Singularities of Schubert Varieties, Tangent Cones and Bruhat Graphs  
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JAMES B. CARRELL  
JOCHEN KUTTLER  

Abstract  
Let $G$ be a semi-simple algebraic group over $\mathbb{C}$, $B$ a Borel subgroup of $G$, $T$ a maximal torus in $B$ and $P$ a parabolic in $G$ containing $B$. In a previous work [7], the authors classified the singular $T$-fixed points of an irreducible $T$-stable subvariety $X$ of the generalized flag variety $G/P$. It turns out that under the restriction that $G$ doesn’t contain any $G_2$-factors, the key geometric invariant determining the singular $T$-fixed points of $X$ is the linear span $\Theta_x(X)$ of the reduced tangent cone to $X$ at a $T$-fixed point $x$. The goal of this paper is to describe this invariant at the maximal singular $T$-fixed points when $X$ a Schubert variety in $G/B$ and $G$ doesn’t contain any $G_2$-factors. We first describe $\Theta_x(X)$ solely in terms of Peterson translates, which were the main tool in [7]. Then, taking a further look at the Peterson translates (with the $G_2$-restriction), we are able to describe $\Theta_x(X)$ in terms of its isotropy submodule and the Bruhat graph of $X$ at $x$. This refinement gives a purely root theoretic description, which should be useful for computations. Finally, still with the $G_2$-restriction, these considerations lead us to a non-recursive algorithm for $X$’s singular locus solely involving only the root system of $(G,T)$ and the Bruhat graph of $X$.  

1. Introduction  
Let $G$ be a semi-simple algebraic group over an arbitrary algebraically closed field $k$, and suppose $T \subset B \subset P$ are respectively a maximal torus, a Borel subgroup and an arbitrary standard parabolic in $G$. Each $G/P$, including $G/B$, is a projective $G$-variety with only finitely many $B$-orbits. Every $B$-orbit contains a unique $T$-fixed point $x \in (G/P)^T$, and these cells define an affine paving of $G/P$. If $x \in (G/P)^T$, then the closure of the $B$-orbit $Bx$ is called the Schubert variety in $G/P$ associated to $x$. This Schubert variety will be denoted throughout by $X(x)$. We will use the well known fact that the $T$-fixed points in $G/B$ are in one to one correspondence with the elements of the Weyl group $W = N_G(T)/T$, so we don’t distinguish between elements of $W$ and fixed points in $G/B$.  
Schubert varieties are in general singular, and it’s an old problem, inspired by a classical paper [8] of Chevalley, to describe their singular loci (or, equivalently, their smooth points). A related problem, with interesting consequences in representation theory, is to determine

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the locus of rationally smooth points of a Schubert variety (cf. [9]). In fact, if $G$ is defined over $\mathbb{C}$ and simply laced (i.e. every simple factor is of type $A$, $D$ or $E$), then all rationally smooth points of any Schubert variety in $G/P$ are in fact smooth (see [7]).

In this paper, we will consider the singular locus of a Schubert variety in an arbitrary $G/P$, where $G$ does not contain any $G_2$-factors. Our results are an outgrowth of [7], where we used Peterson translates (defined below) to characterize the $T$-fixed points in the singular locus of an irreducible $T$-stable subvariety $X \subset G/P$ (a $T$-variety for short). In fact, we showed that, with the $G_2$-restriction, a Cohen-Macaulay point $x \in X$ lying in $X^T$ is in the singular locus of $X$ precisely when one of the following two conditions occurs: either every $T$-stable curve in $X$ containing $x$ lies in the singular locus of $X$, or $\dim \Theta_x(X) > \dim X$, where $\Theta_x(X)$ denotes the linear span of the reduced tangent cone of $X$ at $x$.

To determine the singular locus when $X$ is a Schubert variety, it suffices to describe the $T$-fixed points in the singular of $X$. This is most efficiently accomplished by using the natural ordering on $W = (G/B)^T$: if $x, y \in W$, then $x < y$ if and only if $X(x) \subset X(y)$ but $x \neq y$. This ordering can also be described as the Bruhat-Chevalley order on $W$ (cf. [8]). It has two nice properties. First, if $X = X(w)$, then $X^T$ is the interval $[e, w] = \{x \in W \mid x \leq w\}$. Secondly, if $X$ is smooth (resp. singular) at $y \in X^T$, then it is smooth (resp. singular) at all $z \in X^T$ with $z > y$ (resp. $z \leq y$). Hence the problem of determining the singular locus of $X$ boils down to identifying the maximal singular $T$-fixed points of $X$ in $[e, w]$, i.e. those $y \in [e, w]$ such that $X$ is singular at $y$ and smooth at $z$ if $z \notin [y, w]$. Such a $y$ is called a maximal singularity of $X$.

Among the main objects we consider in this paper are the Peterson translate of $X$. These can be defined as follows. Let $E(X, x)$ denote the set of $T$-stable curves in $X$ which contain the point $x \in X^T$. An element $C$ of $E(X, x)$ is called good if $C$ contains a smooth point of $X$. Then the Peterson translate of $X$ along a good $C \in E(X, x)$ can be defined as

$$\tau_C(X, x) = \lim_{z \to x} T_z(X) \ (z \in C \setminus C^T),$$

where $T_z(X)$ is the Zariski tangent space of $X$ at $z$. Peterson translates are studied extensively in [7]. In particular, one can show they exist, and each Peterson translate $\tau_C(X, x)$ is a $T$-stable subspace of $T_x(X)$ such that $\dim \tau_C(X, x) = \dim X$. Clearly, $X$ is smooth at $x$ if and only if $\tau_C(X, x) = T_x(X)$. Let $TE(X, x) = \sum_{C \in E(X, x)} T_x(C)$ be the subspace of $T_x(X)$ spanned by the tangent lines at $x$ of the elements of $E(X, x)$. Each $T_x(C)$ is a $T$-stable line in $T_x(X)$, and the $T$-weights of these lines are certain elements of the root system $\Phi$ of the pair $(G, T)$. Furthermore, if $T_x(C)$ has weight $\alpha \in \Phi$, then $C^T = \{x, r_\alpha x\}$, where $r_\alpha \in W$ is the reflection corresponding to $\alpha$. A $T$-curve $C \in E(X, x)$ is called short or long according to whether the $T$-weight of its tangent line $T_x(C)$ is short or long in $\Phi$.

One of the key considerations is whether or not $\tau_C(X, x) \subset TE(X, x)$. The following result, which is a combination of Theorems 1.4 and 1.6 of [7], is in part an answer to this.

**Theorem 1.1.** Assume $G$ has no $G_2$-factors, and let $X$ be a $T$-variety in $G/P$ which is Cohen-Macaulay at the point $x \in X^T$. Then $X$ is smooth at $x$ if and only if $\tau_C(X, x) = TE(X, x)$ for at least one good $T$-curve $C$. Moreover,

$$\tau_C(X, x) \subset \Theta_x(X)$$

for at least one good $C$. Moreover,
for every good $C \in E(X, x)$. In fact, if $C$ is good and also short, then $\tau_C(X, x) \subset TE(X, x)$.

**Remark 1.2** In fact, the first assertion is true even if $G$ has $G_2$ factors. We will say more about the necessity of the $G_2$-restriction in Remark 1.4 below.

In view of the above result, the main problem is how to describe $\Theta_x(X)$ for a Schubert variety in $G/P$, provided $E(X, x)$ contains a good curve and $G$ doesn’t have $G_2$ factors. Note that this is weakened version of the classical problem of describing the Zariski tangent space of a Schubert variety at a $T$-fixed point $x$ (cf [2]). Indeed, the linear span of the reduced tangent cone is in general a proper subspace of the tangent space of a Schubert variety. Thus, one of our main goals is to describe $\Theta_x(X)$ when $x$ is a maximal singularity of $X$.

In general, $TE(X, x) \subset \Theta_x(X)$. In fact, we will prove in Theorem 3.1 that if $G$ is simply laced, then for every $T$-variety, $TE(X, x) = \Theta_x(X)$. This was already known for Schubert varieties (see [4] and [6]). The following result seems to be all that is known in the general case (it holds even without the $G_2$-restriction).

