# Smooth Points of $T$-stable Varieties in $G/B$ and the Peterson Map

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## Abstract

Let $G$ be a connected semi-simple algebraic group defined over an algebraically closed field $k$, and let $T \subset B \subset P$ be respectively a maximal torus, a Borel subgroup and a parabolic subgroup of $G$. Inspired by a beautiful result of Dale Peterson describing the singular locus of a Schubert variety in $G/B$, we characterize the $T$-fixed points in the singular locus of an arbitrary irreducible $T$-stable subvariety of $G/P$ (a $T$-variety for short). Peterson's result (cf. The Deformation Theorem, §1) says that if $k = \mathbb{C}$, then a Schubert variety $X \subset G/B$ is smooth at a $T$-fixed point $x$ if and only if it is smooth at every $T$-fixed point $y > x$ (in the Bruhat-Chevalley order on the fixed point set $X^T$) and all the limits $\tau_C(X,x) = \lim_{z\to x} T_z(X)$ $(z \in C \setminus C^T)$ of the Zariski tangent spaces $T_z(X)$ of $X$ coincide as $C$ varies over the set of all $T$-stable curves in $X$ with $C^T = \{x, y\}$, where $y > x$. Using this, Peterson showed that if $G$ is simply laced (and defined over $\mathbb{C}$), then every rationally smooth point of a Schubert variety in $G/B$ is smooth. More generally, the deformation $\tau_C(X,x)$ is defined for any $k$-variety $X$ with a $T$-action provided $C$ is what we call good, i.e. $C$ is a curve of the form $C = \overline{Tz}$, where $z$ is a smooth point of $X \setminus X^T$ and $x \in C^T$. Our first main result (Theorem 1.4) says that if $x \in X$ is an attractive fixed point, then $X$ is smooth at $x$ if and only if there exist at least two good $C$ containing $x$ such that $\tau_C(X,x) = TE(X,x)$, where $TE(X,x)$ denotes the span of the tangent lines of the $T$-stable curves in $X$ containing $x$. In addition, if $X$ is Cohen-Macaulay at $x$ and $\tau_C(X,x) = TE(X,x)$ for even one good $C$, then $X$ is smooth at $x$. Our second main result (Theorem 1.6) says that if $X$ is a $T$-variety in $G/P$, where $G$ is simply laced, then $\tau_C(X,x) \subset TE(X,x)$ for each good $C$. This is not true for general $G$, but when $G$ has no $G_2$ factors, then $\tau_C(X,x)$ is always contained in the linear span of the reduced tangent cone to $X$ at $x$. These results lead to several descriptions of the smooth fixed points of a $T$-variety in $G/P$ and, in particular, they give simple proofs of Peterson's results valid for any algebraically closed field. We also show (cf. Example 7.1) that there can exist $T$-stable subvarieties in $G/B$, where $G$ is simply laced, which have rationally smooth $T$-fixed points in their singular loci.

The first author was partially supported by the Natural Sciences and Engineering Research Council of Canada

The second author was partially supported by the SNF (Schweizerischer Nationalfonds)

## 1. Introduction

Throughout this paper, $k$ will denote an arbitrary algebraically closed field. Let $G$ be a connected semi-simple algebraic group over $k$, and let $T \subset B \subseteq P$ denote respectively a maximal torus, a Borel subgroup and a parabolic subgroup of $G$. The algebraic homogeneous spaces $G/P$ form a class of projective $G$-varieties which frequently arise in algebraic geometry and representation theory. The purpose of this paper is to consider the problem of describing the singular locus of an arbitrary *$T$-variety* in $G/P$, that is an irreducible Zariski closed subset of $G/P$ which is stable under the maximal torus $T$. There are many fundamental examples of such varieties, probably the most important being Schubert varieties, which are the closures of $B$-orbits on $G/P$. As further examples, one has the orbit closures of arbitrary closed connected subgroups of $G$ containing $T$ and the components of their intersections.

The main ideas here are motivated by an unpublished work of Dale Peterson giving an explicit description, when $k = \mathbb{C}$, of the smooth (i.e. nonsingular) $T$-fixed points of a Schubert variety in the flag variety $G/B$ (cf. Theorem 1.3 below). Using this, he also obtained the beautiful fact that when $G$ is simply laced, every rationally smooth point of a Schubert variety in $G/B$ is smooth (cf. the *ADE* Theorem below). Peterson's proofs are based on an elegant result about the fibre of the Nash blow up of a Schubert variety $X$ over a fixed point of $T$. In essence, he showed that all the crucial information is obtained by considering the limits of the Zariski tangent spaces of $X$ along the $T$-stable curves in $X$ containing smooth points. These limits, which are defined carefully below, will be called *Peterson translates* of $X$.

Our results about $T$-varieties are natural extensions of Peterson's in the sense that we will also describe the fixed points in the smooth locus of $X$, but our hypotheses turn out to be much weaker and our arguments work for arbitrary $k$ without assuming $X$ is normal. In fact, we will obtain simple, natural proofs of Peterson's results. Note that since $G/P$ is projective, every $T$-stable closed subset $X$ has fixed points. Knowing the singular fixed points is often sufficient for determining the full singular locus $\mathrm{Sing}(X)$ of $X$. This is the case when $X$ is a Schubert variety, since every $B$-orbit in $G/B$ contains a fixed point. Moreover, a $T$-variety $X$ is globally smooth (i.e. nonsingular) if and only if $\mathrm{Sing}(X) \cap X^T = \emptyset$.

Before stating the main results, namely Theorems 1.4 and 1.6, we will give some background on the singular locus question for Schubert varieties in $G/B$ and state Peterson's results. After that, we will explain our main results. To get started, we need to fix some more notation. Recall that there is a standard identification $w = wB/B$ between the Weyl group $W = N_G(T)/T$ and the fixed point set $(G/B)^T$. Here $wB/B$ denotes $\dot{w}B/B$, where $\dot{w} \in N_G(T)$ is a representative of $w$. Every Schubert variety in $G/B$ has the form $X(w) := \overline{Bw}$ for a unique $w \in W$. The Bruhat-Chevalley order on $W$ is the partial order defined by putting $x < y$ if and only if $x \in X(y)$ but $x \neq y$. Let us put $[x, w] = \{y \in W \mid x \leq y \leq w\}$ and $(x, w] = [x, w] \setminus \{x\}$. Clearly, $[e, w] = (X(w))^T$, where $e$ is the identity element of $W$. Schubert varieties $X$ in an arbitrary $G/P$ admit a similar description: each such is the union of the $B$-orbits of its $T$-fixed points, i.e. $X = BX^T$. Consequently, $X^T$ also admits a relative Bruhat-Chevalley order with properties analogous to those of $W$. If $X$ is a $T$-variety in $G/P$ and $x \in X^T$, we will frequently need to consider the $T$-curves (i.e. irreducible $T$-stable curves) in $X$ containing $x$. This set is denoted by $E(X, x)$. The *Bruhat graph* (or sometimes the moment graph) $\Gamma(X)$ of $X$ is the graph whose vertex set is $X^T$ and whose edge set at a vertex $x$ is $E(X, x)$.

We can now state a first version of Peterson's $ADE$ Theorem. The following form is somewhat more general than what Peterson proved since $k$ is arbitrary and $X$ is a Schubert variety in any $G/P$ (not just $G/B$).

**The** $ADE$**-Theorem** (First Version)**.** *Assume $G$ is simply laced. Then a Schubert variety $X \subset G/P$ is smooth at $x \in X^T$ if and only if the Bruhat graph $\Gamma(X)$ is regular at every vertex $y \geq x$. In other words, $x$ is a smooth point of $X$ if and only $|E(X, y)| = \dim X$ for every $y \in X^T$ such that $y \geq x$.*

We will prove this immediately after stating Theorem 1.6 below. When $X \subset G/B$, this result can be formulated in terms of a weaker notion of smoothness. Recall that a $k$-variety $Y$ is rationally smooth at a point $x$ if and only if there exists an étale neighborhood $U$ of $x$ such that for all $y \in U$, the local étale cohomology of $Y$ at $y$ with coefficients in $\mathbb{Q}_\ell$, $\ell$ a prime different from char$(k)$, is the same as that of a smooth variety (cf. [23]). Note that if $k = \mathbb{C}$, étale cohomology may be replaced by ordinary cohomology with coefficients in $\mathbb{Q}$ (cf. [8]). A celebrated result of Kazhdan and Lusztig [23] says that when char$(k) > 0$, the rationally smooth points of a Schubert variety $X(w) \subset G/B$ have the form $Bx$, where $x \leq w$ is such that the Kazhdan-Lusztig polynomials $P_{y,w} = 1$ for all $y \in [x, w]$. This is in fact equivalent to rational smoothness at $x$ in all characteristics [37]. The connection with the first version comes from a result of Peterson and the first author, namely $P_{y,w} = 1$ for all $y \in [x, w]$ if and only if the Bruhat graph $\Gamma(X(w))$ is regular at all vertices $y \in [x, w]$ [10, THEOREM E]. Given this, the first version of the $ADE$-Theorem can be restated as follows.

**The** $ADE$**-Theorem** (Second Version)**.** *Let $G$ be simply laced. Then every rationally smooth $T$-fixed point of a Schubert variety $X \subset G/B$ is smooth. In particular, every rationally smooth point of $X$ is smooth. More generally, if $k = \mathbb{C}$, the same conclusion holds for all Schubert varieties in $G/P$.*

*Proof.* The $G/B$ case follows from the first version using the preceding remarks. For the $G/P$ case, suppose $k = \mathbb{C}$ and $X \subset G/P$ is rationally smooth at an $x \in X^T$. If $X$ is rationally smooth at $x$, it is also so at each $y \in X^T$ such that $y \geq x$ since $X$ is the closure of a $B$-orbit. By a result of Brion [8, §1.4 Corollary 2], $|E(X, y)| = \dim X$ for all such $y$. Thus the result follows from the first version of the $ADE$-Theorem. □

**Remark 1.1** This second version was originally proved in [17] for Schubert varieties in $SL_n(k)/B$, assuming char$(k) = 0$. The proof in this case is immediate, since in Type $A$, $\dim T_x(X(w)) = |E(X(w), x)|$ for any $x \leq w$ [31]. In fact, by a result of Polo [35], $\dim T_x(X(w))$ is independent of char$(k)$, so the characteristic zero assumption is actually unneeded. The general $G/B$ case of the $ADE$-Theorem was originally conjectured by Peterson and the first author.

**Remark 1.2** As noted above, the problem of determining the locus of rationally smooth points of a Schubert variety $X(w) \subset G/B$ involves only the combinatorics of the Bruhat graph. Its interest lies in the Kazhdan-Lusztig Conjectures (proved by Beilinson-Bernstein and Brylinski-Kashiwara), which relate the $x \leq w$ where $X(w)$ is rationally smooth to the irreducible representations of $G$ if $\mathrm{char}(k) > 0$. The rationally smooth locus in the Schubert case has been treated e.g. in Arabia [1], Boe-Graham [6], Carrell-Peterson [10], Dyer [18], and Kumar [25]. The general case of varieties admitting a torus action is considered in two papers of Brion [8, 9], as well as in Arabia [1]. A number of people have written on the singular locus of an $X(w)$ in the case of Schubert varieties in Type $A$, using the notion of pattern avoidance introduced in [30] (e.g. Billey-Warrington [5], Cortez [16], Kassel-Lascoux-Reutenauer [22] and Manivel [32]). We will make further comments on this topic in §8.

