

#2935

The Equivariant Cohomology Ring of Regular Varieties

MICHEL BRION & JAMES B. CARRELL

MMJ does not
use abstracts

1. Preliminaries

Let \mathfrak{B} be the group of upper triangular 2×2 complex matrices of determinant 1. Let \mathfrak{T} (resp. \mathfrak{U}) be the subgroup of \mathfrak{B} consisting of diagonal (resp. unipotent) matrices. We have isomorphisms $\lambda: \mathbb{C}^* \rightarrow \mathfrak{T}$ and $\varphi: \mathbb{C} \rightarrow \mathfrak{U}$, where

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad \varphi(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

together satisfy the relation

$$\lambda(t)\varphi(u)\lambda(t^{-1}) = \varphi(t^2u). \quad (1)$$

Consider the generators

$$\mathcal{V} = \dot{\varphi}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{W} = \dot{\lambda}(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Lie algebras $\text{Lie}(\mathfrak{U})$ and $\text{Lie}(\mathfrak{T})$, respectively. Then $[\mathcal{W}, \mathcal{V}] = 2\mathcal{V}$, and

$$\text{Ad}(\varphi(u))\mathcal{W} = \mathcal{W} - 2u\mathcal{V} \quad (2)$$

for all $u \in \mathbb{C}$.

In this paper, X will denote a smooth complex projective algebraic variety endowed with an algebraic action of \mathfrak{B} such that the fixed-point scheme $X^{\mathfrak{U}}$ consists of one point o . Both the \mathfrak{B} -variety X and the action will be called *regular*. Then $o \in X^{\mathfrak{T}}$, since $X^{\mathfrak{U}}$ is \mathfrak{T} -stable. Moreover, $X^{\mathfrak{T}}$ is finite by Lemma 1 of [7]. Thus we may write

$$X^{\mathfrak{T}} = \{\zeta_1 = o, \zeta_2, \dots, \zeta_r\}. \quad (3)$$

Clearly $r = \chi(X)$, the Euler characteristic of X .

Let $H_{\mathfrak{T}}^*(X)$ denote the \mathfrak{T} -equivariant cohomology ring of X with complex coefficients. To define it, let \mathcal{E} be a contractible space with a free action of \mathfrak{T} and let $X_{\mathfrak{T}} = (X \times \mathcal{E})/\mathfrak{T}$ (quotient by the diagonal \mathfrak{T} -action). Then

$$H_{\mathfrak{T}}^*(X) = H^*(X_{\mathfrak{T}}).$$

It is well known that the equivariant cohomology ring $H_{\mathfrak{T}}^*(\text{pt})$ of a point is the polynomial ring $\mathbb{C}[z]$, where z denotes the linear form on the Lie algebra of \mathfrak{T}

Received November 27, 2002. Revision received May 29, 2003.

The second author was partially supported by the Natural Sciences and Engineering Research Council of Canada.

such that $z(\mathcal{W}) = 1$. The degree of z is 2. Thus $H_{\mathfrak{T}}^*(X)$ is a graded algebra over the polynomial ring $\mathbb{C}[z] = H_{\mathfrak{T}}^*(\text{pt})$ (via the constant map $X \rightarrow \text{pt}$).

In our situation, the restriction map in cohomology

$$i_{\mathfrak{T}}^*: H_{\mathfrak{T}}^*(X) \rightarrow H_{\mathfrak{T}}^*(X^{\mathfrak{T}})$$

induced by the inclusion $i: X^{\mathfrak{T}} \hookrightarrow X$ is injective (see [10]). By (3),

$$H_{\mathfrak{T}}^*(X^{\mathfrak{T}}) = \bigoplus_{j=1}^r H_{\mathfrak{T}}^*(\zeta_j) \cong \bigoplus_{j=1}^r \mathbb{C}[z],$$

so each $\alpha \in H_{\mathfrak{T}}^*(X)$ defines an r -tuple of polynomials $(\alpha_{\zeta_1}, \dots, \alpha_{\zeta_r})$. That is,

$$i_{\mathfrak{T}}^*(\alpha) = (\alpha_{\zeta_1}, \dots, \alpha_{\zeta_r}). \quad (4)$$

We will define a refined version of this restriction map in Section 4.

ACKNOWLEDGMENTS. The second author thanks Shrawan Kumar for asking the question that led to Theorem 1 and Dale Peterson for describing his result (Theorem 3).

2. The \mathfrak{B} -Stable Curves

Throughout this paper, a curve in X will be a purely one-dimensional closed subset of X ; a curve that is stable under a subgroup G of \mathfrak{B} is called a G -curve. The \mathfrak{B} -curves in X play a crucial role, so we next establish a few of their basic properties.

PROPOSITION 1. *If X is a regular \mathfrak{B} -variety, then every irreducible \mathfrak{B} -curve C in X has the form $C = \overline{\mathfrak{B} \cdot \zeta_j}$ for some index $j \geq 2$. Moreover, every \mathfrak{B} -curve contains o . In particular, there are only finitely many irreducible \mathfrak{B} -curves in X , and they all meet at o .*

Proof. It is clear that if $j \geq 2$ then $C = \overline{\mathfrak{B} \cdot \zeta_j}$ is a \mathfrak{B} -curve in X containing ζ_j that (by the Borel fixed-point theorem) also contains o , since o is the only \mathfrak{B} -fixed point. Conversely, every \mathfrak{B} -curve C in X contains o and at least one other \mathfrak{T} -fixed point. Indeed, since o has an affine open \mathfrak{T} -stable neighborhood X_o in X , the complement $C - X_o$ is nonempty and \mathfrak{T} -stable. It follows immediately that $C = \overline{\mathfrak{B} \cdot x}$ for some $x \in X^{\mathfrak{T}} - o$. \square

Consider the action of \mathfrak{B} on the projective line \mathbb{P}^1 given by

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot z = \frac{z}{t(t+uz)}$$

(the inverse of the standard action). This action has $(\mathbb{P}^1)^{\mathfrak{T}} = \{0, \infty\}$ and $(\mathbb{P}^1)^{\mathfrak{U}} = \{0\}$ and is therefore regular. Note that, if $u \in \mathbb{C}^*$, then

$$\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot \infty = \frac{1}{tu}$$

and so $\varphi(u) \cdot \infty = u^{-1}$.