**Proposition 1.3.** (cf. [6]) If $X$ is a Schubert variety in $G/B$, then the roots which are weights of $T$-lines in $\Theta_x(X)$ lie in the convex hull $\mathcal{H}$ in $\Phi \otimes \mathbb{R}$ of the roots which are $T$-weights of $TE(X, x)$. If $x = e$, then the set of $T$-weights of $T$-lines in $\Theta_x(X)$ is exactly $\mathcal{H} \cap (\Phi \otimes \mathbb{R})$.

**Remark 1.4** This Proposition explains why the $G_2$-restriction in Theorem 1.1 (1) is needed. Consider the singular Schubert variety $X(w)$ in $G_2/B$ corresponding to the reflection $w = r_\beta r_\alpha r_\beta \in W$, where $\alpha$ and $\beta$ stand for the short and long simple roots and $r_\alpha$ is the reflection corresponding to $\alpha$. The $T$-lines in $E(X, e)$ have weights $-\alpha, -\beta,$ and $-(\alpha + \beta)$, so $\Theta_x(X) = TE(X, e)$, since they are the roots in their convex hull. But the $T$-curve $C$ from $w$ to $e$ is good, so (1) certainly cannot hold in $X(w)$.

We now state the first of two characterizations of $\Theta_x(X)$ for a Schubert variety $X$ at a maximal singularity $x$ we will prove in this note.

**Theorem 1.5.** Assume $G$ has no $G_2$-factors, and suppose $X$ is a Schubert variety in $G/P$. Then for any $x \in X^T$ which is either smooth in $X$ or a maximal singularity, we have

$$\Theta_x(X) = \sum_C \tau_C(X, x),$$

where the sum is over all $C \in E(X, x)$ with $C^T = \{x, y\}$, where $y > x$.

The proof is given in §4. Note that if $x$ is maximal (or smooth), all $T$-curves $C$ such that $C^T = \{x, y\}$ and $y > x$ are good. Since there is an algorithm for computing $\tau_C(X, x)$ (cf. [7]), formula (2) gives an explicit method of explicitly computing $\Theta_x(X)$. Using similar methods, the second author has found more results on the tangent cone, which, in particular, give a significant amount of information on the Nash blow up (cf. [11]).

**Remark 1.6** Theorem 1.5 fails for Schubert varieties in $G_2/B$. On the other hand, when $G$ is simply laced, it also follows (much more simply) from the existence of a slice at $x$ ([7, Lemma 4.6]).
Our second characterization uses Theorem 1.5 to get an expression for $\Theta_x(X)$ in which Peterson translates are no longer in the picture. For this, we need to bring in some more concepts.

For any $x \in X^T$, let $B_x \subset B$ be the isotropy subgroup of $x$. That is, $B_x$ is the subgroup of $B$ namely the subgroup generated by $T$ and all root subgroups $U_\alpha$ of $B$ which fix $x$. As usual, a root $\alpha$ such that $U_\alpha \subset B$ is taken to be positive, and we write $\alpha > 0$. The condition that $U_\alpha x = x$ is equivalent to $x^{-1}(\alpha) > 0$. Thus, for any Schubert variety $X$ in $G/P$, $\Theta_x(X)$ is a $B_x$-submodule of $T_x(G/P)$. The isotropy submodule of $\Theta_x(X)$ is the smallest $B_x$-submodule $T_x(X)$ of $T_x(X)$ which contains $TE(X,x)$.

We will show that if $C \in E(X,x)$ is good, then the roots which correspond to $T$-lines in the $T$-module $\tau_C(X,x)/(T_x(X) \cap \tau_C(X,x))$ can be explicitly described in terms of the notion of an orthogonal $B_2$-pair, which is now defined. For each $\gamma \in \Phi$, let $g_\gamma$ denote the $T$-stable line in the Lie algebra $g = \text{Lie}(G)$ of weight $\gamma$. In other words, $g_\gamma$ is the root line of weight $\gamma$.

**Definition 1.7** Let $X = X(w)$ be a Schubert variety in $G/B$, and assume $x < w$. Suppose $\mu$ and $\phi$ are long, positive orthogonal roots such that the following three conditions hold:

1. $g_{-\mu} \oplus g_{-\phi} \subset TE(X(w), x)$ (hence $x < r_\mu x$, $r_\phi x \leq w$),
2. there exists a subroot system $\Phi'$ of $\Phi$ of type $B_2$ containing $\mu$ and $\phi$, and
3. if $\alpha$ and $\beta$ form the unique basis of $\Phi'$ contained in $\Phi^+ \cap \Phi'$ with $\alpha$ short and $\beta$ long, then $r_\alpha x < x$, and $r_\alpha r_\beta x \leq w$.

Then we say that $\{\mu, \phi\}$ form an orthogonal $B_2$-pair for $X$ at $x$.

We now state our second characterization of $\Theta_x(X)$ at a maximal singularity.

**Theorem 1.8.** Assume $G$ has no $G_2$-factors, and suppose $x$ is a maximal singularity of a Schubert variety $X$ in $G/B$. Then for each $T$-weight $\gamma$ of the quotient $\Theta_x(X)/T_x(X)$, there exists an orthogonal $B_2$-pair $\{\mu, \phi\}$ for $X$ at $x$ such that

\[
\gamma = -1/2(\mu + \phi).
\]

In other words, at a maximal singularity of $X$, every $T$-weight of $\Theta_x(X)$ not in $T_x(X)$ is a weight arising from a $B_2$-pair at $x$ as in (3).

This is proved in §5. We also obtain the following necessary and sufficient condition for a $T$-fixed point $x$ of a Schubert variety to be a smooth point, which is also proved in §5.

**Theorem 1.9.** Assume $G$ has no $G_2$-factors, let $X$ be a Schubert variety in $G/B$, and let $x \in X^T$. Then $X$ is smooth at $x$ if and only if the following three conditions hold.

1. $|E(X,x)| = \dim X$, and some $C \in E(X,x)$ is good.
2. We have $T_x(X) = TE(X,x)$, and
3. If $\{\mu, \phi\}$ is an orthogonal $B_2$-pair for $X$ at $x$, and $\gamma = -1/2(\mu + \phi)$, then $g_\gamma \subset TE(X,x)$. Consequently, $r_\gamma x \leq w$. 

Corollary 1.10. There exists a non-recursive algorithm involving only the Bruhat graph and the root system $\Phi$ which classifies the smooth $T$-fixed points of a Schubert variety in $G/B$.

The notion of an orthogonal $B_2$-pair arises from the Schubert variety $X = X(r_\alpha r_\beta r_\alpha)$ in $B_2/B$, where $\alpha$ and $\beta$ are the short and long simple roots in $B_2$. The $T$-fixed point $x = r_\alpha$ is the unique maximal singularity of $X$. Now the weights of $TE(X,x)$ are $\alpha$, $-\beta$ and $-(\beta + 2\alpha)$. Furthermore, $B_x$ is generated by $T$, $U_\beta$, $U_{\alpha + \beta}$ and $U_{2\alpha + \beta}$, so it is easy to see that $TE(X,x)$ is already a $B_x$-submodule of $T_x(X)$. But $\{\beta, \beta + 2\alpha\}$ give an orthogonal $B_2$-pair at $x$ such that $g_\gamma \subset \Theta_x(X)/TE(X,x)$, where $\gamma = -1/2(\mu + \phi) = -(\alpha + \beta)$. (See Example 5.2 and [7] for more details.)

The following figure illustrates the portion of Bruhat graph of a Schubert variety $X$ arising from an orthogonal $B_2$-pair $\{\mu, \phi\}$ at $x$. If $x$ is on a good $T$-curve and there is no edge in $\Gamma(X)$ at $x$ corresponding to a $T$-curve $C$ with $x < r_\gamma x \leq w$, where $\gamma = -1/2(\mu + \phi)$, then $X$ is singular at $x$.