The obvious question of whether all rationally smooth points of arbitrary $T$-stable subvarieties in $G/B$ are smooth when $G$ is simply laced turns out to have a negative answer, as we show in Example 7.1. There exist $T$-orbit closures in $D_4/B$ with rationally smooth but singular $T$-fixed points.

Let us now describe Peterson's key idea. Suppose $X \subset \mathbb{A}^n_k$ is affine, irreducible of dimension $m$, and let $X_0$ be the set of smooth points of $X$. Recall that the Nash blow up $N(X)$ of $X$ is defined to be the closure in $\mathbb{A}^n_k \times G_m(\mathbb{A}^n_k)$ of the graph of the Gauss map $\gamma\colon X_0 \to G_m(\mathbb{A}^n_k)$ taking $x$ to $T_x(X)$, where, by definition, $G_m(\mathbb{A}^n_k)$ is the Grassmannian of $m$-planes in $\mathbb{A}^n_k$. By a general criterion of Nobile [34], a variety $X$ over $\mathbb{C}$ is smooth at $x$ if and only if there is a Zariski neighborhood $U$ of $x$ in $X$ such that $N(U)$ is isomorphic (as a variety) to $U$ under the natural projection $\pi\colon N(X) \to X$. Moreover, when $X$ is normal, this isomorphism obtains as long as $\pi^{-1}(x)$ consists of just one point. Now suppose $X$ is normal and admits a $T$-action. Since the Nash blow up is equivariant, the fibre $\pi^{-1}(x)$ over $x \in X^T$ is $T$-stable, so if $\pi^{-1}(x)^T$ contains only a single point, then $X$ must be smooth at $x$. Now consider a Schubert variety $X = X(w)$ which is smooth at each $y \in (x, w]$. Let $C$ be a $T$-curve in $X$ such that $C^T \subset [x, w]$. Then $X$ is smooth along $C \setminus \{x\}$, so the limit $\tau_C(X, x) = \lim_{z \to x} T_z(X)$ $(z \in C \setminus \{x\})$ is a well defined $T$-stable subspace of $T_x(X)$ having dimension $\dim X$. In particular, $\tau_C(X, x) \in N_x(X)^T$. This is the *Peterson translate* of $X$ along $C$ informally defined at the beginning. What Peterson showed is that the fibre over $x$ consists of a single point exactly when all the $\tau_C(X, x)$ coincide. Thus, since Schubert varieties are normal, their smooth fixed points are characterized as follows.

**Theorem 1.3** (**The Deformation Theorem).** *Suppose $k = \mathbb{C}$. Then $X(w)$ is smooth at an $x \leq w$ if and only if it is smooth at every $y \in (x, w]$ and the $\tau_C(X(w), x)$ coincide for all $C \in E(X(w), x)$ with $C^T \subset [x, w]$.*

Note that by Deodhar's inequality (Proposition 2.5), the number of $C \in E(X, x)$ with $C^T \subset [x, w]$ is at least $\dim X(w) - \dim X(x)$, so checking the hypothesis of the Deformation Theorem may require computing $\tau_C(X, x)$ for a substantial number of $T$-curves. As we will see in §4, Theorem 1.3 is an easy consequence of a general result, Theorem 1.4, which only requires one to compute at most two $\tau_C(X, x)$, and, somewhat surprisingly, doesn't require normality, as does the proof outlined above. It also allows the characteristic of $k$ to be arbitrary.

To describe this generalization, let us enlarge the class of $T$-varieties to include any irreducible variety with an action of an algebraic torus $T$ such that $X^T$ is finite and each $x \in X^T$ has an open $T$-stable affine neighborhood. A *$T$-curve* in a $T$-variety $X$ will by definition be a curve which is the closure (in $X$) of a one-dimensional $T$-orbit. As above, the set of $T$-curves in $X$ containing $x \in X^T$ will be denoted by $E(X, x)$. We will call $C \in E(X, x)$ *good* if $C = \overline{Tz}$, where $z$ is a smooth point of $X$. Note that by Lemma 2.4, $|E(X, x)| \geq \dim X$. Still assuming $x \in X^T$, let

$$TE(X, x) = \sum_{C \in E(X,x)} T_x(C) \subset T_x(X) \tag{1}$$

be the *tangent space* to $E(X, x)$ at $x \in X^T$. Clearly, if $X$ is smooth at $x$, then

$$TE(X, x) = \tau_C(X, x) \tag{2}$$

for all $C \in E(X, x)$, and furthermore, $TE(X, x) = T_x(X)$. Recall that $x \in X^T$ is called *attractive* if every weight of the induced linear $T$-action on $T_x(X)$ lies on one side of a hyperplane in the character group $X(T)$ of $T$ (cf §2). Our first main result is the following.

**Theorem 1.4.** *Let $X$ be a $T$-variety over $k$, and let $x \in X^T$ be attractive. Suppose every $C \in E(X, x)$ is nonsingular and for any two distinct $C, D \in E(X, x)$, $T_x(C)$ and $T_x(D)$ have different $T$-weights as $T$-modules. Consider the following two statements:*

(A) *Equation* (2) *holds for at least two good $C \in E(X, x)$.*

(B) *$X$ is Cohen-Macaulay at $x$ and* (2) *holds for at least one good $C \in E(X, x)$.*

*If either* (A) *or* (B) *holds, then $X$ is nonsingular at $x$.*

**Remark 1.5** A deep result of Ramanathan [36] says that Schubert varieties in $G/P$ are Cohen-Macaulay for any $k$. Our applications of Theorem 1.4 do not require this fact, however, although Cohen-Macaulayness is central to some of our arguments, such as Proposition 6.4.

The hypotheses of Theorem 1.4 are immediately satisfied for any $T$-variety $X \subset G/P$. In particular, $\dim TE(X, x) = |E(X, x)|$ for any such $X$. Recall that if $C \in E(G/P, x)$, then $C = \overline{U_\alpha x}$, where $\alpha$ is a root of $G$ with respect to $T$ and $U_\alpha$ is the associated root subgroup of $G$ [10]. We will call $C$ *long* or *short* according to whether $\alpha$ is long or short. When $G$ is simply laced, every $T$-curve will be called short.

We now consider the question of when condition (A) of Theorem 1.4 holds. It turns out that under the mild restriction that $G$ has no $G_2$ factors, there exists a natural subspace $\Theta_x(X)$ of $T_x(X)$ containing each Peterson translate $\tau_C(X, x)$, where $C$ is a good curve. Let $\mathfrak{T}_x(X) \subset T_x(X)$ denote the reduced tangent cone of $X$ at $x$, and define $\Theta_x(X)$ to be the linear span of $\mathfrak{T}_x(X)$. In general, $\Theta_x(X)$ is a proper $T$-invariant subspace of $T_x(X)$ having minimal dimension $\dim X$, which is achieved exactly when $\mathfrak{T}_x(X)$ is linear. Clearly, one always has

$$TE(X, x) \subset \Theta_x(X). \tag{3}$$

It was shown in [10, 12] that if $G$ is simply laced, then (3) is an equality for all Schubert varieties. (In a sequel, we will extend this fact to all $T$-varieties in $G/P$ provided $G$ is simply laced.) Our second main result is

**Theorem 1.6.** *Assume $G$ has no $G_2$ factors, $X$ is a $T$-variety in $G/P$, $x \in X^T$ and $C \in E(X, x)$ is good. Then*

$$\tau_C(X, x) \subset \Theta_x(X). \tag{4}$$

*Furthermore, if $C$ is short, then*

$$\tau_C(X, x) \subset TE(X, x). \tag{5}$$

*In particular, if $G$ is simply laced, then $\tau_C(X, x) \subset TE(X, x)$.*

In fact, we show in Example 6.6 that Theorem 1.6 fails for $G_2/B$, the problem being the long strings in the root system of $G_2$. The proof of the first version of the $ADE$-Theorem is obtained by combining Theorems 1.4 and 1.6. We now give it.

*Proof of the $ADE$-Theorem.* First consider a Schubert variety $X(w)$ in $G/B$ and suppose $x \leq w$. Letting $\ell(w)$ denote $\dim X(w)$, we will induct on $\ell(w) - \ell(x)$. In fact, each $X(w)$ is smooth at $w$, and by a well known result of Chevalley ([15, Proposition 3, Corollaire]), it is also smooth at each $y < w$ with $\ell(w) - \ell(y) = 1$. Hence, by induction, we may assume $X(w)$ is smooth at all $y \in (x, w]$, $|E(X(w), x)| = \ell(w)$ and $\ell(w) - \ell(x) \geq 2$. By Deodhar's Inequality (Proposition 2.5), there exist at least two $C \in E(X(w), x)$ such that $C^T \subset [x, w]$. Thus $x$ lies on two good $T$-curves. By Theorem 1.6, $\tau_C(X(w), x) = TE(X(w), x)$, so Theorem 1.4 says $X(w)$ is smooth at $x$. The $G/P$ case is proved in exactly the same way. All we need to show is that if a Schubert variety $X \subset G/P$ is smooth at each $y > x$, $x \in X^T$, then $E(X, x)$ contains at least two good $T$-curves. But this follows immediately from Deodhar's Inequality, using the natural projection map $\pi \colon G/B \to G/P$. □

**Remark 1.7** Since Schubert varieties in $G/P$ are Cohen-Macaulay, and since $X$ is smooth at the maximal point of its Bruhat graph, one obtains an even shorter proof from the fact that the maximal point can be joined to $x$ by a path in $\Gamma(X)$ all of whose vertices (except $x$) are greater than $x$.

**Remark 1.8** Exactly the same proof in the nonsimply laced setting shows that if $G$ has no $G_2$ factors, then a Schubert variety $X$ in $G/P$ is smooth at $x \in X^T$ if and only if $\dim \Theta_y(X) = \dim X$ for all $y \in X^T$ such that $y \geq x$. Hence a Schubert variety is globally smooth if and only if each of its reduced tangent cones is linear. As mentioned above, if $G$ is simply laced, then $\Theta_x(X) = TE(X, x)$ for all $T$-varieties in $G/P$. In the nonsimply laced case, $\Theta_x(X)$ is not as well understood. As long as the $G_2$ restriction is in effect, however, the $T$-weights of $\Theta_x(X)$ are in the convex hull of the set of $T$-weights of $TE(X, x)$ for any $x \leq w$. Moreover, if $X \subset G/B$ and $x = e$, then the $\Theta_x(X)$ is the $B$-module span of the set of $T$-lines in $\mathfrak{T}_x(X)$ (cf. [12] for details). In a forthcoming note, the authors will describe $\Theta_x(X)$ at a maximal singular $T$-fixed point $x$ [14].

We now briefly summarize the remainder of the paper. In §2, we establish a number of facts used later, notably that the Bruhat graphs of projective $T$-varieties are connected. In

§3, we define the Peterson map and establish the basic properties used herein. In §4, we prove Theorem 1.4. In §5, we give the main step in the proof of Theorem 1.6, namely that a Peterson translate $\tau_C(X, x)$ of an arbitrary $T$-variety is determined by its behavior on the $T$-surfaces in $X$ containing $C$. In §6, we prove Theorem 1.6 in the $G/B$ case, and then conclude the $G/P$ case. We also derive some corollaries and give Example 6.6, which shows the $G_2$ restriction is necessary. In §7, we make some remarks about the rationally smooth case, assuming $G$ is simply laced. In this section we will assume $k = \mathbb{C}$. As mentioned above, we show there exists a Cohen-Macaulay $T$-variety in $D_4/B$ with a singular rationally smooth point. In §8, we give a (surprisingly pretty) explicit formula for $\tau_C(X(w), x)$ and derive several applications. In the last section, we discuss some of the algorithms for determining $\mathrm{Sing}(X(w))$.