The diagonal action of \mathfrak{B} on $X \times \mathbb{P}^1$ is also regular. By Proposition 1, the irreducible \mathfrak{B} -curves in $X \times \mathbb{P}^1$ are of the form $\overline{\mathfrak{B} \cdot (x, \infty)}$ or $\overline{\mathfrak{B} \cdot (x, 0)}$, where $x \in X^{\mathfrak{T}}$. Only the first type will play a role here. Thus, we put $Z_j = \overline{\mathfrak{B} \cdot (\zeta_j, \infty)}$ and let $\pi_j : Z_j \rightarrow \mathbb{P}^1$ be the second projection. Clearly each π_j is bijective, hence $Z_j \cong \mathbb{P}^1$. In addition,

$$Z_j = \{(\varphi(u) \cdot \zeta_j, u^{-1}) \mid u \in \mathbb{C}^*\} \cup \{(\zeta_j, \infty)\} \cup \{(o, 0)\},$$

so $Z_i \cap Z_j = \{(o, 0)\}$ as long as $i \neq j$. Moreover, restricting π_j gives an isomorphism

$$p_j : Z_j - (\zeta_j, \infty) \rightarrow \mathbb{A}^1.$$

Finally, we put

$$Z = \bigcup_{1 \leq j \leq r} Z_j.$$

Thus, Z is the union of all irreducible \mathfrak{B} -stable curves in $X \times \mathbb{P}^1$ that are mapped onto \mathbb{P}^1 by the second projection $\pi : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

3. The Fundamental Scheme \mathcal{Z}

Let \mathcal{A} denote the vector field on $X \times \mathbb{A}^1$ defined by

$$\mathcal{A}_{(x,v)} = 2\mathcal{V}_x - v\mathcal{W}_x. \quad (5)$$

Obviously, \mathcal{A} is tangent to the fibres of the projection to \mathbb{A}^1 . By (2),

$$(\text{Ad}(\varphi(u))\mathcal{W})_x = -u\mathcal{A}_{(x,u^{-1})}. \quad (6)$$

The contraction operator $i(\mathcal{A})$ defines a sheaf of ideals $i(\mathcal{A})(\Omega_{X \times \mathbb{A}^1}^1)$ of the structure sheaf of $X \times \mathbb{A}^1$. Let \mathcal{Z} denote the associated closed subscheme of $X \times \mathbb{A}^1$. In other words, \mathcal{Z} is the zero scheme of \mathcal{A} .

REMARK 1. The vector field \mathcal{A} discussed here is a variant of the vector field studied in [7]. Both have the same zero scheme.

The properties of \mathcal{Z} figure prominently in this paper. First put

$$X_o = \left\{ x \in X \mid \lim_{t \rightarrow \infty} \lambda(t) \cdot x = o \right\}. \quad (7)$$

Clearly, X_o is \mathfrak{T} -stable, and it follows easily from (1) that X_o is open in X (see [7, Prop. 1] for details). Hence, by the Bialynicki-Birula decomposition theorem [3], X_o is \mathfrak{T} -equivariantly isomorphic to the tangent space $T_o X$, where \mathfrak{T} acts by its canonical representation at a fixed point. The weights of the associated action of λ on $T_o X$ are all negative. Thus we may choose coordinates x_1, \dots, x_n on $X_o \cong T_o X$ that are eigenvectors of \mathfrak{T} ; the weight a_i of x_i is a positive integer (it turns out to be even; see [2]). This identifies the positively graded ring $\mathbb{C}[X_o]$ with $\mathbb{C}[x_1, \dots, x_n]$, where $\deg x_i = a_i$. Now $X \times \mathbb{P}^1$ contains $X_o \times \mathbb{A}^1$ as a \mathfrak{T} -stable affine open subset with coordinate ring $\mathbb{C}[x_1, \dots, x_n, v]$, where v has degree 2.

PROPOSITION 2. *The scheme \mathcal{Z} is reduced and is contained in $X_o \times \mathbb{A}^1$ as a \mathfrak{T} -curve. Its ideal in $\mathbb{C}[X_o \times \mathbb{A}^1] = \mathbb{C}[x_1, \dots, x_n, v]$ is generated by*

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n). \quad (8)$$

These form a homogeneous regular sequence in $\mathbb{C}[x_1, \dots, x_n, v]$, and the degree of each $v\mathcal{W}(x_i) - 2\mathcal{V}(x_i)$ equals $a_i + 2$. The irreducible components of \mathcal{Z} are the \mathcal{Z}_j ($1 \leq j \leq r$), where

$$\mathcal{Z}_j = Z_j - \{(\xi_j, \infty)\}. \quad (9)$$

Each \mathcal{Z}_j is mapped isomorphically to \mathbb{A}^1 by the second projection p . In particular, p is finite and flat of degree r , and \mathcal{Z} has r irreducible components. Any two such components meet only at $(o, 0)$.

Proof. This follows from the results in [7, Sec. 3]. We provide direct arguments for the reader's convenience.

Since $[\mathcal{W}, \mathcal{V}] = 2\mathcal{V}$ and $\mathcal{W}(v) = 2v$ (on \mathbb{A}^1), it follows that \mathcal{A} commutes with the vector field induced by the diagonal \mathfrak{T} -action on $X \times \mathbb{A}^1$. Hence, \mathcal{Z} is \mathfrak{T} -stable.

Next we claim that (9) holds set-theoretically. Let $(x, v) \in X \times \mathbb{A}^1$ with $v \neq 0$. Then $(x, v) \in \mathcal{Z}$ if and only if $\mathcal{W}_x - 2v^{-1}\mathcal{V}_x = 0$, that is, **iff** x is a zero of $\text{Ad}(\varphi(v^{-1}))\mathcal{W}$; equivalently, $\varphi(-v^{-1}) \cdot x \in X^{\mathfrak{T}}$. On the other hand, $(x, 0) \in \mathcal{Z}$ if and only if $\mathcal{V}_x = 0$, that is, **iff** $x = o$. Thus $\mathcal{Z} = Z \cap (X \times \mathbb{A}^1)$ (as sets). Further, $Z \cap (X \times \mathbb{A}^1)$ is an open affine \mathfrak{T} -stable neighborhood of $(o, 0)$ in Z and therefore equals $Z \cap (X_o \times \mathbb{A}^1)$. This implies our claim.

It follows that $\mathcal{Z} \subseteq X_o \times \mathbb{A}^1$ (as schemes), so that the ideal of \mathcal{Z} is generated by $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$. These polynomials are homogeneous of degrees $a_1 + 2, \dots, a_n + 2$; together with v , they have only the origin as their common zero (since o is the unique zero of \mathcal{V}). Hence

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n), v$$

form a regular sequence in $\mathbb{C}[x_1, \dots, x_n, v]$, and v is a nonzero divisor in $\mathbb{C}[\mathcal{Z}]$. As a consequence, the $\mathbb{C}[v]$ -module $\mathbb{C}[\mathcal{Z}]$ is finitely generated and free. In other words, $p: \mathcal{Z} \rightarrow \mathbb{A}^1$ is finite and flat.