**Figure 1:** $\alpha$ and $\beta$ are the short and long simple roots in a $\Phi^+(B_2)$ containing $\{\mu, \phi\}$.

Let us describe the algorithm of Corollary 1.10. Suppose we want to determine whether a Schubert variety $X = X(w)$ is smooth at $x \in X^T$. Consider any single descending path $w > x_1 > x_2 > \cdots > x_m > x$ in $\Gamma(X)$. If $X$ is singular at any $x_i$, then it is singular at $x$. Thus, suppose $X$ is smooth at $x_m$. Then the edge $x_m x$ is a good $T$-curve in $X$, so it suffices to check the conditions of Theorem 1.9 for this $T$-curve. Now (1) reduces to showing $|\{\gamma > 0 \mid r_\gamma x \leq w\}| = \ell(w)$, where $\ell(w)$ is the length of $w$ (and also dim $X(w)$). Verifying (2) amounts to showing that $TE(X,x)$ is $B_x$-stable. This requires verifying that if $g_\gamma \subset TE(X,x)$, then $g_{\gamma + \alpha} \subset TE(X,x)$ for all $\alpha > 0$ such that $x^{-1}(\alpha) > 0$, $\gamma + \alpha \in \Phi$ and $x^{-1}(\alpha + \gamma) < 0$. Condition (3) is verifiable from the Bruhat graph at $x$, so our claim that the algorithm involves only $\Phi$ and $\Gamma(X)$ is
verified. The non-recursivity follows since we only need to consider a single path in $\Gamma(X)$ from $w$ to $x$.

It might also be useful to remark that unlike checking whether a Schubert variety $X$ is smooth at a fixed point $x$, checking for rational smoothness at $x$ via the Bruhat graph requires that one count the number of edges in $\Gamma(X)$ at all vertices $y \geq x$ [4]. Therefore it appears to be easier to identify the smooth points than the rationally smooth points. B. Boe and W. Graham have formulated the following lookup conjecture: a Schubert variety $X$ in $G/P$ is rationally smooth at $x$ if and only if $|E(X,y)| = \dim X$ for all $y$ on an edge of $\Gamma(X)$ containing $x$. Some special cases of the lookup conjecture are verified in [5], but the general conjecture is open. Theorem 1.9 says that as far as smoothness is concerned, one has to examine $\Gamma(X)$ two steps above and one step below $x$. This might be considered somewhat unexpected.

Finally, let us mention that this paper has connections with the work of S. Billey and A. Postnikov [3] and very likely also S. Billey and T. Braden [1]. However, unlike the situation in [3], our results do not say anything in the $G_2$ case, as noted in Remark 1.4.

2. Preliminaries

We will throughout use the terminology and notation of [7], some of which was already introduced in §1. In particular, the $G_2$-hypothesis is always in effect.

Let us first recall some of the standard facts and notations concerning roots, weights, $T$-curves and so on. The $T$-fixed point set of a $T$-variety $X \subset G/P$ is denoted by $X^T$. It’s well known that the mapping $w \rightarrow n_wB$ is a bijection the Weyl group $W = N_G(T)/T$ of $(G,T)$ with $(G/B)^T$, so we assume $W = (G/B)^T$. The projection $\pi : G/B \rightarrow G/P$ is an equivariant closed morphism, so $(G/P)^T$ may be identified with $W/W_P$, $W_P$ being the parabolic subgroup of $W$ associated to $P$. The elements of $W/W_P$ thus parameterize the Schubert varieties in $G/P$.

Every $T$-curve in $E(X,x)$ has the form $C = U_\alpha x$ for a unique root $\alpha \in \Phi$. If $P = B$, then $C^T = \{x, r_\alpha x\}$. If $X$ is a Schubert variety in $G/B$, say $X = X(w)$, then $C = U_\alpha x \subset X$ if and only if both $x, r_\alpha x \leq w$. By [4, LEMMA A], $|E(X,x)| \geq \dim X$ for every $T$-variety $X$. Furthermore, every $T$-curve in $G/P$ is the image of a $T$-curve in $G/B$ under the closed morphism $\pi : G/B \rightarrow G/P$. Also, recall that as $T$-modules, $T_x(G/B) = \bigoplus_{x^{-1}(\gamma) < 0} \mathfrak{g}_\gamma$.

A property of $T$-varieties in $G/P$, used freely throughout the paper is the following: each $T$-fixed point $x \in G/P$ is attractive; that is, all the weights of the tangent space $T_x(G/P)$ lie on one side of a hyperplane in $X(T)$, and in addition, each fixed point $x$ has a $T$-stable open affine neighborhood. Since $X$ is irreducible and any $x \in X^T$ is attractive, the affine open $T$-stable neighborhood of $x$ is unique. It will be denoted by $X_x$. It is well known, and not hard to see, that there is a closed $T$-equivariant embedding of $X_x$ into the tangent space $T_x(X)$ of $X$ at $x$, thanks again to the fact that $x$ is attractive.

Assuming $X_x \subset T_x(X)$, it follows that, for any $T$-stable line $L \subset T_x(X)$, we may choose a linear equivariant projection $T_x(X) \rightarrow L$ and restrict it to $X_x$. Identifying $L$ with $\mathbb{A}^k_k$ we
thus obtain a regular function $f \in k[X_x]$, which is a $T$-eigenvector of weight $-\alpha$ if $L$ has weight $\alpha$. We say $f$ corresponds to $L$, if it is obtained in the described way.
3. Some General Results on $\Theta_x(X)$

In this section, we will establish some general properties of an arbitrary $T$-variety $X$ in $G/P$. For Schubert varieties these properties are well known (see [6]). Let $T_x(X)$ be the reduced tangent cone to $X$ at any $x \in X^T$, so $\Theta_x(X) = \text{span}_k(T_x(X))$.

**Theorem 3.1.** Suppose $G$ has no $G_2$-factors. Let $L = g_{\omega} \subset \Theta_x(X)$ be a $T$-stable line with weight $\omega$. Then the following hold.

(i) If $\omega$ is long, then $L \subset TE(X,x)$. Otherwise, there exist roots $\alpha, \beta$ such that $g_{\alpha}, g_{\beta} \subset TE(X,x)$ and

$$\omega = \frac{1}{2}(\alpha + \beta).$$

(ii) In particular, if $G$ is simply laced, then $\Theta_x(X) = TE(X,x)$.

(iii) If $X$ is a Schubert variety and $L$ does not correspond to a $T$-curve, then $\alpha$ and $\beta$ are long negative orthogonal roots in a copy of $B_2 \subset \Phi$.

**Proof.** Let $z \in k[X_x]$ be a $T$-eigenfunction corresponding to $L$ and let $x_1, x_2, \ldots, x_n \in k[X_x]$ be $T$-eigenfunctions which correspond to the $T$-curves $C_1, C_2, \ldots, C_n$ through $x$. Notice that since $X_x \subset T_x(X)$ each $T$-curve $C \in E(X_x, x)$ is in fact a coordinate line in $T_x(X)$. This follows from the fact that all $T$-curves are smooth and no two $T$-weights of $T_x(X)$ are proportional. Let $\tilde{x}_i$, resp. $\tilde{z}$ denote linear projections $T_x(X) \rightarrow T_x(C_i)$, resp. $T_x(X) \rightarrow L$, which restrict to $x_i, z \in k[X_x]$.