**Acknowledgements**. We would like to thank Dale Peterson for discussions about his result, Michel Brion for his comments on rational smoothness and Sara Billey for her comments on pattern avoidance. We would also like to thank the referees for their valuable suggestions, and in particular for providing a much nicer argument for example 7.1. The second author would like to thank the University of British Columbia and the University of California at San Diego for their hospitality during his work on this paper.

## 2. Preliminaries on $T$-varieties

Throughout this section, $T$ will denote an algebraic torus over the algebraically closed field $k$. Note that we won't repeat all the notation already fixed in the Introduction. Given an algebraic torus $T$, let $X(T)$ be the character group of $T$, and let $Y(T)$ be the dual group of one-parameter subgroups of $T$. Recall the natural nondegenerate pairing $X(T) \times Y(T) \to X(\mathbb{G}_m) \cong \mathbb{Z}$ given by $(\alpha, \lambda) \mapsto \langle \alpha, \lambda \rangle$, where $\langle \alpha, \lambda \rangle$ is defined by the condition $\alpha \circ \lambda(\zeta) = \zeta^{\langle \alpha, \lambda \rangle}$.

An irreducible variety $X$ over $k$ with a regular $T$-action and finite fixed point set $X^T$ is called a *$T$-variety* if every $x \in X^T$ has an open $T$-stable affine neighborhood $U$. In this case, we will say $X$ is *locally linearizable at $x$*. Note that $U$ can always be equivariantly embedded into an affine space $V$ with a linear $T$-action. For each $T$-variety $X$ and each $x \in X^T$, $X_x$ will denote a fixed neighborhood of this form. If $V$ is a $T$-module and $X \subset \mathbb{P}(V)$ is $T$-stable and irreducible, then one sees easily that $X$ is a $T$-variety. More generally, by Sumihiro's Theorem, every irreducible $T$-stable subvariety $X$ of a normal variety $Z$ with a $T$-action is a $T$-variety ([38],[39]).

Recall that every finite $T$-module $V$ decomposes as the direct sum of its $T$-weight subspaces $V_\chi$, where $\chi$ varies through the set $\Omega(V) \subset X(T)$ of $T$-weights of $V$. Let $X$ be a $T$-variety and fix $x \in X^T$. Clearly the Zariski tangent space $T_x(X)$ to $X$ at $x$ is a $T$-module. When $\Omega(T_x(X))$ lies on one side of a hyperplane in $X(T) \otimes \mathbb{Q}$, we say that both $x$ and $X_x$ are *attractive*. Clearly $x$ is attractive if and only if there is a $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \Omega(T_x(X))$, or equivalently, $\lim_{t \to 0} \lambda(t) y = x$ for all $y \in X_x$. Note: if $f \colon \mathbb{G}_m \to X$ is a morphism, then the statement $\lim_{t \to 0} f(t) = x_0$ means that $f$ extends to a morphism $\mathbb{A}^1_k \to X$ with $f(0) = x_0$.

It is well known that if $x$ is attractive, there exists a $T$-equivariant closed embedding $X_x \hookrightarrow T_x(X)$. This implies that every smooth affine $T$-variety with an attractive fixed point is isomorphic to a $T$-module. It also follows that for any $T$-invariant line $L \subset T_x(X)$, the restriction to $X_x$ of a $T$-equivariant linear projection $T_x(X) \to L$ gives rise to a regular function $f\colon X_x \to L \cong \mathbb{A}^1_k$. Note that if the weight of $L$ has multiplicity one, then $f$ is unique up to a scalar, once an embedding $X_x \subset T_x(X)$ is fixed. We will say $f \in k[X_x]$ *corresponds* to $L$ if it is obtained as a projection in this way. In particular, these remarks apply to any $T$-variety $X$ in $G/P$. That is, every $T$-fixed point $x$ is attractive and $T_x(G/P)$ is multiplicity free.

Another fact we will use is:

**Lemma 2.1.** *Let $f\colon Y \to Z$ be a map of affine $T$-varieties such that $Y$ has an attractive $T$-fixed point $y$. Then $f$ is a finite morphism if and only if the fibre over $f(y)$ is a finite set.*

As in the Introduction, $G$ will always denote a connected semi-simple algebraic group over $k$, and $T \subset B \subset P$ will respectively denote a maximal torus, a Borel and a parabolic subgroup of $G$. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $\Phi \subset X(T)$ denote the root system of $G$ with respect to $T$. Thus $\Phi \cup \{0\} = \Omega(\mathfrak{g})$. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the unique line of weight $\alpha$ in $\mathfrak{g}$, and let $U_\alpha$ be the unipotent subgroup of $G$ such that $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$. Put $\Phi^+ = \{\alpha \in \Phi \mid U_\alpha \subset B\}$, and write $\alpha > 0$ or $\alpha < 0$ according to whether $\alpha$ belongs to $\Phi^+$ or $\Phi^- = -\Phi^+$. For each $x \in W = (G/B)^T$, the tangent space $T_x(G/B)$ is a $T$-module, namely

$$T_x(G/B) \cong \bigoplus_{x^{-1}(\alpha)<0} \mathfrak{g}_\alpha.$$

Let $r_\alpha \in W$ denote the reflection associated to $\alpha \in \Phi$. The next lemma describes the $T$-curves in $G/P$ in terms of these reflections (cf. [10]).

**Lemma 2.2.** *Let $x \in (G/P)^T$. Then every $C \in E(G/P, x)$ has the form $C = \overline{U_\alpha x}$ for some $\alpha \in \Phi$. Moreover, $C^T = \{x, r_\alpha x\}$, and each such $C$ is smooth.*

The smoothness follows from the fact that $C$ admits a transitive action of the subgroup of $G$ generated by $U_\alpha$ and $U_{-\alpha}$.

Let $X$ be a $k$-variety with regular $T$-action. Recall that a $T$-curve in $X$ is the closure of a one-dimensional $T$-orbit, and that $E(X, x)$ denotes the set of $T$-curves containing $x \in X^T$. If $X$ has only finitely many $T$-curves and $T$-fixed points, it has an associated *Bruhat graph* $\Gamma(X)$, sometimes called its momentum graph. The vertex set of $\Gamma(X)$ is $X^T$, and two vertices $x, y$ are joined by an edge if and only if there is a $C \in E(X, x)$ with $C^T = \{x, y\}$. For example, the Bruhat graph $\Gamma(G/B)$ is the usual Bruhat graph of the pair $(W, \Phi)$ with vertex set $W$, two vertices $x, y$ being joined by an edge when $xy^{-1} = r_\alpha$ for some $\alpha \in \Phi$, while $\Gamma(X(w))$ has vertex set $[e, w]$ and edges at $x$ corresponding to the reflections $r_\alpha$ such that $r_\alpha x \le w$.

**Lemma 2.3.** *Let $V$ be a $T$-module and $X \subset \mathbb{P}(V)$ a closed connected $T$-stable subvariety with only finitely many $T$-fixed points and $T$-curves. Then the Bruhat graph of $X$ is connected.*

*Proof.* We use induction on $\dim X$. If $\dim X \leq 1$ there is nothing to show. So assume $\dim X > 1$ and consider first the case where $X$ is irreducible. It is clear that $X^T$ isn't empty, so let $x \in X^T$. Let $W \subset V$ be a $T$-stable complement to the $T$-stable line $x$ and put $Z = \mathbb{P}(W) \cap X$. Clearly $Z$ is nonempty and $T$-stable. Moreover, $Z$ is connected (see for example [19, Section 1, Corollary 1]). Since $X$ is irreducible, $\dim Z < \dim X$, and thus $\Gamma(Z)$ is connected by our induction hypothesis. On the other hand, $U = X \setminus Z$ is open affine and $T$-stable, so a $T$-curve in $U$ contains at most one $T$-fixed point, and we conclude that for any $y \in U^T$, each $T$-curve $C \in E(X, y)$ connects $y$ to $\Gamma(Z)$ in $\Gamma(X)$, because $C^T$ has two elements.

For the general case, let $Y \subset X$ be an irreducible component, and let $X'$ be the union of components of $X$ different from $Y$. By induction on the number of components of $X$, $\Gamma(X')$ is connected, as is $\Gamma(Y)$ by what we just said. On the other hand, $Y \cap X'$ is nonempty as $X$ is connected. It therefore contains a $T$-fixed point. Thus $\Gamma(X)$ is the union of two connected graphs meeting in a vertex, so $\Gamma(X)$ is connected as well. □

The following result (cf. [10]) has a number of useful implications. It says, for example, that the Bruhat graph of a $T$-variety $Z$ has at least $\dim Z$ edges at each vertex.

**Lemma 2.4.** *Suppose $Z$ is a $T$-variety and $z \in Z^T$ is isolated. Then*

$$|E(Z, z)| \geq \dim Z.$$

Obviously, if one replaces $\dim Z$ with the local dimension $\dim_z Z$, the lemma also holds for reducible varieties with a $T$-action, provided the action is locally linearizable at each fixed point. Applying this Lemma to Schubert varieties, we get Deodhar's Inequality.

**Proposition 2.5.** *Let $X = X(w)$ be a Schubert variety in $G/B$. Then if $x \in X^T$, then $E(X, x) \geq \ell(w)$. Consequently,*

$$|\{\alpha \in \Phi^+ \mid x < r_\alpha x \leq w\}| \geq \ell(w) - \ell(x). \tag{6}$$

Finally, we make a remark on the tangent bundle of an arbitrary variety $Y$. Denote by $\Omega_{Y/k}$ its sheaf of Kähler differentials over $k$, and (in the notation of Hartshorne [21]) let $\mathbb{V}(\Omega_{Y/k}) = \mathbf{Spec}(S(\Omega_{Y/k}))$ be the corresponding scheme over $X$. Then the fibre of $\mathbb{V}(\Omega_{Y/k})$ over a closed point $y$ in $Y$ is just the Zariski tangent space $T_y(Y)$, so we set $T(Y) = \mathbb{V}(\Omega_{Y/k})_{\mathrm{red}}$. Certainly, if $T$ acts on $Y$, it also acts naturally on $T(Y)$.

## 3. The Peterson map

In this section we will define and study what we call the *Peterson map*, which is a slight generalization of the Peterson translate $\tau_C(X, x)$ introduced in §1 for good $T$-curves. Let $X$ be a $T$-variety, and fix a $C \in E(X, x)$ with open $T$-orbit $C^o$. Note that $C$ is not necessarily assumed to be good. The Peterson map is defined on certain subspaces of the tangent space to $X$ at an arbitrary point $z \in C^o$. Suppose that $X = X_x$ and let $T(X)$ denote the tangent bundle of $X$ defined in the previous section. Assume $M \subset T_z(X)$ is a subspace stable under the isotropy group $S \subset T$ of $z$, and put $\mathbf{M}^o = T \cdot M \subset T(X)|_{C^o}$ Then $\mathbf{M}^o$ is a $T$-stable vector bundle over $C^o$ with fibre $M$ over $z$. Let $\mathbf{M}$ denote the closure of $\mathbf{M}^o$ in $T(X)$. Clearly, $\mathbf{M}$ is independent of the choice of $z$.

