi - \mu \mid \chi, \mu \in \mathcal{F}_u(\omega)\}$. We have already noted that $\mathcal{M} = \mathbb{Z}(\mathcal{F}_u(\omega) - w\omega)$ for any extreme point $w\omega$ of $\mathcal{H}_u(\omega)$. It follows from Lemma 5.2 and Proposition 5.8 that $\mathcal{M}$ does not depend on the choice of an ω associated to P. To motivate what follows, we make a general observation about normality.

Proposition 6.1 *Suppose $X = X(uv_\omega) \subset G/P$ is normal at $w \cdot P$, where $w \in W^P$. Then*

$$\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - w\omega) = \mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big). \tag{2}$$

PROOF: One inclusion of (2) is obvious. The other inclusion uses the fact that $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - w\omega)$ is saturated. So assume $x \in \mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big)$. Then $x = \sum q_i e_i$, where each $q_i \in \mathbb{Q}^+$ and each e_i is a primitive vector on an edge of $\mathcal{H}_u(\omega) - w\omega$ at 0. Since the edges of $\mathcal{H}_u(\omega) - w\omega$ at 0 are also the edges of $\mathcal{F}_u(\omega) - w\omega$ at 0, there exists an $n \in \mathbb{N}$ such that $nx \in \mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - w\omega)$. By saturation, it follows that $x \in \mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - w\omega)$, so the result is proved. □

Consequently, saturation of $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big)$ is a necessary condition for normality at $w \cdot P$. Unfortunately, it is not always a sufficient condition. However, if $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big)$ is saturated, then it follows immediately that X is normal at $w \cdot P$ if and only if $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big) = \mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) -$

$w\omega$). This motivates the following definition. Suppose $X = X(uv_\omega) \subset G/P$. Let us say that ω is *sufficiently positive for X at $w \cdot P \in X^T$* if and only if $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - w\omega) \cap \mathcal{M}\Big)$ is saturated and *sufficiently positive for X* if and only if it is sufficiently positive for all $w \cdot P \in X^T$. We now prove an important lemma.

Lemma 6.2 *Suppose $X \subset G/P$ is a T-orbit closure and $wP \in X^T$. Then there exists a dominant weight ω which is sufficiently positive for X at $w \cdot P$. In particular, there exists a dominant weight ω which is sufficiently positive for X.*

PROOF: By Lemma 5.2, it suffices to assume $w = e$ and $X = \overline{Tu \cdot P}$, where $u \in U_P^-$. Choose a simple expression $u = u_{\beta_1} \cdots u_{\beta_r}$. Then by Proposition 5.8, $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega) = \mathbb{Z}_{\geq 0}\{\beta_1, \ldots, \beta_r\}$. By 5.6, the edges of the parallelotope $\mathbf{P}(\mathcal{F}_u(\omega) - \omega)$ at 0 are the extreme roots among $\beta_1, \ldots, \beta_r$. It follows that the edges of $\mathbf{P}(\mathcal{F}_u(\omega) - \omega)$ at 0 are independent of ω. Since $\mathbf{P}(\mathcal{F}_u(\omega) - \omega) \subset \mathbb{R}_{\geq 0}\{\beta_1, \ldots, \beta_r\}$, it follows that we may, if necessary, replace ω by a sufficiently large multiple $\tilde{\omega} = p\omega$, $p > 0$, so that $\mathbf{P}(\mathcal{F}_u(\omega) - \omega) = \mathbf{P}(\mathcal{F}_u(\tilde{\omega}) - \tilde{\omega}) \subset \mathcal{H}_u(\tilde{\omega}) - \tilde{\omega}$. But this implies $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\tilde{\omega}) - \tilde{\omega}) \cap \mathcal{M}\Big)$ is saturated. Since there are only finitely many T-fixed points, taking p sufficiently large will give an $\tilde{\omega}$ which is sufficiently positive for X. □

To summarize the above results, we state the following theorem which finishes our description the normal orbit closures.