Fix $v_0 \neq 0$ and consider the scheme-theoretic intersection $\mathcal{Z} \cap (X \times v_0)$. This identifies with the zero scheme of $2\mathcal{V}_x - v_0\mathcal{W}_x$ in X , that is, to the zero scheme of $\text{Ad}(\varphi(v_0^{-1}))\mathcal{W}$. The latter consists of $r = \chi(X)$ distinct points, so that $\mathcal{Z} \cap (X \times v_0)$ is reduced. Since $\mathbb{C}[\mathcal{Z}]$ is a free module over $\mathbb{C}[v]$, it follows easily that $\mathcal{Z} \cap (v \neq 0)$ is reduced. But the open subset $\mathcal{Z} \cap (v \neq 0)$ is dense in \mathcal{Z} by (9), and \mathcal{Z} is a complete intersection in \mathbb{A}^{n+1} . Thus, \mathcal{Z} is reduced. This completes the proof. \square

4. The Refined Restriction

We now define our refined restriction on equivariant cohomology. Let $\alpha \in H_{\mathfrak{T}}^*(X)$. Recall from (4) that $i_{\mathfrak{T}}^*(\alpha) = (\alpha_{\zeta_1}, \dots, \alpha_{\zeta_r})$, where each $\alpha_{\zeta_j} \in \mathbb{C}[z]$. We regard each α_{ζ_j} as a polynomial function on \mathcal{Z}_j (isomorphic to \mathbb{A}^1 via p) and hence on

see msp.6

"iff" 2 times o.k.?

$\mathcal{Z}_j - (o, 0)$. Since $\mathcal{Z} - (o, 0)$ is the disjoint union of the $\mathcal{Z}_j - (o, 0)$, this yields an algebra homomorphism

$$\rho: H_{\mathfrak{T}}^*(X) \rightarrow \mathbb{C}[\mathcal{Z} - (o, 0)]$$

such that $\rho(\alpha)(x, v) = \alpha_{\zeta_j}(v)$ whenever $(x, v) \in \mathcal{Z}_j - (o, 0)$. In particular, $\rho(z)(x, v) = v$, so that $\rho(z) = v$. And since $i_{\mathfrak{T}}^*$ preserves the grading, the same holds for ρ .

Note that the value $\alpha_{\zeta_j}(0)$ at the origin is independent of the index j . (For $X_{\mathfrak{T}} = (X \times \mathcal{E})/\mathfrak{T}$ is connected since both X and \mathcal{E} are, so $H^0(X_{\mathfrak{T}})$ is by definition the set of constant functions on $X_{\mathfrak{T}}$; consequently, the component of α in degree 0 gives the same value at each fixed point.) Now let $\mathbb{C}_0[\mathcal{Z}]$ denote the subalgebra of $\mathbb{C}[\mathcal{Z} - (0, o)]$ consisting of all elements that extend continuously to \mathcal{Z} in the classical topology. Then

$$\rho(H_{\mathfrak{T}}^*(X)) \subseteq \mathbb{C}_0[\mathcal{Z}].$$

We will demonstrate that $\rho(H_{\mathfrak{T}}^*(X))$ is in fact the coordinate ring $\mathbb{C}[\mathcal{Z}]$. Toward this end, we compute the image under ρ of an equivariant Chern class.

5. Equivariant Chern Theory

Suppose Y is an algebraic variety with an action of an algebraic group G . Then recall that a vector bundle E on Y is said to be G -linearized if there is an action of G on E lifting that on Y and such that each $g \in G$ defines a linear map from E_y to $E_{g \cdot y}$ for any $y \in Y$. In particular, if $y \in Y^G$ then we have a representation of G in E_y , and hence a representation of $\text{Lie}(G)$. Thus, each $\xi \in \text{Lie}(G)$ acts on E_y by $\xi_y \in \text{End}(E_y)$.

Also recall that the k th equivariant Chern class $c_k^G(E) \in H_G^{2k}(Y)$ is defined to be the k th Chern class of the vector bundle

$$E_G = (E \times \mathcal{E})/G \rightarrow X_G = (X \times \mathcal{E})/G,$$

where \mathcal{E} is a contractible space with a free action of G . For $y \in Y^G$, the restriction $c_k^G(E)_y$ lies in $H_G^*(\text{pt})$. The latter identifies with a subring of the coordinate ring of $\text{Lie}(G)$, and we have

$$c_k^G(E)_y(\xi) = \text{Tr}_{\wedge^k E_y}(\xi_y)$$

for any $\xi \in \text{Lie}(G)$.

Returning to our previous situation, let E be a \mathfrak{B} -linearized vector bundle on X . Then each $(x, v) \in \mathcal{Z}$ is a zero of $v\mathcal{W} - 2\mathcal{V} \in \text{Lie}(\mathfrak{B})$. This yields an element $(v\mathcal{W} - 2\mathcal{V})_x \in \text{End}(E_x)$.

LEMMA 1. *Let E be a \mathfrak{B} -linearized vector bundle on X , and let k be a nonnegative integer. Then, for any $(x, v) \in \mathcal{Z}$ we have*

$$\rho(c_k^T(E))(x, v) = \text{Tr}_{\wedge^k E_x}((v\mathcal{W} - 2\mathcal{V})_x). \quad (10)$$

As a consequence, $\rho(c_k^T(E)) \in \mathbb{C}[\mathcal{Z}]$.

Proof. It suffices to check (10) for $v \neq 0$. Let j be the index such that $x = \varphi(v^{-1}) \cdot \zeta_j$. Letting $\mathcal{W}_{\zeta_j} \in \text{End}(E_{\zeta_j})$ denote the lift of \mathcal{W} at ζ_j , we have

$$\rho(c_k^T(E))(x, v) = c_k^T(E)_{\zeta_j}(v) = v^k \text{Tr}_{\wedge^k E_{\zeta_j}}(\mathcal{W}_{\zeta_j}). \quad (11)$$

Since E is \mathfrak{B} -linearized, this equals

$$\begin{aligned} v^k \text{Tr}_{\wedge^k E_x}((\text{Ad}(\varphi(v^{-1}))\mathcal{W})_x) &= v^k \text{Tr}_{\wedge^k E_x}((\mathcal{W} - 2v^{-1}\mathcal{V})_x) && \text{see msp.9} \\ &= \text{Tr}_{\wedge^k E_x}((v\mathcal{W} - 2\mathcal{V})_x), && \square \end{aligned}$$

which proves (10). □

O.K. to add a
parens in the
beginning here?

We now obtain some of the main properties of ρ .