**Definition 3.1** The correspondence $M \mapsto \tau_C(M,x)$ defined by putting

$$\tau_C(M,x) = \mathbf{M} \cap T_x(X)$$

will be called the *Peterson map*, and $\tau_C(M,x)$ will be called the *Peterson translate* of $M$ along $C$.

**Remark 3.2** Of course, if $M = T_z(X)$, then $\tau_C(M,x)$ is just the Peterson translate $\tau_C(X,x)$ of $X$ along $C$ introduced in the Introduction in the case $C$ is good. The only place we will use this more general formulation is in §8. The fact that $\tau_C(M,x)$ is a linear subspace of $T_x(X)$ follows immediately from the completeness of the Grassmann variety (see Lemma 3.3 below). However, if $X$ is not locally linearizable at $x$, then $\tau_C(X,x)$ as defined above does not necessarily have this property. In fact, it needn't be irreducible, since one can have a $C$ with $C^T = \{x\}$ such that the fibre of $\mathbf{M}$ over $x$ is reducible. For example, let $C$ be a nodal curve which is the image of $\mathbb{P}^1_k$ with a linear $\mathbb{G}_m$-action. Here, $\tau_C(C,x)$ is the union of two distinct lines, which one may write as $\lim_{t\to 0} T_{tz}(C)$ and $\lim_{t\to\infty} T_{tz}(C)$. Of course such a $C$ is singular at $x$.

There is an alternative method of defining $\tau_C(M,x)$ which will be implicitly used in §8. Suppose $X = X_x$ is affine, embedded into some $T$-module $V$. Recall that $G_m(V)$ denotes the Grassmann variety of $m$-dimensional subspaces of $V$. Then $T$ acts naturally on $X \times G_m(V)$ and $(z,M)$ defines an $S$-fixed point in $X \times G_m(V)$. Let $D$ be the orbit closure of $(z,M)$ in $X \times G_m(V)$. With $\pi\colon X \times G_m(V) \to X$ being the first projection, it follows $\pi(D) = C$, since $\pi$ is proper.

**Lemma 3.3.** *With the notation as above, $\tau_C(M,x)$ is an $m$-dimensional linear subspace of $T_x(X)$ invariant under the natural $T$-action on $T_x(X)$. Moreover $D^T = \pi^{-1}(x) \cap D$ consists of exactly one point, namely $(x, \tau_C(M,x))$. In particular, if $\lambda$ is any one parameter subgroup of $T$ with $\lim_{t\to 0}\lambda(t)z = x$, then $\lim_{t\to 0}\lambda(t)M = \tau_C(M,x)$ in $G_m(V)$.*

*Proof.* $\pi$ restricted to $D$ is proper and has finite fibres, so it is a finite covering of $C$, and therefore affine. It follows that $D^T = D \cap \pi^{-1}(x)$ consists of exactly one point, which then corresponds to a $T$-stable subspace $M_0$ of $T_x(X)$ of dimension $m$. Now let $p\colon E \to D$ be the tautological $m$-plane bundle over $D$, i.e. $E = \{(y,W,w) | (y,W) \in D, w \in W\} \subset D \times V$. The map $D \to C$ induces a proper map $E \to C \times V$, whose image is contained entirely in $T(X) \subset X \times V$. In fact its image is exactly $\mathbf{M}$, as defined above, since it is closed and contains $\mathbf{M}^o$ as a dense set. But now it follows that $\tau_C(M,x)$ is just the image of $p^{-1}((x,M_0))$ under this map, i.e. $\tau_C(M,x) = M_0$.

The last statement of the lemma is the fact, that $\lim_{t\to 0}\lambda(t)(z,M) \in D^T$. □

The main properties of the Peterson map are given in the next result.

**Proposition 3.4.** *Suppose the subspace $M$ of $T_z(X)$ defined above has dimension $m$. Then:*

(i) *If $Y$ is a smooth $T$-variety containing $X$ and $\lambda$ is a one parameter group such that $\lim_{t\to 0}\lambda(t)z = x$, then*

$$\tau_C(M,x) = \lim_{t\to 0}\lambda(t)M$$

*in the Grassmannian of $m$-planes in $T(Y)$.*

(ii) *$M$ and $\tau_C(M,x)$ are isomorphic $S$-modules.*
(iii) *If $M = M_1 \oplus \cdots \oplus M_t$ is the $S$-weight decomposition of $M$, then*

$$\tau_C(M,x) = \tau_C(M_1,x) \oplus \cdots \oplus \tau_C(M_t,x)$$

*is the $S$-weight decomposition of $\tau_C(M,x)$.*
(iv) *If $N$ is any $T$-stable subspace of $\tau_C(X,x)$, then there exists an $S$-stable subspace $M \subset T_z(X)$ such that $\tau_C(M,x) = N$.*

*Proof.* For (i), copy the proof of Lemma 3.3 after replacing $X \times G_m(V)$ with the Grassmannian of $m$-planes in $T(Y)$. We next show that $\tau_C(M_i,x)$ and $M_i$ are isomorphic $S$-modules. Now $S$ acts on $M_i$ by a character $\alpha_i$, hence on $T \cdot M_i \subset T(X)$ as well. As $T \cdot M_i$ is dense in $\overline{T \cdot M_i}$, it follows that $sv = \alpha_i(s)v$ for all $v \in \tau_C(M_i,x)$ and $s \in S$. Hence $\tau_C(M,x)$ decomposes as stated proving (iii). Part (ii) is a consequence of this and the comments preceeding the Proposition. For the last statement (iv), it suffices to consider the case where $N$ is a line having $T$-weight say $\alpha$. Let $X$ be equivariantly embedded into a $T$-module $V$, and let $f\colon \mathbb{A}^1_k \to C$ be the extension of an orbit map $\mathbb{G}_m \cong T/S \to C \setminus \{x\}$. Such an $f$ always exists, and $f(0) = x$. Letting $T$ act on $\mathbb{A}^1_k$ with generic isotropy $S$, then $f$ becomes equivariant. Set $\mathbf{M}' = f^*(\mathbf{M}) = \mathbb{A}^1_k \times_C \mathbf{M}$. Then $\mathbf{M}'$ is a closed subbundle of $\mathbb{A}^1_k \times V$. In particular, the set $\Gamma(\mathbf{M}')$ of global sections of $\mathbf{M}'$ is a $T$-stable subspace of $k[\mathbb{A}^1_k] \otimes V$. Here, the action of $T$ is the tensor product of the actions on $k[\mathbb{A}^1_k]$ and $V$. Hence, for $\sigma \in \Gamma(\mathbf{M}')$, we have $(t\sigma)(p) = t(\sigma(t^{-1}p))$ for all $t \in T$ and $p \in \mathbb{A}^1_k$. Notice that the fibre of $\mathbf{M}'$ over 0 is exactly $\tau_C(X,x)$. Now consider the evaluation map $e_0\colon \Gamma(\mathbf{M}') \to \tau_C(X,x)$ sending $\sigma$ to $\sigma(0)$. Clearly $e_0$ is $T$-equivariant. Since $\mathbf{M}'$ is a trivial vector bundle, $e_0$ is surjective. Thus, there is $\sigma \in \Gamma(\mathbf{M}')$ with $e_0(\sigma) \in N \setminus \{0\}$. Since $T$ is linearly reductive, we may choose $\sigma$ to be an eigenvector of the $T$-action on $\Gamma(\mathbf{M}')$. Then $\sigma$ has weight $\alpha$, and $\sigma$ spans a one-dimensional $T$-stable subbundle $E$ of $\mathbf{M}'$. Clearly, $N$ is the fibre of $E$ over 0. Choosing $p \in \mathbb{A}^1_k$ such that $f(p) = z$ and viewing $\sigma(p)$ as an element of $T_z(X)$, we get that $\tau_C(k\sigma(p),x) = N$, so we are done. □

## 4. A smoothness criterion for $T$-varieties

In this section we will prove Theorem 1.4 and then, as a consequence, deduce Peterson's Deformation Theorem (cf. the Introduction). The proof uses the Zariski-Nagata Theorem on the purity of the branch locus. The next lemma is one of the main ingredients.

**Lemma 4.1.** *Let $f\colon X \to Y$ be a finite equivariant morphism of $T$-varieties with $X$ affine and $x \in X^T$ attractive, and $Y$ nonsingular at $f(x)$. Let $Z \subset X$ be the ramification locus of $f$, i.e. the closed subvariety of points at which $f$ is not étale. Then either $Z$ equals $X$, $Z$ is empty or $Z$ has codimension one at $x$.*

*Proof.* Assume that $\operatorname{codim}_x Z \geq 2$. We have to show that $Z$ is empty. First of all, since $x$ is attractive, the image of $f$ is contained in every open $T$-stable affine neighborhood of $f(x)$, hence in $Y_{f(x)}$. Moreover, since $\operatorname{codim}_x Z \geq 2$, $f$ is étale somewhere, and thus $\dim X = \dim Y$. Being finite, $f\colon X \to Y_{f(x)}$ is surjective, and $f(x)$ is an attractive fixed point of $Y_{f(x)}$. Therefore $Y_{f(x)}$ is nonsingular (in particular $Y_{f(x)}$ is isomorphic to a $T$-module). Passing to the normalization $g\colon \tilde{X} \to X$ of $X$, we obtain an equivariant finite map $\tilde{f}\colon \tilde{X} \to Y_{f(x)}$, which is étale in codimension one because $g$ is clearly an isomorphism over $X \setminus Z$. Thus, by the theorem of Zariski-Nagata on the purity of the branch locus [20, X, 3.4], $\tilde{f}$ is étale everywhere. Hence for some point $\tilde{x} \in \tilde{X}$ which maps to $x \in X$ we have $T_{\tilde{x}}(\tilde{X}) \cong T_{f(x)}(Y_{f(x)})$ via $\mathrm{d}f$. This implies that $\tilde{X}$ is attractive (and therefore $\tilde{X}^T = \{\tilde{x}\} = g^{-1}(x)$ as a set), forcing $\tilde{f}$ to be an isomorphism onto $Y_{f(x)}$. Thus $f$ is birational. But being finite, $f$ is also an isomorphism, so we are through. □

This gives the following criterion for smoothness at attractive fixed points, which is our main tool in proving Theorem 1.4. It generalizes the criterion in [27] to ground fields of arbitrary characteristic.

**Theorem 4.2.** *Let $X$ be a $T$-variety and let $x$ be an attractive $T$-fixed point. Suppose there is a nonempty subset $E \subset E(X, x)$ which satisfies the following conditions:*

(i) *Every $C \in E$ is good.*
(ii) $|E(X, x) \setminus E| \leq \dim X - 2$.
(iii) *$\tau_C(X, x) = \tau_D(X, x)$ for all $T$-curves $C, D \in E$.*
(iv) *If $\tau(E)$ denotes the common value of $\tau_C(X, x)$ for $C \in E$, then $T_x(C) \cap \tau(E) \neq 0$ for all curves $C \in E(X, x)$.*

*Then $x$ is a nonsingular point of $X$.*

*Proof.* After replacing $X$ by $X_x$, we may assume that $X$ is affine and, since $x$ is attractive, that $X \subset T_x(X)$ and $x = 0$. Fix an equivariant projection $\tilde{p}\colon T_x(X) \to \tau(E)$, and denote its restriction to $X$ by $p$. We will show that $p$ is finite and étale. This implies $p$ is an isomorphism because $x$ is attractive. Since $T_x(C) \cap \tau(E) \neq 0$ for all curves $C \in E(X, x)$, it follows that there are no $T$-curves in $p^{-1}(0)$. Thus, by Lemma 2.4, $\dim_x p^{-1}(0) = 0$. But clearly every component of $p^{-1}(0)$ has to contain $x$, so $\dim p^{-1}(0) = \dim_x p^{-1}(0) = 0$. Now Lemma 2.1 implies $p$ is finite, as $x$ is attractive.