Theorem 6.3 *Let X be a T-orbit closure in G/P. Then: (1) there exists a dominant weight ω associated to P which is sufficiently positive for X, and (2) if ω is sufficiently positive for X at $w \cdot P \in X^T$, then X is normal at $w \cdot P$ if and only if after X is expressed as $X = wX(uv_\omega)$ with $u \in U_P^-$, we have*

$$\mathbb{Z}_{\geq 0}\{\beta_1, \ldots, \beta_r\} \;=\; \mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - \omega) \cap \mathcal{M}\Big). \tag{3}$$

where $u = u_{\beta_1} \cdots u_{\beta_r}$ is a simple expression for u.

Remark 6.4 It is an open question as to whether there exists an example where $\mathbb{Z}_{\geq 0}\Big((\mathcal{H}_u(\omega) - \omega) \cap \mathcal{M}\Big)$ is not saturated.

Remark 6.5 Theorem 6.3 shows that the local question of normality of $X = w\overline{Tu \cdot P}$ at $w \cdot P$ only depends on the roots that occur in a simple expression of u, not on the order in which the factors u_{β_i} occur. However, we have already noticed in Example 5.9 that the isomorphism type of X depends on the order. The above arguments can be used to show that if X is (globally) normal, then any orbit closure obtained from X by rearranging the factors is still normal.

Corollary 6.6 *Suppose $X = wX(uv_\omega)$ is a T-orbit closure and ω is sufficiently positive for X. Then if $\mathcal{F}_u(\omega) = \mathcal{H}_u(\omega) \cap X_\omega(T)$, X is normal.*

PROOF: Just apply Theorem 6.3. □

Remark 6.7 Suppose that ω is sufficiently positive but $\mathcal{F}_u(\omega)$ is a proper subset of $\mathcal{H}_u(\omega) \cap X_\omega(T)$. Then the inclusion $\mathcal{F}_u(\omega) \subset \mathcal{H}_u(\omega) \cap X_\omega(T)$ induces a T-equivariant dominant morphism $X \to X(uv_\omega)$ of torus orbit closures, where

$X \subset \mathbb{P}(V(\omega))$ is the orbit closure with support $\mathcal{H}_u(\omega) \cap X_\omega(T)$. By the previous corollary, X is normal and hence is a normalization of $X(uv_\omega)$ in $\mathbb{C}(\mathcal{H}_u(\omega) \cap X_\omega(T))$.

We now note that every projective T-orbit closure in G/P for the cases of A_n, B_2 and D_4 is normal. This follows from particular results of Jacqueline Morand about root systems. We first state [12, Proposition 7].

Lemma 6.8 *Let $\gamma_1, \dots, \gamma_n \in X(T)$ be a $\mathbb{Z}$-basis of $X(T)$ and put $\Psi := \{\pm\gamma_{ij} \mid 1 \leq i < j \leq n,\ \gamma_{ij} = \gamma_i + \cdots + \gamma_j\}$. Then, for every subset $\Sigma \subset \Psi$, the semigroup $\mathbb{Z}_{\geq 0}\Sigma$ is saturated in $\mathbb{Z}\Sigma$.*

The point of the proof is as follows. Suppose $M := \mathbb{Z}\Sigma$, $S := \mathbb{Z}_{\geq 0}\Sigma$, and there is an $m \in M \setminus S$ with $km \in S$. Choose s minimal such that $km = \sum_{j=1}^{s} n_j\beta_j$, $\beta_j \in \Sigma$, $n_j \geq 1$. By a result of Wehlau, $\beta_1, \dots, \beta_s$ are linearly independent. The final step is to show under the hypotheses of the lemma, this set can be enlarged to a $\mathbb{Z}$-basis of $X(T)$, which is clearly a contradiction to the existence of m.