Since the (restriction of the) projection $X_x \rightarrow \bigoplus_C T_x(C) = TE(X,x)$ has a finite fibre over 0, $k[X_x]$ is a finite $k[x_1, x_2, \ldots, x_n]$-module by the graded version of Nakayama’s Lemma. In particular $z \in k[X_x]$ is integral over $k[x_1, \ldots, x_n]$, so we obtain a relation

$$z^N = p_{N-1}z^{N-1} + p_{N-2}z^{N-2} + \cdots + p_1z + p_0,$$

where $N$ is a suitable integer and $p_i \in k[x_1, \ldots, x_n]$. Without loss of generality we may assume that every summand on the right hand side is a $T$-eigenvector with weight $N\omega$. Let $P_i \subset k[x_1, \ldots, x_n]$ be a polynomial restricting to $p_i$, having the same weight $(N-i)\omega$ as $p_i$. Then every monomial $m$ of $P_i$ has this weight too. If for all $i$ every such monomial $m$ has degree $m > N-i$, then $p_i z^{N-i}$ is an element of $M^{N+1}$, where $M$ is the maximal ideal of $x$ in $k[X_x]$. This means that $\tilde{z}$ vanishes on the tangent cone of $X_x$, so $L \notin \Theta_x(X)$, which is a contradiction. Thus, there is an $i$ and a monomial $m$ of $P_i$, such that $\deg m \leq d = N-i$. Let $m = cx_1^{d_1}x_2^{d_2} \cdots x_n^{d_n}$, with integers $d_j$ and a nonzero $c \in k$. So $\sum_j d_j \leq d$. Let $\alpha_j$ be the weight of $\tilde{x}_j$. Then we have

$$d\omega = \sum d_j \alpha_j$$

After choosing a new index, if necessary, we may assume that $d_j \neq 0$ for all $j$. Let $(\ , \ )$ be a Killing form on $X(T) \otimes \mathbb{R}$ which induces the length function on $\Phi$. We have to consider two cases. First suppose that $\omega$ is a long root, with length say $l$. Then $(\alpha_j, \omega) \leq l^2$ with equality if and only if $\alpha_j = \omega$. Thus, $d\omega^2 = \sum d_j(\alpha_j, \omega) \leq d \max_j(\alpha_j, \omega) \leq dl^2$ and so there is a $j$ with $\alpha_j = \omega$ and we are done, since this implies $\tilde{z} = \tilde{x}_j$. Hence, $L = C_j$. 
Now suppose \( \omega \) is short, with its length also denoted \( l \). In this case \( (\alpha_j, \omega) \leq l^2 \). Since \( dl^2 = d(\omega, \omega) = \sum_j d_j (\alpha_j, \omega) \) and since \( \sum d_j \leq d \), it follows that all \( \alpha_j \) satisfy \( (\alpha_j, \omega) = l^2 \). If there is a \( j \) such that \( \alpha_j = \omega \), then, as above, we are done. Otherwise for each \( j \), \( \alpha_j \) is long, and \( \alpha_j \) and \( \omega \) are contained in a copy \( B(j) \subset \Phi \) of \( B_2 \). There is a long root \( \beta_j \in B(j) \) with \( \alpha_j + \beta_j = 2\omega \). We have to show that there are \( j_0 \) and \( j_1 \) so that \( \beta_{j_0} = \alpha_{j_1} \). Fix \( j_0 = 1 \) and let \( \alpha = \alpha_1, \beta = \beta_1 \). Then \( (\alpha, \beta) = 0 \). This gives us the result: \( dl^2 = d(\omega, \beta) = 0 + \sum_{j>1} (\alpha_j, \beta) \).

Now if all \( (\alpha_j, \beta) \) are less or equal \( l^2 \), this last equation cannot hold, since \( \sum_{j>1} d_j < M \). We conclude that there is a \( j_1 \) so that \( (\alpha_{j_1}, \beta) = 2l^2 \) (the squared long root length), hence \( \alpha_{j_1} = \beta \), and we are through with (i).

The proof of (ii) is obvious. For (iii), let \( S \) be the slice (cf. [7, Lemma 4.6]) to \( X(w) \) at \( x \). Then, locally, \( X = S \times Bx \), where the weights of \( TE(S, x) \) consist of the roots \( \alpha < 0 \) such that \( x < r_\alpha x \leq w \). Since \( L \not\subset TE(X, x) \), the only possibility is that \( L \subset \Theta_x(S) \) because \( Bx \) is smooth (and so \( TE(Bx, x) = \Theta_x(Bx) \) and \( \Theta_x(X) = \Theta_x(S) \oplus \Theta_x(Bx) \)). No we may apply part (i) to \( S \). \( \square \)

The following generalizes a well known property of Schubert varieties to arbitrary \( T \)-varieties.

**Corollary 3.2.** Suppose \( L \) is a \( T \)-invariant line \( \Sigma_x(X) \). Then \( L \subset TE(X, x) \).

**Proof.** We have already shown that in equation (5), some \( P_j \) contains a monomial of degree at most \( d = i \). Taking homogeneous parts of degree \( N \) in (5), we therefore get a homogeneous polynomial

\[
f = \tilde{z}^N - \sum P_j \tilde{z}^{N_j}.
\]

vanishing on \( \Sigma_x(X) \). Hence \( f(L) = 0 \). But as \( \tilde{z}(L) \neq 0 \), this implies some \( P_j(L) \neq 0 \) as well, which means that \( \tilde{z} \) occurs in a monomial of \( P_j \), hence \( L \subset TE(X, x) \) by the construction of the \( P_j \). \( \square \)

An interesting consequence of Corollary 3.2 is that the linear spans of the tangent cones of two \( T \)-varieties behave nicely under intersections.

**Corollary 3.3.** Suppose the \( G \) is simply laced and that \( X \) and \( Y \) are \( T \)-varieties in \( G/P \). Suppose also that \( x \in (X \cap Y)^T \). Then

\[
\Theta_x(X \cap Y) = \Theta_x(X) \cap \Theta_x(Y).
\]

Consequently, if both \( X \) and \( Y \) are nonsingular at \( x \), then \( X \cap Y \) is nonsingular at \( x \) if and only if \( |E(X \cap Y, x)| = \dim(X \cap Y) \).

**Proof.** The first claim is clear since \( E(X, x) \cap E(Y, x) = E(X \cap Y, x) \). For the second, note that if \( X \) and \( Y \) are nonsingular at \( x \), then

\[
T_x(X) \cap T_x(Y) = \Theta_x(X) \cap \Theta_x(Y)
\]

= \( \Theta_x(X \cap Y) \)

\( \subset T_x(X \cap Y) \)

\( \subset T_x(X) \cap T_x(Y) \)

Hence \( \dim T_x(X \cap Y) = |E(X \cap Y)| \), and the result follows. \( \square \)
For example, it follows that in the simply laced setting, the intersection of a Schubert variety \( X(w) \) and a dual Schubert variety \( Y(v) = \overline{B-v} \) is nonsingular at any \( x \in [v, w] \) as long as \( X(w) \) and \( Y(v) \) are each nonsingular at \( x \).

4. \( \Theta_x(X) \) at a Maximal Singularity

The aim of this section is to prove Theorem 1.5. In fact, we will derive it as a consequence of a general result about the connection between \( \tau_C(X, x) \) and \( \Theta_x(X) \) for an arbitrary \( T \)-variety in \( G/P \) assuming \( x \) is at worst an isolated singularity.

**Theorem 4.1.** Suppose \( X \subset G/P \) is a \( T \)-variety, where \( G \) has no \( G_2 \)-factors. Then for each \( x \in X^T \), we have

\[
\Theta_x(X) \subset \tau(X, x) := \sum_{C \in E(X, x)} \tau_C(X, x).
\]

In particular, if \( x \) is either smooth in \( X \) or an isolated singularity, then

\[
\Theta_x(X) = \sum_{C \in E(X, x)} \tau_C(X, x).
\]

Before proving Theorem 4.1, we will give the proof of Theorem 1.5.