PROPOSITION 3. *The image of the morphism $\rho: H_{\mathfrak{T}}^*(X) \rightarrow \mathbb{C}[\mathcal{Z} - (o, 0)]$ is contained in $\mathbb{C}[\mathcal{Z}]$.*

Proof. Since X is smooth and projective, we have an exact sequence

$$0 \rightarrow zH_{\mathfrak{T}}^*(X) \rightarrow H_{\mathfrak{T}}^*(X) \rightarrow H^*(X) \rightarrow 0. \quad (12)$$

By [7, Prop. 3], the algebra $H^*(X)$ is generated by Chern classes of \mathfrak{B} -linearized vector bundles on X ; it then follows by Nakayama's lemma that $H_{\mathfrak{T}}^*(X)$ is generated (as an algebra) by their equivariant Chern classes. This together with Lemma 1 now implies that $\rho(H_{\mathfrak{T}}^*(X)) \subseteq \mathbb{C}[\mathcal{Z}]$. □

6. The First Main Result

THEOREM 1. *For a smooth projective regular variety X , the homomorphism*

$$\rho: H_{\mathfrak{T}}^*(X) \rightarrow \mathbb{C}[\mathcal{Z}]$$

is an isomorphism of graded algebras.

Proof. To see that ρ is injective, suppose $\rho(\alpha) = 0$. Then each α_{ζ_j} is 0 and so $i_{\mathfrak{T}}^*(\alpha) = 0$, where $i_{\mathfrak{T}}^*$ is the restriction (see (4)). But, as noted previously, $i_{\mathfrak{T}}^*$ is injective, so $\alpha = 0$.

To complete the proof, it suffices to check that the Poincaré series of $H_{\mathfrak{T}}^*(X)$ and $\mathbb{C}[\mathcal{Z}]$ coincide (since both algebras are positively graded, and ρ preserves the grading). Denote the former by $F_X(t)$ and the latter by $F_{\mathcal{Z}}(t)$. By (12), we have an isomorphism

$$H_{\mathfrak{T}}^*(X)/zH_{\mathfrak{T}}^*(X) \cong H^*(X). \quad (13)$$

Since $H_{\mathfrak{T}}^*(X)$ is a free $\mathbb{C}[z]$ -module and z has degree 2, (13) implies

$$F_X(t) = \frac{P_X(t)}{1-t^2}, \quad (14)$$

where $P_X(t)$ is the Poincaré polynomial of $H^*(X)$. On the other hand, since $p: \mathcal{Z} \rightarrow \mathbb{A}^1$ is finite, flat, and \mathfrak{T} -equivariant (where \mathfrak{T} acts linearly on \mathbb{A}^1 with weight 2), it follows that

$$F_{\mathcal{Z}}(t) = \frac{P_{\mathcal{Z}}(t)}{1-t^2},$$

where $P_{\mathcal{Z}}(t)$ is the Poincaré polynomial of the finite-dimensional graded algebra $\mathbb{C}[\mathcal{Z}]/(v)$. We now use the fact that the cohomology ring of X is isomorphic as a graded algebra to the coordinate ring of the zero scheme of \mathcal{V} with the principal grading [7]. So

$$H^*(X) \cong \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n)) \cong \mathbb{C}[\mathcal{Z}]/(v), \quad (15)$$

where the second isomorphism follows from Proposition 2. Thus, $P_X(t) = P_{\mathcal{Z}}(t)$, whence $F_X(t) = F_{\mathcal{Z}}(t)$. \square

For a simple example, let $X = \mathbb{P}^n$. Let $e: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ denote the nilpotent linear transformation defined by

$$v_n \rightarrow v_{n-1} \rightarrow \cdots \rightarrow v_1 \rightarrow v_0 \rightarrow 0,$$

where (v_0, v_1, \dots, v_n) is the standard basis of \mathbb{C}^{n+1} . Also, let $h = \text{diag}(n, n-2, \dots, -n+2, -n)$. Then $[h, e] = 2e$, so we obtain a \mathfrak{B} -action on \mathbb{P}^n . This action is regular with unique fixed point $[v_0] = [1, 0, \dots, 0]$; its neighborhood X_o is the standard affine chart centered at $[v_0]$. Let (x_1, \dots, x_n) be the usual affine coordinates at $[v_0]$. That is, $x_j = z_j/z_0$, where $[z_0, z_1, \dots, z_n]$ are the homogeneous coordinates on \mathbb{P}^n . Then each x_j is homogeneous of degree $2j$ and, by Proposition 2,

$$\begin{aligned} e(x_1) &= x_2 - x_1^2, \quad e(x_2) = x_3 - x_1x_2, \quad \dots, \\ e(x_{n-1}) &= x_n - x_1x_{n-1}, \quad e(x_n) = -x_1x_n. \end{aligned}$$

Thus, the ideal of \mathcal{Z} in $\mathbb{C}[x_1, \dots, x_n, v]$ is generated by $-x_2 + x_1(x_1 + v)$, $-x_3 + x_2(x_1 + 2v)$, \dots , $-x_n + x_{n-1}(x_1 + (n-1)v)$, $x_n(x_1 + nv)$. After we eliminate x_2, x_3, \dots, x_{n-1} , Theorem 1 says that

$$H_{\mathfrak{B}}^*(\mathbb{P}^n) \cong \mathbb{C}[x_1, v]/\left(\prod_{m=0}^n (x_1 + mv)\right).$$

Then, by equivariant Chern theory we have $x_1 = -c_1^{\mathfrak{B}}(L)$, where L is the tautological line bundle on \mathbb{P}^n . This presentation of $H_{\mathfrak{B}}^*(\mathbb{P}^n)$ can be derived directly and is probably well known.

7. The General Case

In this section we will make some comments on regular \mathfrak{B} -varieties, dropping the smoothness assumption. Let Y be a complex projective variety endowed with a \mathfrak{B} -action such that $Y^{\mathfrak{U}}$ is a unique point o . If Y is singular then it is necessarily singular at o , so the \mathfrak{T} -stable neighborhood Y_o of o defined in (7) is singular. Hence the results in Section 3 on the structure of \mathcal{Z} (relative to Y) do not necessarily obtain. To be able to conclude something, let us assume that Y is equivariantly embedded into a smooth projective regular \mathfrak{B} -variety X . Thus the curve \mathcal{Z} (relative to X) is well-defined and enjoys all the properties derived previously. We can therefore define

$$\mathcal{Z}_Y = \mathcal{Z} \cap (Y_o \times \mathbb{A}^1), \quad (16)$$

taking the intersection to be reduced. In other words, \mathcal{Z}_Y is the union of the components of \mathcal{Z} lying in $Y_o \times \mathbb{A}^1$. Now the construction of Section 4 yields a graded homomorphism

$$\rho_Y: H_{\mathbb{Q}}^*(Y) \rightarrow \mathbb{C}_0[\mathcal{Z}_Y].$$

If we make the additional assumption that $H^*(Y)$ is generated by Chern classes of \mathfrak{B} -linearized vector bundles, then the odd cohomology of Y is trivial, so that $H_{\mathbb{Q}}^*(Y)$ is a free module over $\mathbb{C}[z]$, and the restriction $i_{\mathbb{Q}}^*: H_{\mathbb{Q}}^*(Y) \rightarrow H_{\mathbb{Q}}^*(Y^{\mathbb{Q}})$ is injective [10]. Hence, ρ_Y is injective. By exactly the same argument, $\rho_Y(H_{\mathbb{Q}}^*(Y)) \subseteq \mathbb{C}[\mathcal{Z}_Y]$. The obstruction to ρ_Y being an isomorphism is therefore the equality of Poincaré series of these graded algebras.