Lemma 6.8 immediately implies that every semigroup generated by a set of roots in A_n is saturated. This is the case for type D_4 also [12, Prop. 8]. The case of B_2 is easy to check by hand. We can therefore prove

Proposition 6.9 *If G is of type A_n, D_4 or B_2, then every projective T-orbit closure in G/P is normal.*

PROOF: By Lemma 5.2, it suffices to show that every T-orbit closure of the form $X(uv_\omega)$, $u \in U_P^-$, is normal at P. By 5.8, $\mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega)$ is generated by a set of roots, hence is saturated in $\mathcal{M}$ (Lemma 6.8). Therefore $X(uv_\omega)$ is normal at P. □

For the remaining root systems, Wehlau has found examples of non-saturated semigroups generated by roots. Hence non-normal torus orbit closures exist in these cases. We will give some non-normal torus orbits in G_2/B below.

Sturmfels [19, after 2.16] pointed out that one can also prove the normality (in fact, projective normality) of all torus orbit closures in Grassmannians $SL_n(\mathbb{C})/P$ (P a maximal parabolic) from the fact that any semigroup generated by matroid basis monomials is saturated [20].

7 Two dimensional torus orbit closures in G/B

For the remainder of this paper, we consider two-dimensional torus orbit closures in a flag variety. Here it is sufficient to restrict to rank 2 root systems. In fact, if we take $\overline{Tu_{\alpha_1}u_{\alpha_2}u_{\alpha_3} \cdot B} \subset G/B$ for linearly independent negative roots $\alpha_1, \alpha_2, \alpha_3$, then the isotropy group $T_{u_{\alpha_1}u_{\alpha_2}u_{\alpha_3} \cdot B} = \cap_{j=1}^{3} \ker \alpha_j \subset T$ is of codimension 3 in T. Throughout this section, ω denotes the sum of the fundamental weights $\omega_1, \dots, \omega_l$, so that ω is associated to B.

Figure 1 shows the root system B_2 as well as the extremal weights $\mu_1, \dots, \mu_8$ and the non-extremal weights $\nu_1, \dots, \nu_4$ for the highest weight module $V(\omega)$. The lines between vertices correspond to T-invariant curves in G/B for $G = B_2$. The fans pictured on the right show the first lattice vector on each edge.

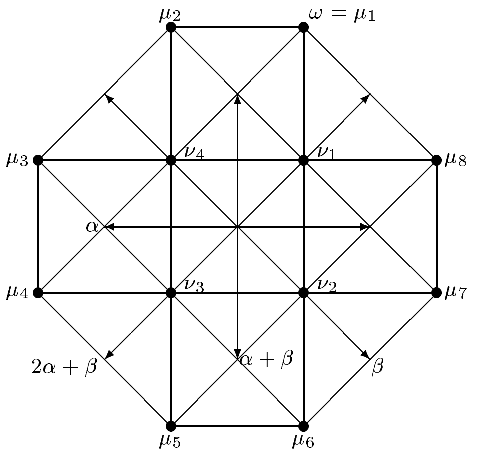
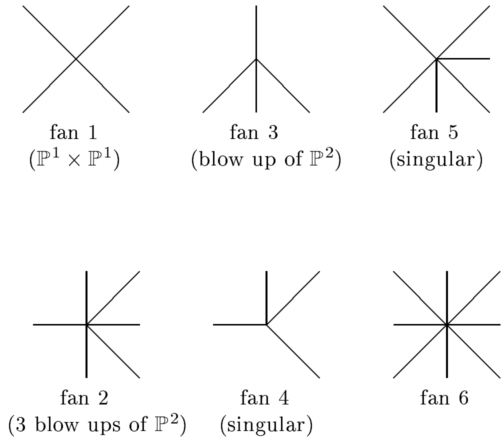


Figure 1



For the root systems $A_1 \times A_1$ and A_2, all torus orbit closures are nonsingular. In Table I we describe torus orbit closures in B_2 of the form $\overline{Tu \cdot B}$ for $u \in U^-$ with explicit decomposition $U^- \simeq U_\alpha \times U_{2\alpha+\beta} \times U_{\alpha+\beta} \times U_\beta$. We have shown that they are normal, however, some are singular. The roots given in the 'orbit type' correspond to the indices of reduced expressions $u = u_{\alpha_{n_1}} \dots u_{\alpha_{n_l}}$. The support $\mathcal{F}_u(\omega)$ is given in the column labeled 'weights', and '$\sharp$ T-fp' lists the number of T-fixed points which correspond to the extreme weights μ_i of the convex hull of $\mathcal{F}_u(\omega)$.