Let $Z$ be the ramification locus of $p$. According to Lemma 4.1, we are done if $\mathrm{codim}_x Z \geq 2$. By assumption (i), if $C \in E$, then $C$ is good and meets the regular locus of $X$. A map between varieties is étale at a regular point $z$ if and only if the image of $z$ is regular as well and the tangent map at $z$ is an isomorphism. It follows that $C^o = C \setminus C^T \subset Z$ if and only if $\mathrm{d}p_z$ has a nontrivial kernel in $T_z(X)$ for some $z \in C^o$, since $X$ and $\tau(E)$ have the same dimension by (i) and the properties of the Peterson translate. Put $L = \ker \mathrm{d}p_z \subset T_z(X)$. Of course $L$ is $S$-stable where $S$ denotes the isotropy group of $z$ in $T$. Hence $\tau_C(L, x)$ is defined and contained in $\tau(E)$, because $C \in E$. On the other hand, if $t \in T$, then $tL$ is the kernel of $\mathrm{d}p_{tz}$ in $T_{tz}(X)$. It follows that $\mathrm{d}p$ vanishes on all of $T \cdot L \subset T(X)$. But by continuity this also means, that $\mathrm{d}p_x(\tau_C(L, x)) = 0$. In other words $\tau_C(L, x) \subset \ker \mathrm{d}p_x \cap \tau(E)$. But $\ker \mathrm{d}p_x$ is a complement of $\tau(E)$, because $p$ is the restriction of a linear projection to $\tau(E)$, so it follows that $\tau_C(L, x) = 0$. This means that $L = 0$, and so $p$ is étale along all of $C^o$.

We conclude that $E \cap E(Z, x)$ is empty. Thus, by condition (ii), $|E(Z, x)| \leq \dim X - 2$. This means that $\dim_x Z \leq \dim X - 2$, thanks again to Lemma 2.4 and ends the proof. □

**Remark 4.3** Notice, that condition (ii) says that $\dim X \geq 2$. Indeed, if $X$ is a $T$-curve itself, then $X$ is good, and $\tau_X(X, x)$ is one-dimensional. But of course $X$ may be singular at a $T$-fixed point.

**Remark 4.4** Note that the last condition (iv) is automatically satisfied for curves $C \in E$ since $\tau_C(C, x) \subset \tau(E)$ for such a curve. Moreover, if all curves in $E(X, x)$ are smooth, then the last condition just says that $T_x(C) \subset \tau(E)$ for all $C \in E(X, x)$.

**Remark 4.5** If $X$ is normal, one does not need to assume $x$ is attractive since the Zariski-Nagata Theorem can be applied directly.

We now prove Theorem 1.4.

*Proof of Theorem 1.4.* Recall that we are assuming that all $C \in E(X, x)$ are smooth and have distinct weights. Under this assumption, if

$$\tau_C(X, x) = TE(X, x)$$

for a good $C \in E(X, x)$, then it follows that $|E(X, x)| = \dim TE(X, x) = \dim X$. Hence if (A) holds, that is $\tau_C(X, x) = \tau_D(X, x) = TE(X, x)$ for two good $T$-curves $C, D$, we may apply Theorem 4.2 with $E = \{C, D\}$ and conclude $X$ is smooth at $x$.

Next, suppose (B) obtains. That is, $X$ is Cohen-Macaulay and $\tau_C(X, x) = TE(X, x)$ holds for at least one good $C \in E(X, x)$. We may as usual assume that $X = X_x \subset T_x(X)$. Let $C_1, C_2, \dots, C_{n-1}$ be the $T$-curves in $E(X, x)$ distinct from $C$, and let $x_1, x_2, \dots, x_{n-1} \in k[X]$ be the corresponding regular functions on $X$, i.e. $x_i$ corresponds to $T_x(C_i) \subset T_x(X)$.

Then clearly $C \subset Z = \mathcal{V}(x_1, x_2, \dots, x_{n-1})$, the set of common zeroes of the $x_i$. For $\dim_x Z \geq 1$, so there is a $T$-curve contained in it. On the other hand, by the choice of the $x_i$, no other $T$-curve except for $C$ can be contained in $Z$. Thus, $Z = C$ by Lemma 2.4. Now, $C$ is nonsingular, so $X$ is smooth at $x$ if the ideal generated by the $x_i$ equals its radical. In other words, it suffices to prove that $A := k[X]/(x_1, x_2, \dots, x_{n-1})$ is a reduced ring. Since $A$ is one-dimensional, it is Cohen-Macaulay [33, p.135]. By definition, the differentials $\mathrm{d}x_i$ are linearly independent on $TE(X, x)$. Since $\tau_C(X, x) = TE(X, x)$ this implies that these differentials $\mathrm{d}x_i$ are independent along all of $C^o$. Thus, $A$ is smooth along $C^o$, if we identify $\mathrm{Spec}(A)$ with $C$ as topological spaces. In other words the set of regular points in $\mathrm{Spec}(A)$ is dense. It follows that $A$ is reduced [33, p.183], so we are done. □

Next, after some preliminary remarks on slices, we will prove Peterson's Deformation Theorem stated in §1. Let $H$ be an algebraic group and let $Y$ be a variety with regular $H$-action. A *slice* $(\mathcal{S}, S)$ to $Hy$ in $y \in Y$ is a locally closed affine subset $\mathcal{S} \subset X$ containing $y$ together with a nontrivial torus $S \subset H_y$ such that $\mathcal{S}$ is $S$-invariant, $Y$ is an $S$-variety, the natural map $H \times \mathcal{S} \to Y$ is smooth at $(e, y)$ and $y$ is isolated in $\mathcal{S} \cap Hy$. $\mathcal{S}$ is called attractive if $y \in \mathcal{S}^S$ is attractive.

If $X$ is a Schubert variety $X(w)$, an explicit attractive slice $\mathcal{S}$ for $X$ at any $x \leq w$ is given as follows.

**Lemma 4.6.** *Let $U$ be the maximal unipotent subgroup of $B$ and $U^-$ the opposite maximal unipotent subgroup, and suppose $x < w$. Then an attractive slice for $X(w)$ is obtained by putting*

$$\mathcal{S} = X(w) \cap (U^- \cap xU^-x^{-1})x$$

*with $S = T$.*

This is well known and an easy consequence of the fact that $U^- \hookrightarrow U^-e \subset G/B$ is an open immersion. For a proof see [15] or [8, Section 1.4]. It is easy to see that the (unique) $T$-stable affine neighborhood $X(w)_x$ of $x$ is in fact isomorphic with $\mathcal{S} \times Bx$ as a $T$-variety . It therefore follows that, if $C \in E(\mathcal{S}, x)$, then $C \in E(X(w)_x, x)$, and $\tau_C(X(w)_x, x) = \tau_C(\mathcal{S}, x) \oplus T_x(Bx)$. In particular $T_x(D) \subset \tau_C(X(w)_x, x)$ for all $D \in E(Bx, x)$.

*Proof of the Deformation Theorem.* Let $X = X(w)$ and suppose $X$ is smooth at every $y \in (x, w]$. Suppose also that all the $\tau_C(X, x)$ coincide as $C$ ranges over the set of $C \in E(X, x)$ such that $C^T \subset [x, w]$. If $\ell(w) - \ell(x) = 1$, there is nothing to prove, since Schubert varieties are nonsingular in codimension one [15]. This fact also follows from the previous Lemma, because a one-dimensional slice is nonsingular, since it is just a $T$-curve. Letting $E$ be the set of $C \in E(X, x)$ such that $C^T \subset [x, w]$ and denoting by $\tau(E)$ the common Peterson translate along curves in $E$, the existence and form of the slice and the hypotheses imply that $T_x(C) \subset \tau(E)$ for all $C \in E(X(x), x) = E(X, x) \setminus E$. Obviously $T_x(C) \subset \tau(E)$ for all $C \in E$ and thus $\tau(E) = TE(X, x)$. Note that all $C \in E$ are contained in the slice given by the last lemma. If $\ell(w) - \ell(x) \geq 2$, then by Deodhar's Inequality (6), $E$ contains more than two elements. The result now follows from Theorem 1.4. □

## 5. A fundamental lemma

Let $X$ denote an arbitrary $T$-variety. In this section, we will show that if $C \in E(X, x)$ is good, then the Peterson translate $\tau_C(X, x)$ is determined by the translates $\tau_C(\Sigma, x)$, as $\Sigma$ ranges over the $T$-stable surfaces containing $C$.

**Lemma 5.1.** *Let $C \in E(X, x)$ be good. Then*

$$\tau_C(X, x) = \sum_{\Sigma} \tau_C(\Sigma, x),$$

*where the sum ranges over all $T$-stable irreducible surfaces $\Sigma$ containing $C$.*

*Proof.* Let $L \subset \tau_C(X, x)$ be a $T$-stable line. Then by Proposition 3.4 there is an $S$-stable line $M \subset T_z(C)$, where $S$ is the isotropy group of an arbitrary $z \in C^o$, such that $\tau_C(M, x) = L$. As $X$ is nonsingular at $z$, there exists an $S$-stable curve $D$ satisfying $M \subset T_z(D)$. Setting $\Sigma = \overline{T \cdot D}$, we obtain a $T$-stable surface containaing $C$, which satisfies $L \subset \tau_C(\Sigma, x)$. □

Although the lemma is almost obvious, it is very useful when $X$ is a $T$-variety in $G/B$, as long as $G$ has no $G_2$-factors. One reason for this is

**Proposition 5.2.** *Suppose $G$ has no $G_2$-factors, and let $\Sigma$ be an irreducible $T$-stable surface in $G/B$. Let $u \in \Sigma^T$. Then $|E(\Sigma, u)| = 2$, and either $\Sigma$ is nonsingular at $u$ or the weights of the two $T$-curves to $\Sigma$ at $u$ are orthogonal long roots $\alpha, \beta$ in a copy of $B_2 \subset \Phi$. In this case, $\Sigma_u$ is isomorphic to a surface of the form $z^2 = xy$ where $x, y, z \in k[\Sigma_u]$ have weights $-\alpha, -\beta, -\frac{1}{2}(\alpha+\beta)$ respectively. In particular, if $G$ is simply laced, then $\Sigma$ is nonsingular.*

*Proof.* It is easy to see that $\Sigma_u = \overline{U_{-\alpha}U_{-\beta}u}$, where $\alpha, \beta \in \Phi$ (cf. [13]). It follows immediately that the two elements of $E(\Sigma, u)$ are $T$-curves $C$ and $D$ of weights $-\alpha, -\beta$ at $u$. Choose functions $x$ and $y$ in $k[\Sigma_u]$ corresponding to $T_u(C)$ and $T_u(D)$ in $T_u(\Sigma)$. Then for any weight $\omega$ of the $T$-representation on $k[\Sigma_u]$, there is a positive integer $N$ such that

$$N\omega \in \mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta. \tag{7}$$