*Proof of Theorem 1.5.* The result is obvious if \( x \) is smooth, so assume it is a maximal singularity. Then there exists a slice representation \( X_x = S \times Bx \), where \( S \) has an isolated singularity at \( x \) and \( E(S, x) \) consists of the \( T \)-curves in \( X \) containing a smooth point of \( X_x \). To get the result, we apply Theorem 4.1 to \( S \) and use the fact that \( \Theta_x(X) = \Theta_x(S) \oplus \Theta_x(Bx) \). Indeed,

\[
\Theta_x(S) \oplus \Theta_x(Bx) = \sum_{C \in E(S, x)} \tau_C(S, x) \oplus TE(Bx, x),
\]

so it suffices to show that \( TE(Bx, x) \subset \tau_C(X, x) \) for any \( C \in E(S, x) \) since clearly \( \tau_C(S, x) \subset \tau_C(X, x) \). Let \( g_\gamma \subset TE(Bx, x) \). Then there is a curve \( D \subset Bx \) with \( g_\gamma = T_x(D) \). In fact, \( D = U_\gamma x \). Thus, the smooth \( T \)-stable surface \( \Sigma = C \times D \subset X_x = S \times Bx \), and Proposition 3.4 of [7] implies \( g_\gamma \subset \tau_C(\Sigma, x) \subset \tau_C(X, x) \).

\( \Sigma = \overline{U_\gamma C} \) is a \( T \)-surface

hard to see

\[ \Box \]

The proof of Theorem 4.1 will use several lemmas. To begin with, let \( R \) be a Noetherian graded commutative ring, with irrelevant ideal \( I = \bigoplus_{d \geq 0} R_d \). Then \( \bigcap_{l \geq 0} I^l = 0 \). Thus, for each \( r \in R \setminus \{0\} \) there is an \( l > 0 \) such that \( r \in I^l \setminus I^{l+1} \). We set \( \text{in}(r) = r + I^{l+1} \) \( \cap I^l \) \( \subset \text{gr} R = \text{gr} I R \), and \( \text{in}(0) = 0 \in \text{gr} R \). Recall that for \( r, s \in R \), \( \text{in}(r) \text{in}(s) = \text{in}(rs) \) or \( \text{in}(r) \text{in}(s) = 0 \). We say \( r \in R \) vanishes on the tangent cone if \( \text{in}(r) \) does, i.e. if \( \text{in}(r) \) is nilpotent. In the case that \( R \) is the coordinate ring of an affine variety \( Z \) with regular \( \mathbb{G}_m \)-action such that \( I \) corresponds to a maximal ideal and hence to an attractive \( \mathbb{G}_m \)-fixed point \( z \), then \( \text{in}(r) \) induces indeed a function on the reduced tangent cone of \( Z \) at \( z \), and \( r \) vanishes on the tangent cone if and only if this function does. In what follows we will
consider closed and $T$-stable subvarieties of $T_x(X)$. We therefore choose a one parameter subgroup $\lambda$ of $T$, such that $\lim_{t \to 0} \lambda(t)v = 0$ for all $v \in T_x(X)$. Then the $G_m$-action by $\lambda^{-1}$ induces a (positive) grading of $k[T_x(X)]$ which carries over to any $T$-stable closed subvariety (note that the grading induced by $\lambda$ would be negative).

For convenience we extend the definition of $\Theta_x(Z)$ also to reducible varieties. Also notice that $\Theta_x(Z)$ may be canonically identified with $T_0(\mathfrak{g}_x(Z)) \subset T_x(Z)$. To set up an induction on the dimension of $X$, we need the following

**Lemma 4.2.** Let $Z \subset T_x(X)$ be a closed $T$-stable subvariety with $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_d$ the decomposition into irreducible components. Then

$$\Theta_0(Z) = \Theta_0(Z_1) + \Theta_0(Z_2) + \cdots + \Theta_0(Z_d).$$

**Proof.** Since every component $Z_i$ of $Z$ is $T$-stable it has to contain 0. Therefore the proof is a simple consequence of the following well known fact: if a variety $Y = A \cup B$ is the union of two closed subvarieties then for every point $x$ in the intersection $A \cap B$ we have $\mathfrak{g}_x(Y) = \mathfrak{g}_x(A) \cup \mathfrak{g}_x(B)$.

Let $Z \subset T_x(X)$ be an irreducible $T$-stable subvariety, and let $L \subset \Theta_0(Z)$ be a $T$-stable line with weight $\omega$. say. Moreover, suppose $\omega$ is short with respect to a Killing form $(\ , \ )$ on $X(T)$. Denote by $z \in k[Z]$ the restriction of a linear $T$-equivariant projection $T_x(X) \to L \cong \mathbb{A}_1^1$. We fix $z$ for the moment.

**Lemma 4.3.** With the preceding notation, let $f \in k[Z]$ correspond to another $T$-equivariant linear projection onto some line $L' \subset T_x(X)$. Then $z$ vanishes on the tangent cone of $V(f)$ if and only if $\text{in}(z)^l = \text{in}(h)\text{in}(f)$ for some positive integer $l$ and a suitable $T$-eigenvector $h \in k[Z]$.

**Proof.** The if is clear, so suppose $z$ vanishes on the tangent cone of $V(f)$. By definition this means that there is an integer $l$ and there are elements $g_1, g_2, \ldots, g_r \in I(V(f))$, the ideal of $V(f)$, such that $\text{in}(z)^l = a_1 \text{in}(g_1) + a_2 \text{in}(g_2) + \cdots + a_r \text{in}(g_r)$ for suitable $a_i \in \text{gr} k[Z]$. Since $\text{in}(z)$ is homogeneous and since $\text{in}(I(V(f)))$ is an homogeneous ideal, we may assume that all of the $a_i$ are homogeneous as well. Moreover the $a_i$ and $g_i$ may be chosen to be $T$-eigenvectors. Omitting any indices $i$ for which $a_i \text{in}(g_i) = 0$ we may lift the $a_i$ equivariantly to $\bar{a}_i \in k[Z]$, such that $\text{in}(\bar{a}_i) = a_i$. Then we have $0 \neq \text{in}(\bar{a}_i) \text{in}(g_i) = \text{in}(a_i g_i)$. Leaving out degrees different from $l$ we may assume that $\sum \text{in}(\bar{a}_i) \text{in}(g_i) = \text{in}(\sum \bar{a}_i g_i)$. Now $\sum \bar{a}_i g_i$ is a $T$-eigenvector $g$ contained in the ideal of $V(f)$. A suitable $n$th power of $g$ is contained in $f k[Z]$, $\text{in}(z)$ is not nilpotent, and due to $\text{in}(z)^l = \text{in}(g)$ also $\text{in}(g)$ is not nilpotent, therefore $\text{in}(g)^n = \text{in}(g^n)$. Replacing $l$ by $nl$ we may assume that $\text{in}(z)^l = \text{in}(g)$ for a $g \in f k[Z]$. In other words $\text{in}(z)^l = \text{in}(h f)$ for a suitable $T$-eigenvector $h \in k[Z]$.

It remains to show that $\text{in}(h f) = \text{in}(h) \text{in}(f)$ which is equivalent to $\text{in}(h) \text{in}(f) \neq 0$. So suppose that $\text{in}(h) \text{in}(f) = 0$. This means that $h \in M^{l-1}$ with $M$ the maximal ideal of 0. Otherwise $\text{in}(h) \text{in}(f)$ would equal $\text{in}(h f)$ by definition. We conclude that $h \in M^n$ for some $n < l - 1$, implying that there is a a homogeneous polynomial $P$ in some linear $T$-homogeneous coordinates $x_1, x_2, \ldots, x_m$ of $T_x(X)$ of the same $T$-weight as $h$, and of degree $n$, such that, restricted to $Z$, $h = P$ modulo $M^{n+1}$. By the definition of $f$ we may even assume that $x_1$ restricted to $Z$ is $f$. Replacing $P$ by any monomial of $P$ and letting $d_i$ be
the degree of $x_i$ in $P$, we see that $l\omega = \alpha_1 + \sum d_i\alpha_i$ with $\alpha_i$ the weight of $x_i$. Applying $(\cdot, \omega)$ on both sides this gives $l(\omega, \omega) = (\alpha_1, \omega) + \sum d_i(\alpha_i, \omega)$. Since $(\alpha_1, \omega) \leq (\omega, \omega)$ for all $i$ this is impossible since $n = \sum d_i < l - 1$. Hence the claim. \hfill \Box

As an easy consequence we get

**Lemma 4.4.** If $Z$ and $z$ are as above, and $f$ corresponds to the projection to any other $T$-stable line of $T_x(X)$ with a short weight, then $z$ does not vanish on the tangent cone of $\mathcal{V}(f)$.