orbit type	weights	$\sharp$ T-fp	$\mathcal{M}$	nonsingular	fan $\sharp$
$\alpha, 2\alpha+\beta$	$\mu_1, \dots, \mu_4; \nu_4$	4	$\mathbb{Z}\Phi$	yes	3
$\alpha, \alpha+\beta$	$\mu_1, \dots, \mu_6; \nu_1, \dots, \nu_4$	6	$\mathbb{Z}\Phi$	yes	2
α, β	$\mu_8, \mu_1, \mu_2, \mu_3; \nu_1, \nu_4$	4	$\mathbb{Z}\Phi$	yes	3
$2\alpha+\beta, \alpha+\beta$	$\mu_1, \mu_4, \mu_5, \mu_6; \nu_1, \dots, \nu_4$	4	$\mathbb{Z}\Phi$	no	4
$2\alpha+\beta, \beta$	$\mu_1, \mu_4, \mu_5, \mu_8; \nu_2, \nu_4$	4	$\mathbb{Z}\{2\alpha+\beta, \beta\}$	yes	1
$\alpha+\beta, \beta$	$\mu_1, \mu_6, \mu_7, \mu_8; \nu_1, \nu_2$	4	$\mathbb{Z}\Phi$	yes	3
$\alpha, 2\alpha+\beta, \alpha+\beta$	$\mu_1, \dots, \mu_6; \nu_1, \dots, \nu_4$	6	$\mathbb{Z}\Phi$	yes	2
$\alpha, 2\alpha+\beta, \beta$	$\mu_1, \dots, \mu_5, \mu_8; \nu_1, \dots, \nu_4$	6	$\mathbb{Z}\Phi$	no	5
$\alpha, \alpha+\beta, \beta$	$\mu_1, \dots, \mu_8; \nu_1, \dots, \nu_4$	8	$\mathbb{Z}\Phi$	yes	6
$2\alpha+\beta, \alpha+\beta, \beta$	$\mu_1, \mu_4, \dots, \mu_8; \nu_1, \dots, \nu_4$	6	$\mathbb{Z}\Phi$	no	5
$\alpha, 2\alpha+\beta, \alpha+\beta, \beta$	$\mu_1, \dots, \mu_8; \nu_1, \dots, \nu_4$	8	$\mathbb{Z}\Phi$	yes	6

TABLE I

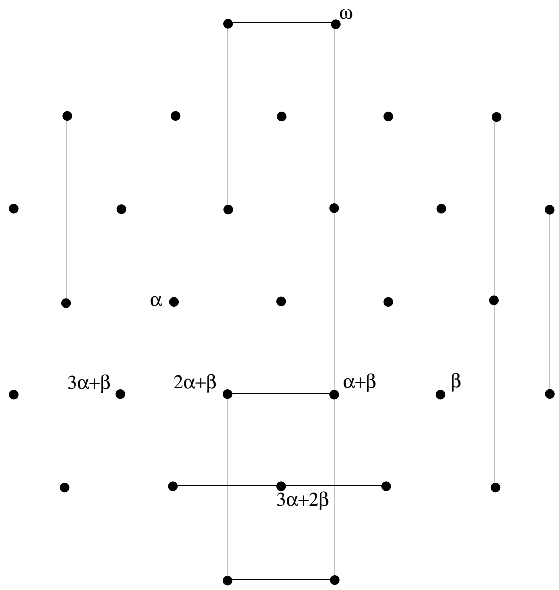


Figure 2

In [4, 1.1], a T-orbit closure $X = \overline{Tu \cdot P} \subset G/P$, $u \in U_P^-$ is called generic, if $X^T = (G/P)^T$ and if $\mathbb{Z}_{\geq 0}(w\mathcal{F}_u(\omega) - \omega) = \mathbb{Z}_{\geq 0}\Phi_P^-$ for each $w \in W$. In the case for $P = B$ this simply means that X is generic if and only if $X^T = (G/B)^T$. To tie the notion of a reduced expression introduced in §5 into this notion, we state the following lemma whose the proof follows from [7, 20.2. Theorem (a)].