Now let $\omega$ be a weight of the dual of $T_u(\Sigma)$, i.e. a weight of the differentials at $u$. These weights all are contained in $\Phi$. Except for the case where $\alpha$, $\beta$ and $\omega$ are contained in a copy of $B_2 \subset \Phi$, (7) actually implies that $\omega = a\alpha + b\beta$ for suitable nonnegative integers $a, b$. Using the multiplicity freeness of the representation of $T$ on $k[\Sigma_u]$, one is done in these cases. For if $f$ is a $T$-eigenvector with weight $\omega$ and $\mathrm{d}f_u \neq 0$, then $f$ equals $x^a y^b$ up to a scalar, and since $\mathrm{d}f_u \neq 0$, it follows $a + b = 1$. This implies that $\omega$ is either $\alpha$ or $\beta$. If this is true for all such $\omega$, then $\Sigma$ is nonsingular at $u$. In the remaining case, there is an $\omega$, such that $\omega, \alpha, \beta$ are contained in a copy of $B_2$. By a similar argument, it turns out that $\Sigma$ is nonsingular at $u$ unless $\alpha, \beta$ are orthogonal long roots in $B_2$ ([13]). There are now two possible cases. Either $\Sigma$ is nonsingular at $u$, or $\omega = \frac{1}{2}(\alpha + \beta)$ is a weight of the dual of the tangent space. By (7) and the structure of $B_2$, this is the only weight distinct from $\alpha$ and $\beta$. Thus $\Sigma_u$ is isomorphic to $z^2 = xy$ where $x, y, z \in k[T_u(\Sigma)]$ correspond to the $T$-invariant lines of weights $\alpha, \beta$ and $\omega$. The claim follows (after replacing $\alpha, \beta, \omega$ with their negatives). □

For more detailed information on $T$-stable surfaces in $G/B$ see [13].

## 6. Peterson translates and the span of the tangent cone

The purpose of this section will be to prove Theorem 1.6. We assume throughout that $G$ has no $G_2$ factors. Recall that $\Theta_x(X)$ denotes the $k$-linear span of the reduced tangent cone $\mathfrak{T}_x(X)$ in $T_x(X)$. Clearly $\mathfrak{T}_x(X)$ is a $T$-stable subvariety of $T_x(X)$, so $\Theta_x(X)$ is a $T$-subspace. Moreover, as noted in (3), $TE(X, x) \subset \Theta_x(X)$.

*Proof of Theorem 1.6.* We will first consider the case where $X$ is an arbitrary $T$-variety in $G/B$. Let $C \in E(X, x)$ be good. We have to show $\tau_C(X, x) \subset \Theta_x(X)$, and, moreover, if $C$ is short, then $\tau_C(X, x) \subset TE(X, x)$. Since $\tau_C(X, x)$ is generated by $T$-invariant surfaces and since $\Theta_x(\Sigma) \subset \Theta_x(X)$ for all surfaces $\Sigma \subset X$ containing $x$, it is enough to assume $X$ is a surface. If $X$ is nonsingular at $x$, then $\tau_C(X, x) = T_x(X)$. Otherwise we know from Proposition 5.2 that $X$ is a cone over $x$, and for a cone, $\Theta_x(X) = T_x(X)$. Since $X$ is nonsingular at $x$ when $C$ is short, the conclusion $\tau_C(X, x) \subset TE(X, x)$ is obvious. The $G/P$ case now follows immediately from the next lemma. □

Suppose $x \in (G/B)^T$ and put $y = \pi(x)$, where $\pi\colon G/B \to G/P$ is the natural map. Note that $\pi$ induces a surjective map $\pi_!\colon E(G/B,x)\backslash E(\pi^{-1}(y),x) \to E(G/P,y)$.

**Lemma 6.1.** *Let $Y \subset G/P$ be closed and $T$-stable, and put $X = \pi^{-1}(Y)$. Then:*

(i) *the projection $\pi\colon X \to Y$ is a smooth morphism, hence $\mathrm{Reg}(X) = \pi^{-1}(\mathrm{Reg}(Y))$;*

(ii) *for all $x \in X^T$, $\mathrm{d}\pi_x\colon T_x(X) \to T_y(Y)$ is surjective and*

$$\mathrm{d}\pi_x(\Theta_x(X)) = \Theta_y(Y),$$

*where $y = \pi(x)$;*

(iii) *$\pi_!(E(X,x)\backslash E(X \cap \pi^{-1}(y),x)) = E(Y,y)$; and*

(iv) *if $C \in E(X,x)$ is good and $\pi(C)$ is a curve, then $\pi(C) \in E(Y,y)$ is good and*

$$\mathrm{d}\pi_x(\tau_C(X,x)) = \tau_{\pi(C)}(Y,y).$$

*Proof.* The lemma follow immediately from the fact that $G/B \to G/P$ is a $T$-equivariant locally trivial fibre bundle. □

Let us next point out some corollaries in the simply laced setting. Recall that in this case, all $T$-curves are short. First, we have

**Corollary 6.2.** *Let $G$ be simply laced. Let the $T$-variety $X \subset G/P$ have dimension at least two and satisfy $|E(X,x)| = \dim X$ at $x \in X^T$. Then $X$ is nonsingular at $x$ if and only if $E(X,x)$ contains at least two good curves. (If $X$ is Cohen-Macaulay, one suffices.)*

*Proof.* This follows immediately from Theorems 1.4 and 1.6. □

**Remark 6.3** Corollary 6.2 is a generalization of the first version of the $ADE$-Theorem that gives a smoothness criterion for e.g. local complete intersections or normal $T$-orbit closures in $G/P$, since both types of varieties are Cohen-Macaulay. The condition that there exists a smooth $T$-fixed point holds whenever $X = \overline{Hx}$, where $x \in (G/P)^T$ and $H$ is any closed subgroup of $G$ containing $T$.

Theorem 1.6 enables one to formulate conditions for locating the smooth $T$-fixed points in $X$, especially when $X$ is Cohen-Macaulay and $G$ is simply laced. For example, let us call a subset $F \subset X^T$ *connected* if for any $x, y \in F$, there is a path in $\Gamma(X)$ joining $x$ and $y$ having all its vertices in $F$. Recall that we showed in Lemma 2.3 that $X^T$ itself is always connected. Applying our two main results again gives

**Proposition 6.4.** *Suppose $G$ is simply laced. Let $X$ be a Cohen-Macaulay $T$-variety in $G/P$, and let $Y \subset X^T$ be the set of points where $|E(X,y)| = \dim X$. Then every connected component of $Y$ containing a smooth point of $X$ consists of smooth points. In particular, if $|E(X,x)| = \dim X$ for all $x \in X^T$ and $X^T$ contains a smooth point, $X$ is smooth.*

**Remark 6.5** Proposition 6.4 is another generalization of the $ADE$-Theorem (version one). If we assume $G$ has no $G_2$ factors and replace $Y$ by the set of $T$-fixed points where the reduced tangent cone of $X$ is linear, then an identical conclusion holds.

We end this section by showing the $G_2$ restriction in Theorem 1.6 is necessary. In fact, the curve along which Theorem 1.6 fails in this example is short.

**Example 6.6** Consider the surface $\Sigma$ given by $z^2 = xy^3$ in $\mathbb{A}^3$. Let $\alpha, \beta, \gamma$ be characters of $T$ satisfying $\gamma = 2\alpha + 3\beta$, and let $T$ act on $\mathbb{A}^3$ by

$$t \cdot (x, y, z) = (t^{\alpha} x, t^{(\alpha+2\beta)} y, t^{\gamma} z).$$

Clearly $\Sigma$ is $T$-stable, and its reduced tangent cone at 0 is by definition $\ker \mathrm{d}z$, hence is linear. The $T$-curve $C = \{x = 0\}$ is good, and along $C^o$, we have $T_v(\Sigma) = \ker \mathrm{d}x$. It follows that $\tau_C(\Sigma, 0) = \ker \mathrm{d}x$, which is not a subspace of $\Theta_0(\Sigma)$. It remains to remark that $\Sigma$ is open in a $T$-stable surface in $G_2/B$, where $T$ is the usual maximal torus and $\alpha$ and $\beta$ are respectively the corresponding long and short simple roots.

## 7. Some results on rational smoothness

A natural question suggested by the second version of the $ADE$-Theorem is whether every rationally smooth $T$-fixed point in an arbitrary $T$-variety in $G/B$ is smooth when $G$ is simply laced. We now answer this in the negative by showing there exists a four-dimensional Cohen-Macaulay $T$-orbit closure in $G/B$, where $G$ is of type $D_4$, having a rationally smooth $T$-fixed point in its singular locus.

**Example 7.1** Let $H$ be the finite subgroup of four by four diagonal matrices over $\mathbb{C}$, satisfying $t_i^2 = 1$ and $t_1t_2t_3t_4 = 1$, where $t_1, t_2, t_3, t_4$ are the diagonal entries. Then $H$ acts on $\mathbb{A}^4_{\mathbb{C}}$ and on its coordinate ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. The invariants of this action are generated by $y_i = x_i^2$ and $z = x_1x_2x_3x_4$. Thus the quotient $\mathbb{A}^4_{\mathbb{C}}//H$ is isomorphic to the hypersurface $Y$ in $\mathbb{A}^5_{\mathbb{C}}$ given by $z^2 = y_1y_2y_3y_4$. Rational smoothness being preserved by finite quotients, $Y$ is rationally smooth, and since it is a hypersurface, $Y$ is Cohen-Macaulay. In addition, it is nonsingular in codimension one and hence normal.

Let $T$ be a four dimensional torus over $\mathbb{C}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_5 \in X(T)$ satisfy $2\alpha_5 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Let $T$ act on $\mathbb{A}^5_{\mathbb{C}}$ with these weights such that the coordinate $y_i$ corresponds to the line with weight $\alpha_i$ ($i \leq 4$), and the coordinate $z$ corresponds to the line with weight $\alpha_5$. Then $Y$ is $T$-invariant and has a dense $T$-orbit. It remains to observe that weights satisfying these relations occur in the positive roots of $D_4$, so there is a $T$-variety $X \subset D_4/B$ containing $Y$ as a dense open subset, which has a nonsmooth rationally smooth $T$-fixed point $x$, as asserted. Moreover $x$ is a normal Cohen-Macaulay point of $X$. In fact, the variety $X$ is normal ([13]), hence toric and therefore globally Cohen-Macaulay.