There is a natural situation where this assumption is satisfied, so we can generalize Theorem 1. Assume that the inclusion $j: Y \rightarrow X$ induces a surjection $j^*: H^*(X) \rightarrow H^*(Y)$. Then $H^*(Y)$ is generated by Chern classes of \mathfrak{B} -linearized vector bundles, and from the Leray spectral sequence we see that $j_{\mathbb{Q}}^*: H_{\mathbb{Q}}^*(X) \rightarrow H_{\mathbb{Q}}^*(Y)$ is surjective. The surjectivity of j^* holds, for example, in the case of Schubert varieties in flag varieties. See also Section 8.

Proceeding as just described, we obtain an extension of Theorem 1.

THEOREM 2. *Suppose that X is a smooth projective variety with a regular \mathfrak{B} -action and that Y is a closed \mathfrak{B} -stable subvariety for which the restriction map $H^*(X) \rightarrow H^*(Y)$ is surjective. Then the map $\rho_Y: H_{\mathbb{Q}}^*(Y) \rightarrow \mathbb{C}_0[\mathcal{Z}_Y]$ yields a graded algebra isomorphism*

$$\rho_Y: H_{\mathbb{Q}}^*(Y) \rightarrow \mathbb{C}[\mathcal{Z}_Y]$$

that fits into a commutative diagram

$$\begin{array}{ccc} H_{\mathbb{Q}}^*(X) & \xrightarrow{\rho} & \mathbb{C}[\mathcal{Z}] \\ \downarrow & & \downarrow \\ H_{\mathbb{Q}}^*(Y) & \xrightarrow{\rho_Y} & \mathbb{C}[\mathcal{Z}_Y], \end{array} \quad (17)$$

where the vertical maps are the natural restrictions. Both vertical maps are surjections.

Proof. By Theorem 1 we know that ρ is an isomorphism, and by our previous remarks we know that ρ_Y is injective. Since the restriction map $\mathbb{C}[\mathcal{Z}] \rightarrow \mathbb{C}[\mathcal{Z}_Y]$ is clearly surjective, it therefore follows that ρ_Y is also. \square

As a corollary, we obtain one of the main results of [7].

COROLLARY 1. *With the notation and assumptions of Theorem 2, there exists a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{Z}]/(v) & \xrightarrow{\psi} & H^*(X) \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathcal{Z}_Y]/(v) & \xrightarrow{\psi_Y} & H^*(Y), \end{array} \quad (18)$$

where ψ and ψ_Y are graded algebra isomorphisms and the vertical maps are the natural restrictions.

Note that $\mathbb{C}[\mathcal{Z}_Y]/(v)$ is the coordinate ring of the schematic intersection of \mathcal{Z}_Y and $X \times 0$ in $X \times \mathbb{A}^1$.

8. Equivariant Cohomology of the Peterson Variety

Let G be a complex semi-simple linear algebraic group. Fix a pair of opposite Borel subgroups B and B^- and let $T = B \cap B^-$, a maximal torus of G . Denote the corresponding Lie algebras by \mathfrak{g} , \mathfrak{b} , \mathfrak{b}^- , and \mathfrak{t} , and let Φ^+ and Φ^- be the roots of the pair (G, T) that arise from \mathfrak{b} and \mathfrak{b}^- , respectively.

Let M be a B -submodule of \mathfrak{g} containing \mathfrak{b} . Then

$$M = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Omega(M)} \mathfrak{g}_\alpha, \quad \text{where } \Omega(M) = \{\alpha \in \Phi^- \mid \mathfrak{g}_\alpha \subset M\}.$$

Hence $\Omega(M)$ is the set of weights of the quotient M/\mathfrak{b} . The B -module $M \subseteq \mathfrak{g}$ yields a homogeneous vector bundle $G \times^B M$ over G/B together with a morphism $G \times^B M \rightarrow \mathfrak{g}$ induced by $(g, m) \mapsto gm$. The fiber of this morphism at an arbitrary $x \in \mathfrak{g}$ identifies with

$$Y_M(x) = \{gB \in G/B \mid g^{-1}x \in M\}.$$

This is a closed subvariety of G/B that is stable under the action of $G_{\mathbb{C}x}$ (the isotropy group of the line $\mathbb{C}x$).

If x is regular and semi-simple, then $Y_M(x)$ is known as a *Hessenberg variety*. That $Y_M(x)$ is nonsingular was shown in [8], where its Poincaré polynomial was also determined.

On the other hand, if x is nilpotent then the $Y_M(x)$ give a broad class of projective varieties containing, for example, the variety $\mathcal{B}_x \subset G/B$ of Borel subgroups of G whose Lie algebra contains x . If x is also regular then it defines a canonical \mathfrak{B} -action on G/B stabilizing $Y_M(x)$. The cohomology of the $Y_M(x)$ has been obtained by Dale Peterson (unpublished). In fact, according to Peterson's result, the $Y_M(x)$ satisfy the hypotheses of Theorem 2.

To describe this cohomology, we first make a convenient choice for x . Let $\alpha_1, \dots, \alpha_l \in \Phi^+$ denote the simple roots, and fix a nonzero e_{α_j} in each T -weight space $\mathfrak{g}_{\alpha_j} \subset \mathfrak{b}$. Then for x choose the principal nilpotent $e = \sum_{j=1}^l e_{\alpha_j}$. Let $h \in \mathfrak{t}$ be the unique element for which $\alpha_j(h) = 2$ for each index j . Then the pair $(e, h) \in \mathfrak{b} \times \mathfrak{t}$ determines a regular \mathfrak{B} -action on G/B with unique \mathfrak{B} -fixed point $o = B$ that stabilizes $Y_M(e)$ for any M . Note that the semi-simple element h is also regular.