Lemma 7.1 *If $u = u_{\alpha_1} \dots u_{\alpha_k}$ is reduced of maximal length, then $\overline{Tu \cdot B} \subset G/B$ is generic.*

We next consider torus orbit closures in type G_2. In the next example we find a non-normal torus orbit closure which is generic in the sense of Flaschka and Haine [5].

Example 7.2 Consider $X = \overline{Tu \cdot B}$ in G_2/B with $u = u_{3\alpha+\beta}u_{3\alpha+2\beta}u_{\alpha+\beta}u_\beta$. The vertices of the parallelotope $P(\mathcal{F}_u(\omega) - \omega)$ are 0, $3\alpha + \beta$, β, and $3\alpha + 2\beta$. The lattice $\mathcal{M}$ being $\mathbb{Z}\Phi$, X is not normal because $2\alpha + \beta \in P(\mathcal{F}_u(\omega) - \omega) \cap \mathcal{M}$, but $2\alpha + \beta \notin \mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - \omega)$. The number of T-fixed points, that is the number of vertices in $\mathcal{F}_u(\omega)$, is 10. Now let $P \supset B$ be the parabolic subgroup associated to the short simple root α. X maps equivariantly under the projection $G/B \to G/P$ to a non-normal torus orbit closure Y in G_2/P. Using Figure 2 one finds that $|Y^T| = 6$, so Y is generic in the sense of [5, 2.1. Remark (iii)].

In the same way as in 7.2 we can show that $\overline{Tu \cdot B} \subset G_2/B$ is non-normal at B for $u = u_\alpha u_{3\alpha+\beta}u_{3\alpha+2\beta}$, $u = u_\alpha u_{3\alpha+2\beta}u_\beta$ et al. For $u = u_\alpha u_{3\alpha+\beta}u_{3\alpha+2\beta}u_\beta$ the torus orbit closure $\overline{Tu \cdot B}$ is normal at B, but non-normal at $r_\beta \cdot B$, since $2\alpha + \beta \in P(\mathcal{F}_u(\omega) - r_\beta\omega) \cap \mathbb{Z}\Phi$, but $2\alpha + \beta \notin \mathbb{Z}_{\geq 0}(\mathcal{F}_u(\omega) - r_\beta\omega)$.

In Table II we list the roots of the reduced expressions $u = u_{\alpha_{n_1}} \dots u_{\alpha_{n_l}} \in U^-$ such that $\overline{Tu \cdot B}$ is generic. Here we use the decomposition

$$U^- \simeq U_\alpha \times U_{3\alpha+\beta} \times U_{2\alpha+\beta} \times U_{3\alpha+2\beta} \times U_{\alpha+\beta} \times U_\beta.$$

Curiously, there are generic torus orbit closures $\overline{Tu \cdot B}$ where u is reduced of less than maximal length.

generic orbit type
$\alpha, 2\alpha+\beta, \alpha+\beta, \beta$
$\alpha, 3\alpha+2\beta, \alpha+\beta, \beta$
$\alpha, 3\alpha+\beta, 2\alpha+\beta, 3\alpha+2\beta, \beta$
$\alpha, 3\alpha+\beta, 3\alpha+2\beta, \alpha+\beta, \beta$
$\alpha, 2\alpha+\beta, 3\alpha+2\beta, \alpha+\beta, \beta$
$\alpha, 3\alpha+\beta, 2\alpha+\beta, 3\alpha+2\beta, \alpha+\beta, \beta$

TABLE II

There are also normal but singular torus orbit closures: for example $\overline{Tu \cdot B}$ where $u = u_\alpha u_{\alpha+\beta} u_\beta$, or $u = u_\alpha u_{3\alpha+\beta} u_{3\alpha+2\beta} u_\beta$ (cf. Remark 3.9). Moreover, the normalization of $\overline{Tu \cdot B}$ where $u = u_\alpha u_{3\alpha+2\beta} u_\beta$ is seen to be singular by using the fan of §4.

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