Proposition 6.4 makes it clear that an essential reason rationally smooth Schubert varieties in the simply laced case are smooth is that every Schubert variety contains at least one smooth $T$-fixed point. More generally, when $k = \mathbb{C}$, we have

**Proposition 7.2.** *If $G$ is simply laced and over $\mathbb{C}$, then every rationally smooth Cohen-Macaulay $T$-variety $Z$ in $G/P$ having a smooth $T$-fixed point is globally smooth. In particular, a rationally smooth toric variety in $G/P$ with a smooth $T$-fixed point is smooth.*

*Proof.* It follows from [8, 1.4, Corollary 2] that if $z$ is a rationally smooth fixed point, then $|E(Z,z)| = \dim Z$. Hence we may apply Proposition 6.4 since the Bruhat graph of $Z$ is connected. The second statement is just the fact that toric varieties are Cohen-Macaulay. □

**Remark 7.3** An obvious problem is to determine conditions under which a general $T$-variety is globally rationally smooth. For example, when $k = \mathbb{C}$, a Schubert variety in $G/B$ is rationally smooth provided either its classical Poincaré polynomial is symmetric or its Bruhat graph is regular [10, THEOREM A]. Note that the Bruhat graph of a $T$-variety $X$ is regular exactly when $|E(X)| = \dfrac{1}{2}|X^T| \dim X$, where $E(X)$ denotes the set of all $T$-curves in $X$. These two (equivalent) conditions can easily be extended to Schubert varieties $X$ in $G/P$. For example, if $\Gamma(X)$ is regular, then the Bruhat graph of the Schubert variety $Y = \pi^{-1}(X) \subset G/B$ is clearly also regular, hence $Y$ is rationally smooth. Moreover, if the Poincaré polynomial of $X$ is symmetric, then that of $Y$ is symmetric also. Indeed, this follows from the Leray spectral sequence, since $X$ is simply connected and the fibres of $\pi$ are smooth. But if $Y$ is rationally smooth, then so is $X$, since when $k = \mathbb{C}$, rational smoothness is invariant under smooth morphisms.

To summarize, we state

**Proposition 7.4.** *Assuming $k = \mathbb{C}$, a Schubert variety in $G/P$ is rationally smooth if and only if either its Poincaré polynomial is symmetric or its Bruhat graph is regular.*

It would be interesting to know whether either one of these conditions is sufficient for rational smoothness, provided we restrict attention just to $T$-varieties in $G/P$. For some relevant results, see [9, Theorem 1].

## 8. The Peterson map for Schubert varieties in $G/B$

The main purpose of this section is to give an explicit computation of the Peterson map $M \mapsto \tau_C(M,x)$ for a Schubert variety $X(w) \subset G/B$. In fact, taking advantage of the fact that Schubert varieties are $B$-stable, we will obtain a somewhat more elegant formulation, namely as a map from $T$-invariant subspaces $M$ of $T_y(X)$ to $T$-invariant subspaces of $T_x(X)$, where $C^T = \{x, y\}$ and $y > x$. What will result is a simple formula for $\tau_C(M,x)$ in terms of root strings in $\Omega(M) \subset \Omega\big(T_y(X)\big)$. Furthermore the assumption that $C$ is good will not be necessary here. We will also compute some examples in $B_2/B$ and give several applications. Finally, we will compare some of the algorithms for finding the singular loci of Schubert varieties with what can be obtained from Peterson translates. The notation introduced in §3 will be used throughout, and, for convenience, we will assume $G$ is adjoint.

As usual, let $X$ denote $X(w)$, and let $C \in E(X,x)$ denote a $T$-curve such that $C^T = \{x, y\} \subset [x, w]$. After writing $C = \overline{U_\alpha y}$ for a unique $\alpha \in \Phi$, note that since $y > x$, it follows that $\alpha > 0$, so $U_\alpha \subset B$. Hence the additive group $U_\alpha$, which acts transitively on $C \backslash \{x\}$, stabilizes both $X$ and $x$. Since $G$ is adjoint, $S := \ker(\alpha)$ is a codimension one subtorus of $T$, and moreover, $S$ is the generic isotropy group of $C$. Thus, if $z \in C \backslash C^T$, any $S$-stable subspace $M \subset T_z(X)$ is the translate of a unique $S$-stable subspace of $T_y(X)$ under an element of $U_\alpha$. If this subspace is also $T$-stable, e.g. is an $S$-weight subspace of $T_y(X)$, then $\mathbf{M}$ (cf. §3) is also $U_\alpha T$-equivariant, and $M$ is the translate of $\tau_C(M,y)$ by an element of $U_\alpha$. Therefore, $\tau_C$ can be viewed as defined on $T$-subspaces of $T_y(X)$. In particular, we may write

$$\tau_C(T_y(X),x)=\tau_C(X,x).$$

Another way of saying this is that $\tau_C(X,x)$ is the fibre over $x$ of $\overline{U_\alpha T_y(X)}$. Moreover, $T_y(X)=\tau_C(X,y)$ is the fibre over $y$.

Now suppose $M\subset T_y(X)$ is an $S$-weight space of dimension $\ell$. Then $M$ and $\tau_C(M,x)$ are $T$-modules. In addition, $\tau_C(M,x)$ is also a $U_\alpha$-module. Now $M$ and $\tau_C(M,x)$ are abstractly isomorphic as $S$-modules (due to Proposition 3.4), although they aren't necessarily isomorphic $T$-modules. Since $M$ has only one $S$-weight, it follows that $\Omega(M)$ and $\Omega\big(\tau_C(M,x)\big)$ are contained in a single $\alpha$-string in $\Phi$. But $U_{-\alpha}$ fixes $y$, so $T_y(G/B)$ is a $\mathfrak{g}_{-\alpha}$-module, although, in general, $T_y(X)$ need not be one. We claim there exists a unique $\mathfrak{g}_{-\alpha}$-submodule $M^{-\alpha}$ of $T_y(G/B)$ such that $M^{-\alpha}\cong M$ as $S$-modules. That $M^{-\alpha}$ exists is clear. If $M$ isn't $\mathfrak{g}_{-\alpha}$-stable, then define $M^{-\alpha}$ to be the unique $U_{-\alpha}$-fixed point on the $T$-curve $\overline{U_{-\alpha}M}$ in the Grassmannian $G_\ell\big(T_y(G/B)\big)$. Then $M^{-\alpha}$ is a $U_{-\alpha}T$-module having only one $S$-weight, hence there is a unique $\beta\in\Phi$ such that $\Omega(M^{-\alpha})$ is a saturated (partial) string of the form

$$\Omega(M^{-\alpha})=\{\beta,\beta-\alpha,\dots,\beta-(\ell-1)\alpha\},\tag{8}$$

where $\ell$ is, as above, the dimension of $M$, and $\beta$ is the root in the $\alpha$-string determined by $M$ such that $y^{-1}(\{\beta,\beta-\alpha,\dots,\beta-(\ell-1)\alpha\})\subset\Phi^-$ but $y^{-1}(\beta-\ell\alpha)>0$. Notice that $\beta$ is uniquely determined by $\ell$ and the $\alpha$-string, because each $U_{-\alpha}$-module contains a line of fixed points, and each $\alpha$-string of $\Omega(T_y(G/B))$ contains exactly one root $\gamma$ corresponding to a trivial $U_{-\alpha}$-module (so $\beta=\gamma+(\ell-1)\alpha$).

Recall that $\dot r_\alpha\in N_G(T)$ denotes a representative of the reflection $r_\alpha$.

**Proposition 8.1.** *Let $M$ be an $S$-weight subspace of $T_y(X)$. Then*

$$\tau_C(M,x)=\mathrm{d}\dot r_\alpha(M^{-\alpha}),$$

*where $\mathrm{d}\dot r_\alpha\colon T_y(G/B)\to T_x(G/B)$ is the differential of $\dot r_\alpha$ at $y$. Consequently,*

$$\Omega\big(\tau_C(M,x)\big)=r_\alpha\Omega(M^{-\alpha}).$$

*Proof.* Since $M^{-\alpha}$ is a $U_{-\alpha}T$-module, $\mathrm{d}\dot r_\alpha(M^{-\alpha})$ is a $U_\alpha T$-submodule of $T_x(G/B)$ isomorphic to $M$ as an $S$-module since $r_\alpha$ preserves $\alpha$-strings. But, as in (8), two $S$-isomorphic $U_\alpha T$-submodules of $T_x(G/B)$ with one $S$-weight coincide since both have their $T$-weights contained in the same $\alpha$-string, i.e. their $T$-weights comprise a single saturated string of the form $r_\alpha(\beta)+j\alpha$, $1\le j\le r$. Thus $\tau_C(M,x)=\mathrm{d}\dot r_\alpha(M^{-\alpha})$. □

**Corollary 8.2.** *Let $T_y(X)=M_1\oplus\cdots\oplus M_t$ be the decomposition of $T_y(X)$ into $S$-weight spaces, and let $T_y(X)^\alpha=M_1^\alpha\oplus\cdots\oplus M_t^\alpha$ be the corresponding $U_{-\alpha}T$-module. Then*

$$\tau_C(T_y(X),x)=\mathrm{d}\dot r_\alpha(M_1^\alpha)\oplus\cdots\oplus\mathrm{d}\dot r_\alpha(M_t^\alpha).$$

*Consequently,*

$$\Omega\big(\tau_C(T_y(X),x)\big) = r_\alpha\Omega(M_1^\alpha) \cup \cdots \cup r_\alpha\Omega(M_t^\alpha). \tag{9}$$

*Proof.* Just apply Propositions 8.1 and 3.4. □

**Remark 8.3** One can also obtain $\tau_C(M,x)$ by first applying $\mathrm{d}\dot{r}_\alpha$ to $M$ and then taking the unique $U_\alpha$-fixed point on $\overline{U_\alpha \mathrm{d}\dot{r}_\alpha(M)}$.

**Remark 8.4** It is clear that the above formula for $\tau_C(X,x)$ also applies to $G/P$ for any parabolic $P \supset B$. One can either mimic the proof of Proposition 8.1 or apply Lemma 6.1.

Proposition 8.1 gives an *a priori* test whether or not $\tau_C(X,x) = \tau_D(X,x)$ for two good $T$-curves $C, D \in E(X,x)$. Suppose $C = \overline{U_\alpha y}$ and $D = \overline{U_\beta z}$, where $\alpha, \beta > 0$. Then $\tau_C(X,x) = \tau_D(X,x)$ if and only if

$$r_\alpha\Omega\big(TE(X,y)^\alpha\big) = r_\beta\Omega\big(TE(X,z)^\beta\big), \tag{10}$$

or, equivalently,

$$TE(X,y)^\alpha = \mathrm{d}\dot{r}_\alpha \mathrm{d}\dot{r}_\beta TE(X,z)^\beta. \tag{11}$$

Now the right hand side of (11) is a $\mathfrak{g}_{r_\alpha(\beta)}$-module, hence in order that $\tau_C(X,x) = \tau_D(X,x)$, the left hand side must be one too.

**Corollary 8.5.** *Let $C$ and $D$ be as above. If $\tau_C(X,x) = \tau_D(X,x)$, then $\Omega\big(TE(X,y)^\alpha\big)$ is the set of weights of a $\mathfrak{g}_{r_\alpha(\beta)}$-module. In other words, the weights of $TE(X,y)^\alpha$ occur in saturated $r_\alpha(\beta)$-strings.*

We now give several applications of formula (9). Let $\Phi(x,w)$ denote $\Omega\big(TE(X(w),x)\big)$. It is well known that

$$\Phi(x,w) = \{\gamma \in \Phi \mid x^{-1}(\gamma) < 0,\ r_\gamma x \le w\}, \tag{12}$$

but there is no explicit algorithm for computing $\Phi(x,w)$ for an arbitrary pair $x < w$. However, if $X$ is smooth at $x$, we have

**Corollary 8.6.** *Let $X(w)$ be smooth at two adjacent vertices $y > x$ of $\Gamma(X(w))$, and write $x = r_\alpha y$ where $\alpha > 0$. Then*

$$\Phi(x,w) = r_\alpha\Omega\big(TE(X(w),y)^\alpha\big). \tag{13}$$

In particular, this always applies when $x = r_\alpha w$ and $\ell(x) = \ell(w) - 1$ (since $X$ is smooth at $x$ and there is a $T$-curve $C$ in $X$ with $C^T = \{x,w\}$). Let us illustrate this case with an example.