*Proof.* By the last Lemma we know, that if $z$ vanishes on the tangent cone of $\mathcal{V}(f)$, there is a $T$-eigenvector $h \in k[Z]$ such that $\text{in}(z) = \text{in}(h)^{l}\text{in}(f)$. Choosing a monomial as in the proof of the previous Lemma, we get a relation $l\omega = \alpha_1 + \sum d_i\alpha_i$ with $\sum d_i = l - 1$. But $(\alpha_1, \omega) < (\omega, \omega)$, because $\alpha_1$ is short, and $(\alpha_i, \omega) \leq (\omega, \omega)$ for all $i$, so no such relation exists. \hfill \Box

For reasons which will become clear in the proof of the Theorem, we now restrict our attention to varieties $Z$ in $T_x(X)$ such that $T_0(Z)$ contains exactly one $T$-stable line with a short weight.

**Lemma 4.5.** If $L \subset \Theta_0(Z)$ is the only line in $T_0(Z)$, and if $C \in E(Z, 0)$ is any $T$-curve, then $L \subset T_p(Z)$ for all $p \in C^\circ = C \setminus \{0\}$.

*Proof.* Choose any equivariant embedding $Z \subset T_0(Z)$. Then, if $C = L$ as a subset of $T_0(Z)$, there is nothing to show. Otherwise $C$ is a coordinate line of $T_0(Z)$ having a long $T$-weight $\alpha$, say. If $L \not\subset T_p(Z)$ for a $p \in C^\circ$ there is a $T$-eigenfunction $f$ in the ideal of $Z$ in $k[T_0(Z)]$, such that $df_p(L) \neq 0$. We may assume that $k[T_0(Z)] = k[z, x_1, x_2, \ldots, x_n]$ with $z$ as above corresponding to $L$, and the $x_i$ corresponding to the long lines of $T_0(Z)$. Then we write $f = P_0 + P_1 z + P_2 z^2 + \cdots + P_d z^d$ with the $P_i$ $T$-eigenvectors and polynomials in the $x_i$ only. Without loss of generality $P_1 z^l$ has the same weight as $f$. It follows that $df_p = dP_0, p + P_1(p)dz_p$ because $z$ vanishes on $C$. By assumption $P_1(p)$ is nonzero, implying that there is a monomial of the form $x^l$ contained in $P_1$, where $x$ is the coordinate corresponding to $C$ and $l \geq 1$. Thus, the $T$-weight of $f$ is $l\alpha + \omega$. On the other hand $P_0$ is nonzero. For if $P_0 = 0$, then $f$ is divisible by $z$, and therefore $f = h z$ for some $h$. But $Z$ is irreducible and clearly $z$ does not vanish on $Z$, so $h$ vanishes on $Z$. Now $z$ and $h$ vanish in $p$ forcing $df_p$ to be zero as well, a contradiction. With $P_0$ being nonzero it follows that there is a monomial in the $x_i$ of weight $l\alpha + \omega$. This clearly shows that $\omega = (l\alpha + \omega) - l\alpha$ is contained in the $Z$-submodule of $X(T)$ generated by all long weights of $T_0(Z)$. The next lemma shows that this is impossible and therefore ends the proof. \hfill \Box

**Lemma 4.6.** Let $\Gamma$ be a $Z$-submodule of $X(T)$ generated by long roots. If the Killing form $F$ is normalized such that $(\omega, \omega) = 1$ is the short root length, then the function $f : \Gamma \to \mathbb{Q}$ given by $f(\gamma) = (\gamma, \gamma)$ has actually values in $2\mathbb{Z}$.

*Proof.* If $\alpha$, $\beta$ are long roots, then $(\alpha, \beta) \in \mathbb{Z}$. Indeed, $(\alpha, \beta) \in \{0, \pm 1, \pm 2\}$ by general properties of root systems. Hence, $(\gamma, \delta) \in \mathbb{Z}$ for all $\gamma, \delta \in \Gamma$, as well. Now $f(\gamma + \delta) = f(\gamma) + f(\delta) + 2(\alpha, \delta) \in 2\mathbb{Z}$, if $f(\gamma)$ and $f(\delta)$ are even integers. The result follows by induction.
on the length of a shortest representation \( \gamma = \sum n_i \alpha_i \) with \( n_i \in \mathbb{Z} \) and \( \alpha_1, \alpha_2, \ldots \) the long generators of \( \Gamma \). The length of such a representation is just \( \sum |n_i| \). So, if \( n_1 \) is nonzero and positive, then \( \gamma = \alpha_1 + (n_1 - 1)\alpha_1 + \sum_{i>2} n_i \alpha_i \). The induction hypothesis for \( \alpha_1 \) and \((n_1 - 1)\alpha_1 + \sum_{i>2} n_i \alpha_i \) give the result for \( \gamma \) by the above arguments. If \( n_1 \) is negative we may use \(-\gamma\), since \( f(\gamma) = f(-\gamma) \). Finally, if \( n_1 \) is zero, we may replace \( \alpha_1 \) with any other \( \alpha_i \) such that \( n_i \neq 0 \). \( \square \)

We are now in a position to prove the Theorem.

**Proof of Theorem 4.1.** We proceed by induction on \( \dim Z \) for an irreducible \( T \)-stable subvariety \( Z \subset X \subset T_x(X) \). Of course there is nothing to show when \( \dim Z \leq 1 \). If \( \dim Z > 1 \), let \( L \subset \Theta_0(Z) \) be any \( T \)-stable line that has a short weight \( \omega \), say. Let \( z \) be a corresponding function of \( k[Z] \). Suppose there is another line with short weight in \( T_0(Z) \). By the previous lemma, if \( f \) is a corresponding function \( z \) does not vanish on the tangent cone of \( V(f) \). Thanks to Lemma 4.2, \( z \) does not vanish on the tangent cone of at least one irreducible component \( Z' \) of \( V(f) \). In particular this implies that \( L \) is contained in \( \Theta_0(Z') \).

By induction \( L \subset \tau(Z',0) \subset \tau(Z,0) \). This concludes the case that there is a short line in \( T_0(Z) \) different from \( L \). So suppose \( L \) is the only line in \( T_0(Z) \) with a short weight. Then \( L \subset T_{0}(Z) \) for all \( p \in C^\alpha \) and any curve \( C \in E(Z,0) \). For each such \( C \) it then follows that \( L \subset \tau_C(Z,0) \). By Theorem 3.1 all the lines in \( \Theta_0(Z) \) with long \( T \)-weights are tangent to \( T \)-curves, so they are contained in \( \tau(Z,0) \). \( \square \)

### 5. Proof of Theorems 1.8 and 1.9

The goal of this section is to study the \( T \)-weights in \( \tau_C(X,x) \) for a Schubert variety in \( G/B \) and to eventually prove Theorems 1.8 and 1.9. As usual, we will suppose throughout that \( G \) does not contain any \( G_2 \)-factors. Let \( X = X(w) \) and assume \( C \) is a good \( T \)-curve in \( X \) such that \( C^T = \{x,y\} \), where \( y > x \). Thus we can write \( C = \overline{U_{\mu}x} \), where \( \mu > 0 \), and it follows that \( y = r_\mu x > x \). Since \( \tau_C(X,x) \subset TE(X,x) \) if \( \mu \) is short, we can ignore this case and suppose \( \mu \) is long. Recall also that if \( g_\gamma \subset \Theta_x(X) \) and \( \gamma \) is long, then \( g_\gamma \subset TE(X,x) \).

To begin, we need a result similar to Theorem 3.1 for \( \tau_C(X,x) \).