Let $\mathfrak{T} \subset \mathfrak{B}$ denote the maximal torus in \mathfrak{B} with Lie algebra $\mathbb{C}h$. Since h is regular, $Y_M(e)^{\mathfrak{T}} = (G/B)^{\mathfrak{T}} \cap Y_M(e)$. It is well known that $(G/B)^{\mathfrak{T}} = \{n_w B \mid w \in W\}$, where $W = N_G(T)/T$ is the Weyl group of (G, T) and n_w is a representative of w . But $n_w B \in Y_M(e)^{\mathfrak{T}}$ if and only if $n_w^{-1}e \in M$; that is, $w^{-1}(\alpha_j) \in \Omega(M) \cup \Phi^+$ for each j .

Formulated in these terms, Peterson's result can be stated as follows.

see msp.15

"To describe this cohomology" o.k.?

THEOREM 3. *For any B -submodule M of \mathfrak{g} containing \mathfrak{b} , the restriction map*

$$H^*(G/B) \rightarrow H^*(Y_M(e))$$

is surjective. Hence, by Theorem 2,

$$H_{\mathbb{Z}}^*(Y_M(e)) \cong \mathbb{C}[\mathcal{Z}_{Y_M(e)}].$$

Moreover, $\mathcal{Z}_{Y_M(e)}$ is a complete intersection. Thus, the cohomology ring $H^(Y_M(e))$ satisfies Poincaré duality. Its Poincaré polynomial is given by the product formula*

$$P(Y_M(e), t) = \prod_{-\alpha \in \Omega(M)} \frac{1 - t^{ht(\alpha)+1}}{1 - t^{ht(\alpha)}}, \quad (19)$$

where $ht(\alpha)$ denotes the sum of the coefficients of $\alpha \in \Phi^+$ over the simple roots.

If $M = \mathfrak{g}$ then (19) is, of course, a well-known product formula for the Poincaré polynomial of the flag variety.

The ideals $I(\mathcal{Z}_{Y_M(e)})$ admit explicit expressions related to the geometry of the nilpotent variety of \mathfrak{g} . Let $U^- \subset G$ denote the unipotent radical of B^- . Since the natural map $\mu: U^- \rightarrow G/B$, $\mu(u) = uB$, is T -equivariant with respect to the conjugation action of T on U^- , we may equivariantly identify U^- and the open cell $(G/B)_o$. Let \mathfrak{u}^- denote the Lie algebra of U^- , and let $\Pi_*: \mathfrak{g} \rightarrow \mathfrak{u}^-$ denote the projection. To get this explicit picture, we first note that $L_{u^{-1}*}\mathcal{V}_u = \Pi_*(u^{-1}e)$, where L_u denotes left translation by $u \in U^-$ and L_{u*} is its differential (cf. [6, Sec. 2.1]). Consequently,

$$L_{u^{-1}*}\mathcal{A}_{(u,v)} = 2\Pi_*(u^{-1}e) - vL_{u^{-1}*}\mathcal{W}_u. \quad (20)$$

Defining $F_\alpha(u)$ as the component of $L_{u^{-1}*}\mathcal{A}_{(u,v)}$ in \mathfrak{g}_α , it follows that

$$I(\mathcal{Z}) = (F_\alpha \mid \alpha \in \Phi^-).$$

More precisely, we can write

$$u^{-1}e = e + k(u) + \sum_{\alpha \in \Phi^-} v_\alpha(u)e_\alpha,$$

where $k \in \mathfrak{t} \otimes \mathbb{C}[U^-]$ and all $v_\alpha \in \mathbb{C}[U^-]$. Then $F_\alpha = 2v_\alpha - vw_\alpha$, where we have put $w_\alpha(u) = (L_{u^{-1}*}\mathcal{W}_u)_\alpha$. Hence,

$$H_{\mathbb{Z}}^*(G/B, \mathbb{C}) \cong \mathbb{C}[U^- \times \mathbb{A}^1]/(2v_\alpha - vw_\alpha \mid \alpha \in \Phi^-).$$

Since the functions v_β ($\beta \in \Phi^- - \Omega(M)$) cut out $Y_M(e)_o \times \mathbb{A}^1$, it follows that $I(\mathcal{Z}_{Y_M(e)})$ is the ideal of the variety cut out by the $2v_\alpha - vw_\alpha$ (where $\alpha \in \Phi^-$) and the v_β (where $\beta \in \Phi^- - \Omega(M)$).

The case where

$$M = \mathfrak{b} \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ \alpha(h) > 2}} \mathfrak{g}_{-\alpha} \quad (21)$$

is of particular interest. Then, as shown by Kostant (see [11, (9)]), the Givental–Kim and Peterson formulas for the flag variety quantum cohomology [9; 12] may be interpreted as asserting an isomorphism of graded rings

$$\mathbb{C}[Y_M(e)_o] \cong QH^*(G^\vee/B^\vee),$$

where G^\vee and B^\vee denote the Langlands duals of G and B (respectively) and $QH^*(G^\vee/B^\vee)$ is the complex quantum cohomology ring of G^\vee/B^\vee . This led Kostant to call $Y_M(e)$ the *Peterson variety*. Notice that, by Theorem 3, $P(Y_M(e), t) = (1+t)^{n-1}$.

More recently, Tymoczko [13] has shown that the varieties $Y_M(e)$ admit affine cell decompositions when G is of classical type.

9. An Alternative Proof of Theorem 1

We now give another proof of Theorem 1 that is independent of (15). Hence we will also obtain another proof of (15). Denote by $[X^\mathfrak{T}]_{\mathfrak{T}} \in H_{\mathfrak{T}}^*(X)$ the equivariant cohomology class of the \mathfrak{T} -fixed-point set. We compute its image under $\rho: H_{\mathfrak{T}}^*(X) \rightarrow \mathbb{C}[\mathcal{Z}]$.

LEMMA 2. $\rho([X^\mathfrak{T}]_{\mathfrak{T}})$ is the restriction to \mathcal{Z} of the Jacobian determinant of the polynomial functions $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$ in the variables x_1, \dots, x_n .

Proof. Let T_X be the tangent bundle to X . Then \mathcal{W} yields a \mathfrak{T} -invariant global section of T_X with zero scheme $X^\mathfrak{T}$. Thus, we have $c_n^\mathfrak{T}(T_X) = [X^\mathfrak{T}]_{\mathfrak{T}}$ in $H_{\mathfrak{T}}^*(X)$. On the other hand, T_X carries a \mathfrak{B} -linearization and so, by Lemma 1, we have

$$\rho(c_n^\mathfrak{T}(T_X))(x, v) = \text{Tr}_{\wedge^n T_X} (v\mathcal{W}_x - 2\mathcal{V}_x).$$

This is the Jacobian determinant of $v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$. □

We will also need the following easy result of commutative algebra; we will provide a proof, for lack of a reference.