**Example 8.7** Suppose $G$ is of type $B_2$, and let $w = r_\alpha r_\beta r_\alpha$, where $\alpha$ is the short simple root and $\beta$ is the long simple root. As usual, put $X = X(w)$. We will compute $\Phi(x,w)$ at the two codimension one fixed points $x = r_\alpha r_\beta$ and $x = r_\beta r_\alpha$. For the reader's convenience, the Bruhat graph of $X$ is given in FIGURE 1.

Now

$$\Phi(w,w) = \{\alpha, \alpha+\beta, 2\alpha+\beta\} \subset \{\alpha, \alpha+\beta, 2\alpha+\beta, -\beta\} = \Omega(T_w(G/B)).$$

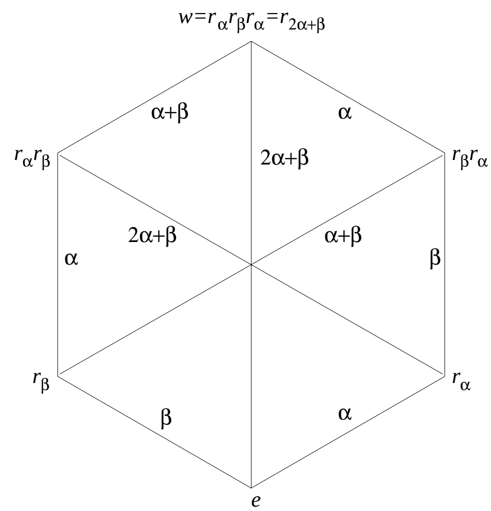


FIGURE 1. The Bruhat graph $\Gamma(X(w))$ in type $B_2$. The edges are labeled by the positive root which is defined by the $T$-curve corresponding to a given edge.

First let $x = r_\alpha r_\beta$. Then $x = r_{\alpha+\beta} w$, so we first compute the $\alpha+\beta$-strings in $\Phi(w,w)$. One of these is $2\alpha+\beta, \alpha$, while the other one is simply $\alpha+\beta$. Next, move these strings into the weights of a $U_{-(\alpha+\beta)}T$-submodule of $T_w(G/B)$, which gives $\{\alpha+\beta\} \cup \{\alpha, -\beta\}$. Finally, reflect these two sets by $r_{\alpha+\beta}$ getting

$$\Phi(x,w) = r_{\alpha+\beta}(\{\alpha, -\beta, \alpha+\beta\}) = \{\alpha, \beta+2\alpha, -(\alpha+\beta)\}.$$

In the other case $x = r_\beta r_\alpha = r_\alpha w$, it is immediate that $\Phi(x,w) = r_\alpha \Phi(w,w) = \{\beta, \alpha+\beta, -\alpha\}$ since $\dot{r}_\alpha$ leaves $X$ stable.

To continue this example, we compute the singular locus of $X$.

**Example 8.8** By the codimension one property, $X$ is smooth at $r_\alpha r_\beta$ and $r_\beta r_\alpha$. Next consider $r_\alpha$. In order to verify whether or not $r_\alpha$ is smooth, it suffices, by Theorem 1.4, to compare $TE(X, r_\alpha)$ and $\tau_C(X, r_\alpha)$ for one of the good curves. Inspecting FIGURE 1, one sees that $\Phi(r_\alpha, w) = \{\alpha, -\beta, -(\beta+2\alpha)\}$. Notice from the previous example that $\Phi(r_\beta r_\alpha, w)$ is already the set of weights of a $\mathfrak{g}_{-\beta}$-module, so if $C = \overline{U_\beta r_\beta r_\alpha}$, we get

$$\Omega\big(\tau_C(X, r_\alpha)\big) = r_\beta\{\beta, \alpha+\beta, -\alpha\} = \{-\beta, \alpha, -(\alpha+\beta)\}.$$

Hence $\tau_C(X, r_\alpha) \neq TE(X, r_\alpha)$, so $X$ must be singular at $r_\alpha$. It follows that the Schubert variety $X(r_\alpha)$ is contained in the singular locus of $X$. However, it is immediate that $X$ is smooth at $r_\beta$ since it is smooth at $r_\alpha r_\beta$ and the good $T$-curve $\overline{U_\alpha r_\alpha r_\beta}$ containing $r_\beta$ is short. Therefore the singular locus of $X$ is $X(r_\alpha)$.

If $X(w)$ is smooth at $y$ but not at $x = r_\alpha y < y$, and if $\alpha$ is short, Theorem 1.6 still guarantees that $r_\alpha \Omega\big(TE(X(w), y)^\alpha\big) \subset \Phi(x, w)$, assuming the $G_2$-restriction. However, the previous example shows the containment needn't hold for a long curve.

The case where $G = SL_n(k)$ is special since by [31] one knows

$$\Phi(x, w) = \Omega\big(T_x(X)\big)$$

for every $x \leq w$. It follows easily from this that in type $A$, the Bruhat graph of a Schubert variety $X(w)$ has the property that the number of edges $N(x)$ at a vertex $x$ is a nonincreasing function, i.e. $x \leq y \leq w$ implies $N(x) \geq N(y)$. In the general case, the Bruhat graph does not have this monotonicity property.

## 9. Concluding Remarks

As illustrated in Example 8.8, Theorem 1.4 suggests a procedure for finding $\mathrm{Sing}(X)$ for an arbitrary Schubert variety $X = X(w)$ in $G/B$, where $G$ is any semi-simple group. Starting at a codmension one point $y$ of $X^T$, which is smooth in $X$ by Chevalley's result, consider any $C \in E(X, y)$ with $C^T = \{x, y\}$ where $x < y$. In other words, $C$ is an edge in $\Gamma(X)$ from $y$ down to $x$. Clearly $C$ is good, and, by the Cohen-Macaulay critierion, $X$ is smooth at $x$ if and only if $\tau_C(X, x) = TE(X, x)$. If $|\Phi(x, w)| = |E(X, x)| > \dim X$, then $x$ is of course singular, so suppose $|\Phi(x, w)| = \dim X$. Then, if $G$ has no $G_2$-factors, $x$ is smooth if $C$ is short (which is always the case when $G$ is simply laced). If $C$ is long or if there are $G_2$-factors, then one can use Proposition 8.1 to calculate $\tau_C(X, x)$ to determine whether or not it coincides with $TE(X, x)$. If so, $x$ turns out to be a smooth point, and then one repeats the procedure. Notice that a singular point $x$ is a maximal point of $\mathrm{Sing}(X) \cap X^T$ if and only if all $y$ with $w \geq y > x$ and $\ell(y) - \ell(x) = 1$ are smooth points of $X$. In that case, $X(x)$ is an irreducible component of $\mathrm{Sing}(X)$. The implementation of this procedure is exponential in time since computing the Bruhat graph is. However, aside from computing the Bruhat graph, the additional step of comparing $\tau_C(X, x)$ and $TE(X, x)$ is a polynomial time procedure. Note that this procedure can be readily extended to any $G/P$ (cf. Remark 8.4).

There are other well known procedures for finding the singular locus of a Schubert variety in a general $G/B$, due e.g. to S. Kumar [25], V. Lakshmibai [28] and V. Lakshmibai, P. Littelmann and P. Magyar [29]. All of these algorithms require working down the Bruhat graph as remarked above, hence are exponential in time. In particular, Kumar showed that to any $x, w \in W$ with $x < w$, there is an element $c_{w,x}$ in the nil-Hecke ring of $W$ such that $X(w)$ is smooth at $x$ if and only if

$$c_{w,x} = (-1)^{\ell(w)-\ell(x)} \prod_{\alpha \in \Phi(x,w)} \alpha^{-1}.$$

Suppose $s_1, \ldots, s_p$ are simple reflections such that $w = s_1 \cdots s_p$, and let $\alpha_1, \ldots \alpha_p$ be the corresponding simple roots. Then

$$c_{w,x} = (-1)^{\ell(w)} \sum s_1^{\epsilon_1} \alpha_1^{-1} s_2^{\epsilon_2} \alpha_2^{-1} \cdots s_p^{\epsilon_p} \alpha_p^{-1},$$

where the sum is over all sequences $(\epsilon_1, \cdots, \epsilon_p)$ of 0's and 1's such that $x = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_p^{\epsilon_p}$. Note that in this sum, each $s_i$ where $\epsilon_i = 1$ acts on all the $\alpha_j^{-1}$ with $j \geq i$. In Example 8.8, when $x = r_\alpha$, the sequences are $(1, 0, 0)$ and $(0, 0, 1)$, so

$$\begin{aligned} c_{w,r_\alpha} &= (-1)^{\ell(w)}\big(\frac{1}{r_\alpha(\alpha) r_\alpha(\beta) r_\alpha(\alpha)} + \frac{1}{\alpha\beta r_\alpha(\alpha)}\big) \\ &= \frac{2}{\alpha\beta(2\alpha+\beta)}. \end{aligned}$$

Hence $r_\alpha$ is a singular point.

The determination of the singular loci for the Schubert varieties in $G_2/B$ is carried out in [25] and also in Kumar's forthcoming opus on Kac Moody algebras [26, Chapter 12]. Also, see [25] or [3, cf. pp.91-97] for further comments. Since in order to write down $c_{w,x}$, one requires a reduced expression $w = s_1 \cdots s_p$, plus all the subexpressions $x = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_p^{\epsilon_p}$, and since one still has to incorporate paths in $\Gamma(X(w))$ in order to find the irreducible components of $\mathrm{Sing}(X(w))$, this algorithm is difficult to implement in large ranks. On the other hand, S. Billey has written a program to use the $c_{w,x}$'s to calculate the singular loci of all Schubert varieties in $F_4/B$, but the computations are too difficult to implement in $E_6/B$.

In [29], Lakshmibai-Seshadri paths are used to find standard monomial bases for Schubert varieties in the classical types, which are then used to also determine the tangent spaces $T_x(X(w))$. The methods of [29] also extend to exceptional types and to the Kac-Moody setting. A number of comments on singular loci are contained in [loc.cit.]. Lakshmibai [28] has also given a formula for the tangent spaces $T_e(X(w))$ of Schubert varieties at the $B$-fixed point $e$, and her other work giving detailed formulas for $T_x(X(w))$ for classical types is summarized in detail in [3].

As mentioned in the Introduction, the singular loci (= rational singular loci) of Schubert varieties in Type $A_n$ in terms of pattern avoidance has been treated by a number of authors. Recently, Billey and Warrington [5] gave a polynomial time algorithm (in fact, $O(n^4)$) for determining the irreducible components of $\mathrm{Sing}(X(w))$. Pattern avoidance conditions for global smoothness have also been given in the classical types by Billey [2], and this has been extended to the exceptional groups by Billey and Postnikov [4]. Using other methods, Billey was able to determine the globally smooth Schubert varieties in $E_6, E_7, E_8$. Some of the data for these is summarized in the following table:

| $n$ | $\lvert W \rvert$ | # smooth |
|---|---|---|
| 6 | $51,840$ | $2,356$ |
| 7 | $2,903,040$ | $10,734$ |
| 8 | $696,729,600$ | $47,870$ |

Note the curious fact that 10,734/2,356 is about 4.556, while 47,870/10,734 is about 4.459.

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