**Lemma 5.1.** Suppose \( \gamma \) is a short root such that \( g_\gamma \subset \tau_C(X,x) \). If \( g_\gamma \not\subset TE(X,x) \), then there exists a long root \( \phi \) orthogonal to \( \mu \) such that \( g_{-\phi} \subset TE(X,x) \), and

\[
\gamma = \frac{1}{2}(\mu + \phi).
\]

In addition, the roots \( \gamma, \mu, \phi \) lie in a copy of \( B_2 \) contained in \( \Phi \). When \( g_\gamma \subset TE(X,x) \), there exists a \( T \)-surface in \( S \subset X \) which is smooth at \( x \) containing \( C \) and the \( T \)-curve corresponding to \( \gamma \).

**Proof.** This follows from Lemma 5.1 and Proposition 5.2 of [7]. \( \square \)

We will see below that if \( g_\gamma \not\subset TE(X,x) \), then \( \phi > 0 \). The notion of an orthogonal \( B_2 \)-pair arises from the following illuminating example worked out in detail in [7, Example 8.4].
Example 5.2 Let $G$ be of type $B_2$, and let $w = r_3 r_2 r_1$, where $\alpha$ is the short simple root and $\beta$ is the long simple root. Put $X = X(w)$. The singular set of $X$ is $X(r_1)$, so $x = r_1$ is $X$’s unique maximal singular point. There are two good $T$-curves at $x$, namely $C = U_{-\beta}x$ and $D = U_{-2\alpha}x$. Suppose $y = r_\beta x$ and $z = r_{2\alpha+\beta} x$. Then

$$T_y(X) = g_\alpha \oplus g_{\alpha + \beta} \oplus g_\beta \quad \text{and} \quad T_z(X) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_{2\alpha + \beta}.$$  

Thus, by the algorithm of [7, §3],

$$\tau_C(X,x) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_{-\beta} \quad \text{and} \quad \tau_D(X,x) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_{-(2\alpha + \beta)}.$$ 

Note that the weight at $x$ that does not give a $T$-curve, namely $-(\alpha + \beta)$, is in both Peterson translates. The next result extends this example to the general case.

Remark 5.3 We will use the algorithm in [7, §3] in several places to compute a Peterson translate $\tau_C(X,x)$. Let us briefly summarize how this works. Suppose $C = U_{-\mu} x$, where $\mu > 0$ and $y = r_\mu x$. Consider the weights of the form $\nu + k\mu$ in $T_y(X)$, and form a (possibly partial) $\mu$ string consisting of roots of the form $\kappa - j\mu$, where $0 \leq j \leq r$, such that $y^{-1}(\kappa - j\mu) < 0$ for each $j$, but $y^{-1}(\kappa - (r + 1) \mu) > 0$. Then the roots $r_\mu(\kappa - j\mu)$ occur as weights in $\tau_C(X,x)$, and every weight occuring in $\tau_C(X,x)$ arises in this way.

Recall that $(\ , \ )$ is a $W$-invariant inner product on $X(T) \otimes \mathbb{R}$. Assuming $\gamma$ is as in the last Lemma, we now say more about $g_\gamma$.

Theorem 5.4. Suppose $\gamma$ is a short root such that $g_\gamma \subset \tau_C(X,x)$. If either $(\gamma, \mu) \geq 0$, or in the equation (6) one has $\phi < 0$, then $g_\gamma \subset TE(X,x)$. On the other hand, if $g_\gamma \not\subset TE(X,x)$, then the following statements hold:

(a) $\gamma < 0$,

(b) $(\gamma, \mu) < 0$, hence $\delta := \gamma + \mu \in \Phi$,

(c) if $x^{-1}(\delta) < 0$, then $g_\delta \subset \tau_C(X,x) \cap TE(X,x)$ (and, of course, conversely), and

(d) $\phi > 0$.

Remark 5.5 Example 5.2 shows that one can have $g_\gamma \subset \tau_C(X,x) \cap TE(X,x)$ yet still have $(\gamma, \mu) < 0$.

Proof. If $(\gamma, \mu) \geq 0$, it follows immediately from Lemma 5.1 that $g_\gamma \subset TE(X,x)$. Suppose $\gamma$ and has the form (6), where $\phi < 0$, and put $\delta = \gamma + \mu$. Since $(\gamma, \mu) < 0$, $\delta \in \Phi$. Moreover, since $\phi < 0$, we have $\delta > 0$. Now if $\gamma > 0$, then $r_\gamma x < x$, since $x^{-1}(\gamma) < 0$. Thus $g_\gamma \subset TE(X,x)$ if $\gamma > 0$.

Next, suppose $\gamma < 0$. We will consider the two cases $x^{-1}(\delta) < 0$ and $x^{-1}(\delta) > 0$ separately. Assume first that $x^{-1}(\delta) < 0$. Since $\tau_C(X,x)$ is a $g_\mu$-submodule of $T_x(X)$ (cf. [7, §3]) and $g_\gamma \subset \tau_C(X,x)$, we therefore know that

$$g_\delta \oplus g_\gamma \subset \tau_C(X,x).$$

Since $\mu$ is long and there are no $G_2$-factors, Proposition 8.1 [7] implies

$$g_\delta \oplus g_\gamma \subset T_y(X).$$
Since \( \gamma < 0 \), we therefore get the inequality \( y < r_\gamma y \leq w \), and hence \( X \) is also nonsingular at \( r_\gamma y \). Moreover, since \( \phi < 0 \) and \( x^{-1}(\phi) = y^{-1}(\phi) > 0 \), it also follows that \( g_{-\phi} \subset T_E(X,y) \), which equals \( T_y(X) \) since \( X \) is smooth at \( y \). Since there are no \( G_2 \) factors, \( \mu, \delta, -\phi \) constitute a complete \( \gamma \)-string occurring as \( T \)-weights of \( T_y(X) \). Letting \( E \) be the good \( T \)-curve in \( X \) such that \( E^T = \{ y, r_\gamma y \} \), we have \( \tau_E(X,y) = T_y(X) \), so the string \( \mu, \delta, -\phi \) also has to occur as \( T \)-weights of \( T_{r_\gamma y}(X) \). In particular, \( g_{-\phi} \subset T(E,X,r_\gamma y) = T_{r_\gamma y}(X) \), and hence \( r_\phi r_\gamma y \leq w \). But this means

\[
r_\gamma x = r_\gamma r_\mu y = r_\gamma r_\mu r_\gamma r_\gamma y = r_\phi r_\gamma y \leq w,
\]

so \( g_\gamma \subset T(E,X,x) \).

Next, assume \( x^{-1}(\delta) > 0 \). Since \( \mu \) is long, \( r_\mu(\delta) = \delta - \mu = \gamma \), hence \( y^{-1}(\delta) = x^{-1}(\gamma) < 0 \). Thus, since \( \delta > 0 \), \( g_\delta \subset T_y(X) \). Furthermore,

\[
y^{-1}(-\gamma) = -x^{-1}r_\mu(\gamma) = -x^{-1}(\delta) < 0,
\]

so \( g_{-\gamma} \subset T_y(X) \). It follows that \( r_\gamma y < y \). As \( -\phi > 0 \), \( U_{-\phi}r_\gamma y \subset X \) as well. We claim \( U_{-\phi}r_\gamma y \neq r_\gamma y \), which then proves that \( r_\phi r_\gamma y \leq w \). But

\[
(r_\gamma y)^{-1}(-\phi) = y^{-1}(r_\gamma(-\phi)) = y^{-1}(\mu) < 0,
\]

hence we get the claim. Finally, we note that \( r_\phi r_\gamma r_\mu = r_\gamma \), so it follows that \( r_\gamma x \leq w \). Therefore, if \( \phi < 0 \), we get \( g_\gamma \subset T(E,X,x) \).