LEMMA 3. Let P_1, \dots, P_n be polynomial functions in x_1, \dots, x_n that are weighted homogeneous for a positive grading defined by $\deg x_j = a_j$. If the origin is the unique common zero to P_1, \dots, P_n , then the Jacobian determinant $J(P_1, \dots, P_n)$ is not in the ideal (P_1, \dots, P_n) .

Proof. We begin with the special case where P_1, \dots, P_n are homogeneous. We argue by induction on n , the result being evident for $n = 1$. Let d_1, \dots, d_n be the degrees of P_1, \dots, P_n . We have the Euler identities

$$\begin{aligned} x_1 \frac{\partial P_1}{\partial x_1} + \dots + x_n \frac{\partial P_1}{\partial x_n} &= d_1 P_1, \\ &\vdots \\ x_1 \frac{\partial P_n}{\partial x_1} + \dots + x_n \frac{\partial P_n}{\partial x_n} &= d_n P_n, \end{aligned}$$

which we view as a system of linear equalities in x_1, \dots, x_n . The determinant of this system is the Jacobian $J(P_1, \dots, P_n) = J$. For $1 \leq i, j \leq n$, let $J_{i,j}$ be its maximal minor associated with the i th line and the j th column. Then we have

see msp.19

"The determinant of this system" o.k.?

$$x_i J = \sum_{j=1}^n (-1)^{j-1} d_j P_j J_{ij}. \quad (22)$$

Assume that $J \in (P_1, \dots, P_n)$ and write

$$J = f_1 P_1 + \dots + f_n P_n, \quad \text{with } f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n].$$

Using (22) for $i = 1$, it follows that $(x_1 f_1 - d_1 J_{11}) P_1$ is in the ideal (P_2, \dots, P_n) . But P_1, \dots, P_n form a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$, since they are homogeneous and the origin is their unique common zero. Therefore, $x_1 f_1 - d_1 J_{11} \in (P_2, \dots, P_n)$. In other words, $J_{11} \in (x_1, P_2, \dots, P_n)$.

After a linear change of coordinates, we can assume that (x_1, P_2, \dots, P_n) is a regular sequence. For $2 \leq i \leq n$, let

$$Q_i(x_2, \dots, x_n) = P_i(0, x_2, \dots, x_n).$$

Then, in $\mathbb{C}[x_2, \dots, x_n]$ we have

$$J_{11}(0, x_2, \dots, x_n) \in (Q_2, \dots, Q_n).$$

Now $J_{11}(0, x_2, \dots, x_n) = J(Q_2, \dots, Q_n)$, and Q_2, \dots, Q_n are homogeneous polynomial functions of x_2, \dots, x_n having the origin as their unique common zero. But this contradicts the inductive assumption, which completes the proof in the homogeneous case.

Consider now the case where P_1, \dots, P_n are quasi-homogeneous for the weights a_1, \dots, a_n . Let y_1, \dots, y_n be indeterminates; then the functions

$$(y_1, \dots, y_n) \mapsto P_i(y_1^{a_1}, \dots, y_n^{a_n})$$

($1 \leq i \leq n$) are homogeneous polynomials with the origin as their unique common zero. Their Jacobian determinant is

$$\left(\prod_{i=1}^n a_i y_i^{a_i-1} \right) J(P_1, \dots, P_n)(y_1^{a_1}, \dots, y_n^{a_n}).$$

By the first step of the proof, this function of (y_1, \dots, y_n) is not in the ideal generated by the $P_i(y_1^{a_1}, \dots, y_n^{a_n})$. Thus, $J(P_1, \dots, P_n)(x_1, \dots, x_n)$ cannot be in (P_1, \dots, P_n) . \square

Proof of Theorem 1. The functions $\mathcal{V}(x_1), \dots, \mathcal{V}(x_n)$ satisfy the assumption of Lemma 3 because (a) they are quasi-homogeneous for the grading defined by the action of \mathfrak{T} and (b) o is the unique zero of \mathcal{V} . Thus, the Jacobian determinant of these functions is not in the ideal that they generate. By Lemma 2, it follows that $\rho([X^{\mathfrak{T}}]_{\mathfrak{T}})$ is not divisible by v in $\mathbb{C}[\mathcal{Z}]$ for $\mathbb{C}[\mathcal{Z}]/(v) = \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n))$.

The $\mathbb{C}[\mathcal{z}]$ -linear map $\rho: H_{\mathfrak{T}}^*(X) \rightarrow \mathbb{C}[\mathcal{Z}]$ defines a map

$$\bar{\rho}: H^*(X) \cong H_{\mathfrak{T}}^*(X)/(z) \rightarrow \mathbb{C}[\mathcal{Z}]/(v),$$

a graded ring homomorphism. It suffices to prove that $\bar{\rho}$ is an isomorphism. Note that the spaces $H^*(X)$ and $\mathbb{C}[\mathcal{Z}]/(v)$ have dimension r , so it suffices to check the injectivity of $\bar{\rho}$.

The image in $H^*(X)$ of $[X^{\mathbb{Z}}]_{\mathbb{Z}}$ is $r[\text{pt}]$, where $[\text{pt}]$ denotes the cohomology class of a point. Thus, $\bar{\rho}([\text{pt}]) \neq 0$. Let $(\alpha_1, \dots, \alpha_r)$ be a basis of $H^*(X)$ consisting of homogeneous elements, and let $(\beta_1, \dots, \beta_r)$ be the dual basis for the intersection pairing $(\alpha, \beta) \mapsto \int_X (\alpha \cup \beta) \cap [X]$. Then the homogeneous component of degree $2n$ in each product $\alpha_i \cup \beta_j$ equals $[\text{pt}]$ if $i = j$ and is zero otherwise. Assume that $\bar{\rho}(t_1\alpha_1 + \dots + t_r\alpha_r) = 0$ for some complex numbers t_1, \dots, t_r . Multiplying by $\bar{\rho}(\beta_j)$ and taking the homogeneous component of degree $2n$, we obtain $t_j\bar{\rho}([\text{pt}]) = 0$, whence $t_j = 0$. Thus $\bar{\rho}$ is injective, and the proof is complete. \square

10. The Equivariant Push-Forward

Next we describe the equivariant push-forward map

$$\int_X : H_{\mathbb{Z}}^*(X) \rightarrow H_{\mathbb{Z}}^*(\text{pt}) = \mathbb{C}[z]$$

associated to the map $X \rightarrow \text{pt}$. Note that \int_X is $\mathbb{C}[z]$ -linear and homogeneous of degree $-2n$. Denote by J the restriction to \mathcal{Z} of the Jacobian determinant of the polynomial functions

$$v\mathcal{W}(x_1) - 2\mathcal{V}(x_1), \dots, v\mathcal{W}(x_n) - 2\mathcal{V}(x_n)$$

in the variables x_1, \dots, x_n . Then J is homogeneous of degree $2n$, as follows from Lemma 2.