Now suppose \( g_\gamma \not\subset T(E,X,x) \). Then (a) is immediate and (b) follows from (6). Since \( \tau_C(X,x) \) is a \( g_{-\gamma} \)-submodule of \( T_x(X) \), \( g_\delta \subset \tau_C(X,x) \) since \( x^{-1}(\delta) < 0 \). Then \( \gamma \) is given by (6), so \( (\delta, \mu) \geq 0 \) (since \( \mu \) is long). Thus, Lemma 5.1 implies \( g_\delta \subset T(E,X,x) \). On the other hand, if \( x^{-1}(\delta) > 0 \), then \( g_\delta \not\subset T_x(X) \). This establishes (c). The assumption that \( g_\gamma \not\subset T(E,X,x) \) immediately implies that \( \phi \) is positive giving (d).

\[ \square \]

**Remark 5.6** Let \( X \) be a Schubert variety, and suppose \( x \in X^T \) is a maximal singularity where \( |E(X,x)| = \dim X \). In this case, the second author has shown that the multiplicity \( \tau_x(X) \) of \( X \) at \( x \) is exactly \( 2^d \), where

\[
d = \left| \{ \alpha \in x(\Phi^-) \mid g_\alpha \subset \tau_C(X,x) \text{ and } r_\alpha x \neq w \} \right|
\]

for any good \( C \in E(X,x) \) ([11]).

**Theorem 5.7.** Suppose \( C = U_{-\mu}x \) is a good \( T \)-curve, where \( \mu > 0 \), and let \( y = r_\mu x \). Assume \( g_\gamma \subset \tau_C(X,x) \) but \( g_\gamma \not\subset T_x(X) \). Then there exists a positive root \( \phi \) such that \( \{ \mu, \phi \} \) is an orthogonal \( B_2 \)-pair for \( X \) at \( x \) such that \( \gamma = -1/2(\mu + \phi) \). Conversely, suppose that for some \( \phi > 0 \), \( \{ \mu, \phi \} \) is an orthogonal \( B_2 \)-pair for \( X \) at \( x \), and \( \gamma = -1/2(\mu + \phi) \). Then \( g_\gamma \subset \tau_C(X,x) \).

**Proof.** Suppose \( g_\gamma \subset \tau_C(X,x) \) but \( g_\gamma \not\subset T_x(X) \). By Lemma 5.1 and Theorem 5.4, there exists a long positive root \( \phi \) orthogonal to \( \mu \) such that \( \gamma = -1/2(\mu + \phi) \). Put \( y = r_\mu x \), and note \( X \) is smooth at \( y \). To show that \( \{ \mu, \phi \} \) is an orthogonal \( B_2 \)-pair, we have to consider two cases.
by assumption, \( g \) assertion of Theorem 5.7, \( \text{Proof of Theorem 1.8.} \) Suppose \( x \leq w \). But \( g \not\in T_x(X) \) implies \( x^{-1}(\alpha) < 0 \), since if \( x^{-1}(\alpha) > 0 \), then the fact that \( \gamma = -\phi + \alpha \) would say \( g \subset T_x(X) \). Hence \( r_\alpha x < x \).

Since \( r_\mu (\alpha) = -\gamma \), it follows that \( y^{-1}(\alpha) = x^{-1}(\gamma) > 0 \), so \( g_\gamma \not\subset T_y(G/B) \). But \( y^{-1}(\gamma) = x^{-1}(-\alpha) > 0 \), hence \( g_\gamma \not\subset T_y(G/B) \). Hence, by the algorithm for computing the Peterson translate in \( \Theta \) and the fact that \( g_\gamma \subset \tau_C(X,x) \), we infer that \( \mu_\gamma \subset T_y(G/B) \). Therefore, \( r_\alpha y = r_\alpha r_\mu x \leq w \), as was to be shown.

**Case 2.** \( \phi \) is simple. Here \( \alpha = \gamma + \mu \) is the short simple root, and \( r_\mu (\gamma) = \alpha \). As in Case 1, \( x^{-1}(\alpha) < 0 \), so \( r_\alpha x < x \). Now \( y^{-1}(\alpha) = x^{-1}(\gamma) > 0 \), so \( r_\alpha y < y \) and hence \( g_\alpha \not\subset T_y(G/B) \).

Also, \( y^{-1}(\gamma) = x^{-1}(\alpha) < 0 \), so \( g_\gamma \subset T_y(G/B) \). Thus the algorithm for \( \tau_C(X,x) \) says that \( g_\alpha \subset \tau_C(X,x) \).

But as \( g_\gamma \subset \tau_C(X,x) \), too, we have to conclude that \( g_\gamma \subset T_y(G/B) \). But then \( \gamma \) and \( \alpha \) comprise a \( \mu \) string. Hence \( r_\alpha y \leq w \). But since we are in a \( B_2 \) where \( \alpha \) and \( \phi \) are the simple roots, \( r_\gamma r_\mu = r_\alpha r_\phi \) Hence \( r_\alpha r_\phi x \leq w \), so Case 2 is finished.

To prove the converse, we need to consider Cases 1 and 2 again with the assumption that \( x^{-1}(\alpha) < 0 \), which follows from the condition that \( r_\alpha x < x \). The argument is, in fact, very similar to the above, but we will outline it anyway. Assume first that \( \mu = \beta \), i.e. \( \mu \) is simple. As \( r_\alpha r_\mu x \leq w \), we see that \( r_\alpha y \leq w \). But \( y^{-1}(-\alpha) = x^{-1}(\gamma) < 0 \), consequently \( g_\alpha \subset T_y(G/B) \).

Also, \( y^{-1}(\gamma) = x^{-1}(-\alpha) > 0 \), so \( g_\gamma \not\subset T_y(G/B) \). But then \( \gamma \) and \( \alpha \) make up a \( \beta + 2\alpha \)-string in \( B_2 \), \( g_\alpha + g_\gamma \subset \tau_C(X,x) \) also. This finishes the proof.

We now prove Theorems 1.8 and 1.9.

**Proof of Theorem 1.8.** Suppose \( g_\gamma \subset \Theta_x(X) \). Since \( x \) is either smooth or a maximal singularity, Theorem 1.5 \( g_\gamma \subset \tau_C(X,x) \) for some good \( C \). If \( C \) is short, then \( \tau_C(X,x) \subset T_E(X,x) \), by Theorem 1.1, hence \( \tau_C(X,x) \subset T_x(X) \). Thus we can suppose \( C \) is long. But then, by Theorem 5.7, either \( g_\gamma \subset T_x(X) \) or there exists a \( B_2 \)-pair \( \mu, \phi \) for \( X \) at \( x \) such that \( \gamma = -1/2(\mu + \phi) \). Hence Theorem 1.8 is proven.

**Proof of Theorem 1.9.** Suppose \( C \in E(X,x) \) is good and \( \dim T_E(X,x) = \dim T_x(X) = \dim X \). If \( C \) is short, then \( X \) is smooth at \( x \) by Theorem 1.1. Hence we may suppose \( C \) is long. Suppose there exists a \( T \)-line \( g_\gamma \) in \( \tau_C(X,x) \) which is not in \( T_x(X) \). Then by Theorem 5.7, there is an orthogonal \( B_2 \)-pair \( \mu, \phi \) for \( X \) at \( x \) for which \( \gamma = -1/2(\mu + \phi) \). But then by assumption, \( g_\gamma \subset T_E(X,x) \). This contradicts the choice of \( g_\gamma \), so \( \tau_C(X,x) \subset T_x(X) = T_E(X,x) \). Hence, by Theorem 1.1 again, \( X \) is smooth at \( x \).

For the converse, suppose \( X \) is smooth at \( x \). Then conditions (1) and (2) of Theorem 1.8 clearly hold. Suppose \( \mu, \phi \) is a \( B_2 \)-pair for \( X \) at \( x \) and \( \gamma = -1/2(\mu + \phi) \). By the converse assertion of Theorem 5.7, \( g_\gamma \subset \tau_C(X,x) \), where \( C \in E(X,x) \) is the \( T \)-curve of weight \( \mu \) at \( x \). Since \( x \) is smooth, \( \tau_C(X,x) = T_E(X,x) \), so \( g_\gamma \subset T_E(X,x) \).
References


James B. Carrell
Department of Mathematics
University of British Columbia
Vancouver, Canada V6T 1Z2
carrell@math.ubc.ca

Jochen Kuttler
Mathematisches Institut
Universität Basel
CH-4051 Basel
Switzerland
kuttler@math.unibas.ch