THEOREM 4. *For any $f \in \mathbb{C}[\mathcal{Z}]$, the function*

$$v \mapsto \sum_{(x,v) \in \mathcal{Z}} \frac{f(x,v)}{J(x,v)}$$

is polynomial. Furthermore, for any $\alpha \in H_{\mathbb{Z}}^(X)$,*

$$\left(\int_X \alpha \right) (v) = \sum_{(x,v) \in \mathcal{Z}} \frac{\rho(\alpha)(x,v)}{J(x,v)}. \quad (23)$$

Proof. Since $\mathbb{C}[\mathcal{Z}]$ is a graded free $\mathbb{C}[v]$ -module of rank r , we may choose a homogeneous basis f_1, \dots, f_r with $f_1 = 1$. As $J \notin (v)$, we may also assume $f_r = J$. If $f \in \mathbb{C}[\mathcal{Z}]$, let $\varphi(f)$ be the r th coordinate of f in this basis. Then the map

$$\mathbb{C}[\mathcal{Z}] \times \mathbb{C}[\mathcal{Z}] \rightarrow \mathbb{C}[v], \quad (f, g) \mapsto \varphi(fg)$$

is a nondegenerate bilinear form, since it reduces modulo (v) to the duality pairing on $\mathbb{C}[\mathcal{Z}]/(v) = \mathbb{C}[x_1, \dots, x_n]/(\mathcal{V}(x_1), \dots, \mathcal{V}(x_n))$. If (g_1, \dots, g_r) is the dual basis with respect to (f_1, \dots, f_r) for this bilinear form, then g_1, \dots, g_r are homogeneous and satisfy $\deg(f_i) + \deg(g_i) = 2n$ for all i . As a consequence, the kernel of φ is generated by f_1, \dots, f_{r-1} and also by g_2, \dots, g_r as a $\mathbb{C}[z]$ -module. Since $J = f_r$, we have $\varphi(Jg_1) = \dots = \varphi(Jg_{r-1}) = 0$, so that $\varphi(Jf_2) = \dots = \varphi(Jf_r) = 0$ whereas $\varphi(Jf_1) = \varphi(J) = 1$.

Let

$$\mathrm{Tr}: \mathbb{C}[\mathcal{Z}] \rightarrow \mathbb{C}[v]$$

be the trace map for the (finite, flat) morphism $p: \mathcal{Z} \rightarrow \mathbb{A}^1$. Then

$$\mathrm{Tr}(f)(v) = \sum_{(x,v) \in \mathcal{Z}} f(x, v)$$

for all $v \in \mathbb{A}^1$; in particular, $\mathrm{Tr}(1) = r$. Since Tr is homogeneous of degree 0 and $\mathbb{C}[\mathcal{Z}]$ -linear, its kernel is a graded complement of the $\mathbb{C}[v]$ -module $\mathbb{C}[v] = \mathbb{C}[v]f_1$ in $\mathbb{C}[\mathcal{Z}]$. It follows that this kernel is generated by f_2, \dots, f_r . Thus, we have

$$\mathrm{Tr}(g) = r\varphi(Jg)$$

for all $g \in \mathbb{C}[\mathcal{Z}]$. This equality holds then for all $g \in \mathbb{C}[\mathcal{Z}][v^{-1}]$ and hence for all rational functions on \mathcal{Z} . Note that J restricts to a nonzero function on any component of \mathcal{Z} , so $1/J$ is a rational function on \mathcal{Z} . We thus have

$$\mathrm{Tr}(f/J) = r\varphi(f)$$

for any $f \in \mathbb{C}[\mathcal{Z}]$; this implies the first assertion. The second assertion follows from the localization theorem in equivariant cohomology. \square

11. Some Concluding Remarks

If $\alpha \in H_{\mathbb{C}}^*(X)$ is a product of equivariant Chern classes, then (23) is the equivariant Bott residue formula for regular \mathfrak{B} -varieties (cf. [4]).

Using similar methods, one can also extend (23) to obtain a formula for the equivariant Gysin homomorphism associated to an equivariant morphism of regular smooth projective B -varieties (cf. [1] for the case of flag varieties).

More generally, consider a smooth projective variety X with an action of an arbitrary torus T such that X^T is finite. Then the precise version of the localization theorem given in [10] yields a reduced affine scheme whose coordinate ring is the equivariant cohomology ring of X (see [5] for details and applications). The scheme \mathcal{Z} in this paper gives a more explicit picture, but the requirement of a regular \mathfrak{B} -action is harder to satisfy. It would be nice to be able to relax the regularity assumption and so allow X^{ul} to have positive dimension.

References

- [1] E. Akyildiz and J. B. Carrell, *An algebraic formula for the Gysin homomorphism from G/B to G/P* , Illinois J. Math. 31 (1987), 312–320.
- [2] ———, *A generalization of the Kostant–Macdonald identity*, Proc. Nat. Acad. Sci. U.S.A. 86 (1989), 3934–3937.
- [3] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) 98 (1973), 480–497.
- [4] R. Bott, *Vector fields and characteristic numbers*, Michigan Math. J. 14 (1967), 231–244.

- [5] M. Brion, *Poincaré duality and equivariant (co)homology*, Michigan Math. J. 48 (2000), 77–92.
- [6] J. B. Carrell, *Bruhat cells in the nilpotent variety and the intersection rings of Schubert varieties*, J. Differential Geom. 37 (1993), 651–668.
- [7] ———, *Deformation of the nilpotent zero scheme and the intersection ring of invariant subvarieties*, J. Reine Angew. Math. 460 (1995), 37–54.
- [8] F. DeMari, C. Procesi, and M. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. 332 (1992), 529–534.
- [9] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. 168 (1995), 609–641.
- [10] R. M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. 131 (1998), 25–84.
- [11] B. Kostant, *Flag manifold quantum cohomology, the Toda lattice and the representation with highest weight*, New Series Selecta Math. 2 (1996), 43–91.
- [12] D. Peterson, *Quantum cohomology of G/P* , unpublished manuscript.
- [13] J. Tymoczko, *Decomposing nilpotent Hessenberg varieties over classical groups*, preprint, arXiv:math.AG/0211226.

M. Brion
Université de Grenoble I
Département de Mathématiques
Institut Fourier
UMR 5582 du CNRS
38402 Saint-Martin d’Hères Cedex
France

Michel.Brion@ujf-grenoble.fr

J. B. Carrell
Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Z2
Canada
carrell@math.ubc.ca